Polynomial Integrals of the Geodesics Equations in Two-Dimensional Case

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Let M^2 be two-dimensional surface with the Riemannian metric

$$ds^2 = g_{11}(x,y)dx^2 + 2g_{12}(x,y)dxdy + g_{22}(x,y)dy^2. \tag{1}$$

Geodesics equations of a given metric can be treated as a system of Euler-Lagrange equations

$$\frac{d}{dt}L_{\dot{x}} - L_x = 0, \qquad \frac{d}{dt}L_{\dot{y}} - L_y = 0 \tag{2}$$

with the Lagrangian

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}g_{11}(x,y)\dot{x}^2 + g_{12}(x,y)\dot{x}\dot{y} + \frac{1}{2}g_{22}(x,y)\dot{y}^2. \tag{3}$$

Geodesic flow of the metric (1) is Liouville integrable, if it possesses a smooth first integral F functionally independent of the Lagrangian (3). In the present work we consider the problem of the existence of the polynomial integral of the system (2), (3) of the first degree

$$F_1 = b_0(x,y)\dot{x} + b_1(x,y)\dot{y}, \tag{4}$$

the second degree

$$F_2 = b_0(x,y)\dot{x}^2 + 2b_1(x,y)\dot{x}\dot{y} + b_2(x,y)\dot{y}^2 \tag{5}$$

and the third degree

$$F_3 = b_0(x,y)\dot{x}^3 + 3b_1(x,y)\dot{x}^2\dot{y} + 3b_2(x,y)\dot{x}\dot{y}^2 + b_3(x,y)\dot{y}^3. \tag{6}$$

For an integral (4), (5) or (6) the existence conditions are obtained as the compatibility conditions of an overdetermined system of linear homogeneous first-order equations in the functions $b_i(x,y)$. Here these conditions are expressed in terms of the invariants

$$I_1(x,y) = \frac{J_1}{j_0 J_0^3}, \qquad I_2(x,y) = \frac{J_2}{j_0 J_0^2} \tag{7}$$

of the equivalence transformations of the family of equations (2), (3) defined by

$$\tilde{t} = k(t+t_0), \qquad \tilde{x} = \varphi(x,y), \qquad \tilde{y} = \psi(x,y), \qquad k, t_0 = \mathrm{const}.$$

In (7) the value J_0 up to a constant multiplier coincides with the main (scalar) curvature K of the surface M^2,

$$j_0 = g_{11}g_{22} - g_{12}^2, \qquad J_1 = g_{22}J_{0x}^2 - 2g_{12}J_{0x}J_{0y} + g_{11}J_{0y}^2.$$

All results on the integrals (4)–(6) are obtained in assumption of the non-degeneracy of the surface M^2. It means, when the conditions

$$j_0 \neq 0, \qquad J_0 \neq 0, \qquad J_1 \neq 0 \tag{8}$$

hold. The geometrical sense of the first two conditions (8) is obvious (non-degeneracy of the matrix $g_{ij}(x,y)$ and nonzero curvature of the surface). The sense of the third condition (8) is not so evident. The question is what properties has the degenerate surface M^2 with the curvature K, which satisfies the relation

$$g_{22}(x,y)\left(\frac{\partial K}{\partial x}\right)^2 - 2g_{12}(x,y)\frac{\partial K}{\partial x}\frac{\partial K}{\partial y} + g_{11}(x,y)\left(\frac{\partial K}{\partial y}\right)^2 = 0. \tag{9}$$