

NILPOTENT SUB-RIEMANNIAN PROBLEM ON THE ENGEL GROUP

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The following sub-Riemannian problem is considered:

$$\begin{aligned} \dot{q} &= u_1 X_1 + u_2 X_2, \quad q = (x, y, z, v)^T \in M = \mathbf{R}^4, \quad (u_1, u_2) \in \mathbf{R}^2, \\ X_1 &= \left(1, 0, -\frac{y}{2}, 0\right)^T, \quad X_2 = \left(0, 1, \frac{x}{2}, \frac{x^2 + y^2}{2}\right)^T, \\ q(0) &= q_0 = (0, 0, 0, 0)^T, \quad q(t_1) = q_1, \\ l &= \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min. \end{aligned}$$

It arises as a nilpotent approximation to nonholonomic systems in four-dimensional space with two-dimensional control, for instance for the system describing motion of a mobile robot with a trailer on a plane.

Vector fields at the controls X_1, X_2 generate four-dimensional nilpotent Lie algebra called the Engel algebra [1]. $X_1, X_2, X_3 = [X_1, X_2], X_4 = [X_1, X_3]$ are basis left invariant fields on the Engel group M [2]. The system is completely controllable by Rashevskii–Chow theorem [3]. Existence of optimal solutions is implied by Filippov theorem.

Pontryagin’s maximum principle has been applied. Projections of abnormal extremals on the plane XY are straight lines. Family of all normal extremals is parametrized by the phase cylinder of pendulum

$$C = T_{q_0}^* M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1; c, \alpha \in \mathbf{R}\},$$

where H is the Hamiltonian function.

Adjoint subsystem of the Hamiltonian system is reduced to the equation of pendulum:

$$\ddot{\theta} = -\alpha \sin \theta, \quad \alpha = \text{const}.$$

The cylinder C has the stratification by value of the energy integral. Every subset of the cylinder corresponds to the particular type of trajectories of the pendulum. Hamiltonian system has been integrated in every case [4], thus exponential mapping is defined as:

$$\text{Exp}: N \rightarrow M, \quad N = C \times \mathbf{R}_+.$$

Discrete symmetries of exponential mapping have been considered in order to find the first Maxwell time which gives upper bound for the cut time (i. e., the time of loss of global optimality) along extremal trajectories:

$$t_{\text{cut}}(\lambda) \leq t_{\text{MAX}^1}(\lambda).$$

Moreover, the first conjugate time (i. e., the time of loss of local optimality) along the trajectories has been investigated [5]. The function that gives the upper bound of the cut time provides the lower bound of the first conjugate time:

$$t_{\text{MAX}^1}(\lambda) \leq t_{\text{conj}^1}(\lambda).$$

So the first Maxwell time defines the decomposition of the preimage and the image of the exponential mapping into corresponding subdomains. Hadamard theorem about global diffeomorphism has been applied to prove that restriction of the exponential mapping for these subdomains is a diffeomorphism. Finally the following theorem has been proved.

Theorem. *For any $\lambda \in C$*

$$t_{\text{cut}}(\lambda) = t_{\text{MAX}^1}(\lambda)$$

On the basis of the results obtained, a software for numerical computation of a global solution to the sub-Riemannian problem on the Engel group has been developed.

References

1. *R. Montgomery.* A tour of subriemannian geometries, their geodesics and applications. Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002.
2. *Yu.L. Sachkov.* Controlability and symmetries of invariant systems on Lie groups and homogeneous spaces, in Russian. Fizmatlit Publishers, 2007.
3. *A.A. Agrachev, Yu.L. Sachkov.* Control Theory from the Geometric Viewpoint. Springer, 2004.
4. *Yu.L. Sachkov, A.A. Ardentov.* Extremal trajectories in a nilpotent sub-Riemannian problem on the Engel group. Sbornik: Mathematics. 2011. 202, 11. 31–54.
5. *Yu.L. Sachkov, A.A. Ardentov.* Conjugate points in nilpotent sub-Riemannian problem on the Engel group. Journal of Mathematical Sciences. 2013. 195, 3. 369–390