Probability and analysis on convex bodies

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joint work with **Emanuel Milman** (Technion, Haifa)

Main object of our study

Uniform distributions on convex bodies

Log-concave distributions

Definition

Probability measure μ on $\mathbb{R}^d$ is log-concave if and only if

$$\mu(tA + (1-t)B) \geq \mu^t(A)\mu^{1-t}(B), \quad 0 \leq t \leq 1,$$

A, B are compact sets in $\mathbb{R}^d$.

Equivalently

μ is supported on some linear k-dimensional subspace $E \subset \mathbb{R}^d$ where it has the form

$$\mu = e^{-V} \cdot \mathcal{H}^k,$$

where V is convex and $\mathcal{H}^k$ is the Hausdorff measure on E.

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Poincaré inequality

We say that a probability measure μ satisfies the Poincaré inequality if

$$\mathrm{Var}_\mu f = \int f^2 d\mu - \left(\int f d\mu\right)^2 \leq C_\mu \int |\nabla f|^2 d\mu.$$

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Basic properties

1) Stability under bounded perturbations (**Holley, Stroock, 80's**)
   If $\nu = e^f \cdot \mu$, then $C_\nu \leq e^{\sup f - \inf f} C_\mu$
2) Tensorisation
   $$C_\mu \leq \max_{1 \leq i \leq N} C_{\mu_i},$$
   where $\mu = \mu_1 \times \mu_2 \cdots \times \mu_N$
3) Exponential concentration
   There exists C depending on C_μ such that for every 1-Lipschitz function f
   $$\mu\Big(\Big|f - \int f d\mu\Big| > t\Big) \leq e^{-ct}.$$
   In particular, $\int e^{\rho|x|} d\mu < \infty$ for some $\rho > 0$.

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Examples

Brascamp-Lieb inequality

Theorem

(Brascamp, Lieb) *Let $\mu = e^{-V}dx$ be a log-concave measure with twice continuously defferentiable V. Then*

$$\mathrm{Var}_\mu f \leq \int \langle (D^2V)^{-1}\nabla f, \nabla f\rangle \, d\mu.$$

Corollary

If $D^2V \geq K \cdot \mathrm{Id}$, then

$$C_\mu \leq \frac{1}{K}.$$

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(Bobkov) *Any log-concave measure satisfies the Poincaré inequality.*

Convex bodies

Theorem

(**Payne, Weinberger, 50's**) *Let K be a convex body. Then its Poncaré constant C_K satisfies*

$$C_K \leq \frac{4}{\pi^2} \cdot \mathrm{diam}^2(K).$$

This theorem

1) can not be improved (think of thin cylinder) in the class of ALL convex bodies

2) gives very rough estimate for concrete examples!

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l_p-balls

$$B_p = \{x : |x_1|^p + |x_2|^p + \cdots |x_d|^p \leq 1\},\ 1 \leq p \leq \infty.$$

Theorem

(Sodin; Schechtman & Zinn; Latala & Wojtaszczyk ...)

$$C_{B_p} \leq \frac{c(p)}{d^{\frac{2}{p}}}.$$

This is much better than the general Payne-Weinberger estimate.

Motivating problem: KLS conjecture

Conjecture

Kannan, Lovasz, Simonovits (1995). There exists an universal number c such that for every uniform distribution μ on a convex body K satisfying

$$\mathbb{E}_\mu x_i = 0, \ \mathbb{E}_\mu (x_i x_j) = \delta_{ij}.$$

one has

$$C_\mu \leq c.$$

We call such bodies **isotropic**.

Other related conjectures and results

1) **Hyperplane conjecture.** (**Bourgain**). There exists universal $c > 0$ such that for any convex set $K \subset \mathbb{R}^d$ of $\mathrm{Vol(K)} = 1$ there exists a hyperplane L such that $\mathrm{Vol(K \cap L) > c}$.
2) **Thin-shell conjecture.** This is KLS conjecture for concrete function $f = |X|$. Can be put in the following way:

$$\mathbb{E}_\mu((|X| - \sqrt{d})^2) \leq c$$

for some universal constant c and every uniform distibution μ on an isotropic convex body $K \subset \mathbb{R}^d$.
3) **Central limit theorem for convex bodies**. Estimates of the thin-shell type $\Longrightarrow$ central limit theorem for convex bodies (initiated by in works of **Sudakov** (70's), recent result: **Klartag, 2006**).

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Starting point: Brascamp-Lieb inequality for manifolds

Theorem

Let M be complete Riemannian manifold with metric g and V be twice continuously differentiable function on M. Assume that $\mu = e^{-V} \cdot \mathrm{Vol}_g$ is a probability measure and its **Bakry-Emery** *tensor $B = D^2 V + \mathrm{Ricci}_g$ is positive. Then*

$$\mathrm{Var}_\mu f \leq \int g(B^{-1}\nabla f, \nabla f)\, d\mu.$$

Approach:

Given a convex set K find a (complete) Riemannian metric g on a convex set K such that

$$B \geq c \cdot g_0,$$

where g_0 is the standard Euclidean metric on K. Then $C_K \leq \frac{1}{c}$.

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Brascamp-Lieb inequality for manifolds with boundary

Consider compact manifold with boundary ∂M. Notations: n is the unit normal vector to ∂M, II is the second fundamental form of ∂M, H_g is the mean curvature of ∂M, $H = H_g - \partial_n V$ its *generalized mean curvature*. If $H > 0$ we set:
$\mu^+ = \frac{1}{H} e^{-V} \cdot Vol_{g|_{\partial M}}$.

