

OPTIMAL R&D POLICY IN A MODEL BASED ON EXHAUSTIBLE RESOURCES

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We study an infinite-horizon optimal control problem arising from an endogenous growth model in which both production and research require an exhaustible resource. The model is a development of the earlier considered problem [2] (going back to Jones [4, 5]) of optimal extraction and use of a finite stock of some resource:

$$Y(t) = A(t)^\kappa L^Y(t)^\alpha R_1(t)^{1-\alpha} \quad \text{where } \alpha \in (0, 1) \text{ and } \kappa > 0, \quad (1)$$

$$\dot{A}(t) = A(t)^\theta L^A(t)^\eta R_2(t)^\beta \quad \text{where } \eta \in (0, 1], \quad \beta \in [0, 1 - \eta], \text{ and } \theta \in (0, 1]. \quad (2)$$

Here $Y(t)$ is the output produced at time t and $A(t)$ is the current knowledge stock. The resource is divided between production ($R_1(t)$) and research ($R_2(t)$). The total amount of the extracted resource cannot exceed the initial supply $S_0 > 0$ of the resource:

$$\int_0^\infty [R_1(t) + R_2(t)] dt \leq S_0. \quad (3)$$

The population (total labor supply) is fixed at a certain level $L > 0$. Part of the labor $L^Y(t)$ is employed in production, while the other part $L^A(t) \in [0, L]$ is allocated to research:

$$L^A(t) + L^Y(t) \equiv L. \quad (4)$$

We consider a discounted logarithmic utility function of the output as a measure of *welfare*:

$$J_0(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^\infty e^{-\rho t} \ln Y(t) dt \rightarrow \max, \quad (5)$$

where $\rho > 0$ is a subjective discount rate.

As our study has shown [2], in the nonexceptional case of $(1 - \beta)\theta < 1$, the labor and resource allocated (optimally) to research gradually decrease and ultimately vanish. Accordingly, the expansion of the knowledge stock is limited and stops or virtually stops at some moment and the output depletes to zero in the long run. However, experience suggests that there may occur jumps (transitions) from one technological trajectory to another. So we develop the above model further in order to take account of the possibility of such a jump. Namely, we assume that the moment of a jump T is a random variable such that

$$P(T < t + \Delta t \mid t \leq T) = \nu L^A(t) \Delta t + o(\Delta t), \quad o(\Delta t)/\Delta t \rightarrow 0 \text{ as } \Delta t \rightarrow 0. \quad (6)$$

Then the probability density function for the random variable T is

$$\nu L^A(t) e^{-\nu \mathfrak{L}(t)}, \quad \text{where } \mathfrak{L}(t) = \int_0^t L^A(s) ds \text{ for } 0 \leq t < \infty,$$

provided that $\mathfrak{L}(\infty) = \int_0^\infty L^A(s) ds = \infty$. If $\mathfrak{L}(\infty) < \infty$, then there is a positive probability that the jump will not occur at all, i.e. $p(T = \infty) = e^{-\nu \mathfrak{L}(\infty)} > 0$.

It is important to note that the process described by relations (1)–(4) is now of finite duration with probability $1 - e^{-\nu \mathcal{L}(\infty)}$. Hence the integral in (5) must be taken only over the interval $[0, T]$ rather than over $[0, \infty)$. However, some estimate of the knowledge stock accumulated by the moment $t = T$ should also be taken into account because the accumulated knowledge $A(T)$ augments the productivity of the production means and hence increases the welfare on the remaining time interval $[T, \infty)$. Thus, we come to the following functional measuring the welfare:

$$J_T(A(\cdot), L^A(\cdot), R_1(\cdot)) = \int_0^T e^{-\rho t} \ln Y(t) dt + e^{-\rho T} V(A(T)).$$

To determine the value of the accumulated knowledge $A(T)$, we consider an auxiliary simple optimization problem on the interval $[T, \infty)$, which yields

$$V(A(T)) = C + \frac{\kappa}{\rho} \ln A(T),$$

where $C = C(\rho, \kappa, L)$ is a constant.

In this situation it is natural to aim at maximizing the expectation of the utility functional $J_T(A(\cdot), L^A(\cdot), R_1(\cdot))$ considered as a function of the random variable T . After some transformations, we reduce the problem to an equivalent infinite-horizon optimal control problem. Using standard results, we show the existence of an optimal control in the resulting problem. Then we apply the recent version of the Pontryagin Maximum Principle [1] (see also [3]) and analyze the solutions of the arising Hamiltonian system of the PMP. In particular, it is interesting to compare the behavior of optimal controls in this problem with that in problem (1)–(5).

References

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