# CLASSIFICATION OF BINARY FORMS WITH CONTROL PARAMETER

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Let  $V_n(u)$  be the space of binary forms, whose coefficients depend on the control parameter.

$$f(x,y;u) = \sum_{i=0}^{n} a_i(u)x^iy^{n-i}$$
, where  $a_i$  are holomorphic functions.

The pseudogroup  $G:=\mathrm{SL}_2 \leftthreetimes (\mathcal{F}(u) \times \mathrm{T}(u))$  acts on  $V_n(u)$ :

1) «semisimple part» SL<sub>2</sub>

$$\operatorname{SL}_2 \ni A \colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A^{-1} \begin{pmatrix} x \\ y \end{pmatrix};$$

2) «functional part»  $\mathcal{F}(u)$ 

$$u \mapsto \varphi(u)$$
;

3) «torus» T(u)

$$f \mapsto \lambda(u)f$$
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When are two given binary forms f and  $\widetilde{f}$  with control parameter G-equivalent?

# Geometric interpretation:

binary form  $f \longleftrightarrow \operatorname{set}$  of projective points

$$f(x, y; u) = (\alpha_1(u)y - \beta_1(u)x) \cdot \ldots \cdot (\alpha_n(u)y - \beta_n(u)x)$$

 $\Rightarrow$  the set of zeros of f is the non-ordered set

$$\{P_1(u),\ldots,P_n(u)\}$$

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 $\Rightarrow$  we get famous algebraic

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When are two given binary forms f and  $\tilde{f}$  of degree n with complex coefficients  $SL_2$ -equivalent?

- Bool (1841) n = 3;
- Cayley, Eisenstein (1851) n=4; debut of classical invariant theory;
- Cayley, Hermite (1860) n = 5;
- Gordan, Shioda, Hilbert, Dixmier, Lazard, etc. (1980–2000)  $n \le 10$ , n = 12;
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k-jet  $[f]_b^k$  of function  $f: \mathbb{C}^3 \to \mathbb{C}$  is a segment of the Teylor series of function f in point b up to the members of order k.

k-jet space  $J^k$  is the set of all k-jets for all functions in all points.

Canonical coordinates:

$$(x, y, u, h, h_x, h_y, h_u, h_{xx}, h_{xy}, h_{yy}, h_{xu}, h_{yu}, h_{uu} \ldots),$$

$$h_x([f]_b^k) = f_x(b), h_{xy}([f]_b^k) = f_{xy}(b), \text{ etc.}$$



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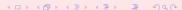
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Binary forms with control parameter can be considered as solutions of the *Euler differential equation* 

$$\mathcal{E} := \{xh_x + yh_y = nh\} \subset J^1.$$

Prolongation of the Euler equation:

$$\mathcal{E}^{(1)} = \left\{ \mathcal{E}, \frac{d}{dx} \mathcal{E}, \frac{d}{dy} \mathcal{E}, \frac{d}{du} \mathcal{E} \right\} \subset J^2,$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{xx} \frac{\partial}{\partial h_x} + h_{xy} \frac{\partial}{\partial h_y} + h_{xu} \frac{\partial}{\partial h_y} + \dots$$



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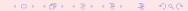
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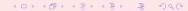
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The action of the pseudogroup G on 0-jet space  $J^0$  prolongs to the action on all prolongations  $\mathcal{E}^{(k-1)} \subset J^k$ :

$$g \circ [f]_b^k = [g \circ f]_{gb}^k.$$

#### Definition

- Differential invariant of the action of pseudogroup G of order k is G-invariant function on manifold  $\mathcal{E}^{(k-1)}$ , which is polynomial in derivatives  $h_{\sigma}$ ,  $h^{-1}$  and  $(h_x h_{yu} h_y h_{xu})^{-1}$ .
- Invariant derivation is a combination of total derivations, which commutes with the action of group G.



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## Algebra of differential invariants

### Theorem

Differential invariant algebra of the action of pseudogroup G is freely generated by differential invariant

$$H := \frac{h_{xx}h_{yy} - h_{xy}^2}{h^2}$$

of order 2 and by invariant derivations

$$\nabla_1 := \frac{h_y}{h} \frac{d}{dx} - \frac{h_x}{h} \frac{d}{dy} \quad \text{and} \quad \nabla_2 := \frac{h^2}{h_x h_{yu} - h_y h_{xu}} \cdot \frac{d}{du}$$



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### Definition

Binary form  $f \in V_n(u)$  is said to be regular, if the restrictions of the invariants H,  $H_1$  and  $H_2$  on form f are functionally independent in points of some domain  $\Omega \subset \mathbb{C}^3$ .

Consider the regular binary form f. Then the restrictions of invariants  $H_{11}$ ,  $H_{12}$  and  $H_{22}$  on form f can be extended through the restrictions of the invariants H,  $H_1$  and  $H_2$  on f:

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Consider the regular binary form f. Then the restrictions of invariants  $H_{11}$ ,  $H_{12}$  and  $H_{22}$  on form f can be extended through the restrictions of the invariants H,  $H_1$  and  $H_2$  on f:

$$H_{11} = A(H, H_1, H_2), \ H_{12} = B(H, H_1, H_2), \ H_{22} = C(H, H_1, H_2).$$



## Classification theorem

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Two regular binary forms f and  $\tilde{f}$  with control parameter are G-equivalent, iff

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" $\Leftarrow$ " Let  $(A,B,C)=(\widetilde{A},\widetilde{B},\widetilde{C})$  for two regular forms f and  $\widetilde{f}$ 

Consider "invariant coordinate systems"

$$S := (H(f), H_1(f), H_2(f)), \ \widetilde{S} := (H(\widetilde{f}), H_1(\widetilde{f}), H_2(\widetilde{f})).$$

Let us take two jets

$$[f]_a^4$$
 and  $[\widetilde{f}]_b^4$ 

with the same coordinates in S and  $\widetilde{S}$  correspondingly.

Values of all differential invariants of the 4-th order in these jets coincide. Hence, these jets are *G*-equivalent, and

$$\exists \ g \in G: \quad g(a) = b \quad \text{and} \quad [g \circ f]_b^4 = [\widetilde{f}]_b^4.$$

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