Theorem

*(***K., Milman***) Assume that $B > 0$ on M.*
If $\mathrm{II} \geq 0$, then

$$\mathrm{Var}_\mu(f) \leq \int_M g(B^{-1}\nabla f, \nabla f) d\mu.$$

If $H > 0$, then

$$\mathrm{Var}_\mu(f) \leq \int_M g(\mathrm{B}^{-1}\nabla f, \nabla f) d\mu + \mathrm{Var}_{\mu^+}(f|_{\partial M}).$$

Interesting byproduct: infinitesimal Brunn-Minkowski inequality

Theorem

(**Colesanti, 2007**) *Let $M \subset \mathbb{R}^d$ be a convex body. Then the following inequality holds for any $f \in C^1(\partial M)$:*

$$\int_{\partial M} Hf^2 d\mu - \frac{d-1}{d} \frac{\left(\int_{\partial M} f d\mu\right)^2}{\mathrm{Vol}(M)} \leq \int_{\partial M} \langle \mathrm{II}^{-1} \nabla_{\partial M} f, \nabla_{\partial M} f \rangle d\mu,$$

where $\mu = \mathcal{H}^{d-1}|_{\partial M}$

Kolesnikov, Milman (2013): generalization to manifolds.
Colesanti: proof by differentiating the BM inequality; in Euclidean case this inequality is equivalent to the BM inequality. Thus we have a kind on **infinitesimal** version of the BM inequality on manifolds. An integrated version can be obtained by construction of some related **curvature flow** (work in progress ...).

Non-Euclidean metrics on convex sets

Diagonal metrics

Consider probability measure $\mu = e^{-V} dx$ on $(\mathbb{R}^+)^d$ and the following metric

$$g = \sum_{i=1}^{d} \frac{dx_i^2}{x_i}.$$

Theorem

$$\mathrm{Var}_\mu f \le \int \langle \mathrm{B}^{-1} \nabla f, \nabla f \rangle \, d\mu,$$

provided $\mathrm{B} = D^2 V + \mathrm{diag}\left\{ \frac{V_{x_i}}{2x_i} + \frac{1}{4x_i^2} \right\} \ge 0.$

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Corollary

*(**Klartag**) Assume that μ is the normalized Lebesgue measure on a convex set $\Omega \subset [0,\infty)^d$. Assume that every outer normal of Ω has only non-negative coordinates. Then*

$$\mathrm{Var}_\mu f \leq 4 \int \sum_{i=1}^{d} x_i^2 f_{x_i}^2 \, d\mu.$$

This result implies the **thin shell estimate** for **unconditional** sets (**Klartag**).

Conformal change of metrics

Let $B_p = \{|x_1|^p + \cdots + |x_d|^p \leq 1\}$ be the unit l_p-ball, $1 < p \leq 2$.

Theorem

*(**K., Milman**). The space $\mu = e^{-\sum_{i=1}^d |x_i|^p} I_{B_p}$, $g = e^{-\sum_{i=1}^d |x_i|^p} \sum_{i=1}^d dx_i^2$ has the Bakry Emery tensor B satisfying*

$$B \geq \frac{c_p}{d^{\frac{2}{p}}} g_0.$$

This gives another proof of the KLS conjecture for B_p.

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This gives another proof of the KLS conjecture for B_p.

Consider a convex set $K \subset B_{\sqrt{d}/2} \subset \mathbb{R}^d$ containing the origin. Applying conformal metric

$$g = e^{-\frac{1}{2d}\sum_{i=1}^d x_i^2} \sum_{i=1}^{d} dx_i^2$$

and the Brascamp-Lieb inequality for manifolds with boundary we get the following estimate.

Theorem

(K., Milman) *There exists universal c such that every C^1-function f satisfies the following inequality*

$$\mathrm{Var}_\mu f \leq c \int_K |\nabla f|^2 d\mu + \frac{1}{\mathrm{Vol}(K)} \int_{\partial K} \frac{1}{H}(f - C)^2 \, d\mathcal{H}^{d-1},$$

where C is arbitrary constant, $H = H_0 + \frac{\langle x, n\rangle}{d}$, and H_0 is the Euclidean mean curvature of ∂K.

Hessian metrics

These are metrics of the form $g = D^2\Phi$, where Φ is a convex function. Motivation comes from the **optimal transportation** theory and the theory of the **Monge-Ampére** equation.

Theorem

(K) *Let μ and ν be log-concave probability measure on $\mathbb{R}^d$ and $\nabla\Phi$ be the quadratic optimal transportation of μ onto ν. Then the space $(g = D^2\Phi, \mu)$ has a positive Bakry-Emery tensor.*

Suggestion of Klartag

A special form of the **Monge-Ampére** equation

$$e^{-\Phi} = \det D^2\Phi$$

and the related transportation problem must have interesting applications to convex bodies.

This is the so-called **Kähler-Einstein equation** well-known in differential geometry.

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Klartag, Eldan (2010) applied Hessian metrics to prove the following statement:

$$\text{this shell conjecture} \Longrightarrow \text{hyperplane conjecture.}$$

Eldan (2011) has shown that positive solution to **thin shell conjecture** implies positive solution to **KLS conjecture** up to $\log d$-factor.

Another example: Cauchy distribution

$$\mu_\beta = \frac{1}{Z}(1+|x|^2)^{-\beta},\ \beta > \frac{d}{2}.$$

Bobkov and Ledoux (2009) have shown

$$\mathrm{Var}_{\mu_\beta} f \le \frac{C}{\beta} \int |\nabla f|^2 (1+|x|^2) d\mu_\beta.$$

This is the standard Poincaré inequality on the projective space, because Cauchy distribution is the Riemannian volume on RP^n.