

CLASSIFICATION OF BINARY FORMS WITH CONTROL PARAMETER

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The aim of the talk is to classify binary forms, whose coefficients depend on control parameter, with respect to the action of some pseudogroup. We solve this problem in two steps. Firstly, we consider the action of our pseudogroup on the infinite prolongation of the differential Euler equation and find differential invariant algebra of this action. Secondly, using methods from geometric theory of differential equations, we prove that three dependencies between basic differential invariants and their invariant derivatives uniquely define the equivalent class of binary forms with control parameter.

Let us consider the space $V_n(u)$ of binary forms, whose coefficients depend on the control parameter:

$$f(x, y; u) = \sum_{i=0}^n a_i(u) x^i y^{n-i}, \quad \text{where } a_i \text{ are holomorphic functions.}$$

The pseudogroup $G := \mathrm{SL}_2 \ltimes (\mathcal{F}(u) \times \mathrm{T}(u))$ acts on the space $V_n(u)$ in the following way:

1) “semisimple part” SL_2 acts by linear transformations of the coordinates (x, y) :

$$\mathrm{SL}_2 \ni A: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A^{-1} \begin{pmatrix} x \\ y \end{pmatrix};$$

2) “functional part” $\mathcal{F}(u)$ acts by holomorphic transformations of the control parameter: $u \mapsto \varphi(u)$;

3) “torus” $\mathrm{T}(u)$ acts by multiplications on the holomorphic functions on the control parameter: $f \mapsto \lambda(u)f$.

Consider space $\mathbb{C}^3$ with coordinates (x, y, u) and k -jet space J^k of functions on it (all necessary definitions and facts can be found in [1]). Denote by $(x, y, u, h, h_x, h_y, h_u, \dots)$ the coordinates in k -jet space.

Binary forms with control parameter can be considered as solutions of the *Euler differential equation*

$$\mathcal{E} := \{xh_x + yh_y = nh\} \subset J^1$$

(see also [2]). The action of the pseudogroup G on 0-jet space J^0 prolongs to the action on all prolongations $\mathcal{E}^{(k-1)} \subset J^k$ (see [1]).

Definition 1. *Differential invariant of the action of pseudogroup G of order k is G -invariant function on manifold $\mathcal{E}^{(k-1)}$, which is polynomial in derivatives h_σ , h^{-1} and $(h_x h_{yu} - h_y h_{xu})^{-1}$ (see Theorem 1).*

Remark. Function $h_x h_{yu} - h_y h_{xu}$ is “total Poisson bracket” $\{h, h_u\}$. Hence this function is a differential semi-invariant of pseudogroup G (see [3]).

Definition 2. *Invariant derivative is a combination of total derivatives, which commutes with the action of group G .*

Theorem 1. *Differential invariant algebra of the action of pseudogroup G on the manifold $\mathcal{E}^{(\infty)}$ is freely generated by differential invariant*

$$H := \frac{h_{xx} h_{yy} - h_{xy}^2}{h^2}$$

of order 2 and by invariant derivatives

$$\nabla_1 := \frac{h_y}{h} D_x - \frac{h_x}{h} D_y \quad \text{and} \quad \nabla_2 := \frac{h^2}{h_x h_{yu} - h_y h_{xu}} \cdot D_u$$

(where D_x, D_y, D_u are total derivative operators with respect to variables x, y, u correspondingly).

Definition 3. Binary form $f \in V_n(u)$ is said to be *regular*, if the restrictions of the invariants H, H_1 and H_2 on form f are functionally independent in points of some domain $\Omega \subset \mathbb{C}^3$ (here indexes denote the corresponding invariant derivatives ∇_1 and ∇_2).

Consider the regular binary form f . Then the restrictions of invariants H_{11}, H_{12} and H_{22} on form f can be extended through the restrictions of the invariants H, H_1 and H_2 on f :

$$H_{11} = A(H, H_1, H_2), \quad H_{12} = B(H, H_1, H_2), \quad H_{22} = C(H, H_1, H_2).$$

The triple (A, B, C) is said to be *triple of dependencies* of form f .

Theorem 2. Two regular binary forms f and $\tilde{f}$ with control parameters are G -equivalent iff the triples of dependencies coincide:

$$(A, B, C) = (\tilde{A}, \tilde{B}, \tilde{C}).$$

The author is supported by RFBR, grand mol_a-14-01-31045.

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