Local Conformal Flatness of Left-Invariant 3D Contact Structures

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In this talk I want to address the problem of finding the locally flat left-invariant contact structures on a three dimensional Lie Group up to conformal transformations, that is I will determine the ones locally conformally equivalent to the Heisenberg algebra $\mathbb{H}_3$. In particular I will show how to build the Fefferman metric associated to a generic three dimensional contact structure (not necessarily left-invariant) and by means of this construction I will give the explicit formula for the (unique) conformal invariant associated to such a structure. Next, specializing the study to the left-invariant case, I will give a complete list of the locally conformally flat structures which may appear and I will find the explicit form of the maps $\varphi\colon M \to \mathbb{R}$ which flatten our structures, and I will show that they are essentially (i.e. up to multiplication by a constant) unique.

Theorem. *Let (M, Δ, g) be a left-invariant 3D contact structure. Then it is locally conformally flat if and only if its canonical frame satisfies one of the following*

$$i)\ \begin{cases} [f_2, f_1] = f_0 + c_{12}^2 f_2, \\ [f_1, f_0] = \frac{2}{9}\left(c_{12}^2\right)^2 f_2, \\ [f_2, f_0] = 0. \end{cases} \qquad ii)\ \begin{cases} [f_2, f_1] = f_0 + c_{12}^1 f_1, \\ [f_1, f_2] = 0, \\ [f_2, f_0] = -\frac{2}{9}\left(c_{12}^1\right)^2 f_2. \end{cases}$$

or

$$iii)\ \begin{cases} [f_2, f_1] = f_0, \\ [f_1, f_0] = \kappa f_2, \\ [f_2, f_0] = -\kappa f_1, \qquad \kappa < 0. \end{cases}$$

Where κ is the curvature of the structure.

Open question 1. Give a complete classification (i.e. not just the locally conformally flat ones) of left-invariant three dimesional contact structures, up to real rescalings.

Open question 2. Give satisfactory criteria to determine whether a given three dimesional contact structure (not necessarily left-invariant) is locally conformally flat or not.

References

1. *A. A. Agrachev*, Exponential mappings for contact sub-Riemannian structures. J. Dynamical and Control Systems, 1996, v. 2, pp. 321–358.
2. *A. A. Agrachev, D. Barilari*, Sub-Riemannian structures on 3D Lie groups, J. Dynamical and Control Systems, 2012, v. 18, pp. 21–44.
3. *A. L. Castro, R. Montgomery*, The chains of left-invariant Cauchy-Riemann structures on $SU(2)$, Pacific J. Math, 238, 2008, no. 1, pp. 41–71.
4. *F. A. Farris*, An intrinsic construction of Fefferman's CR metric, Pacific J. Math, 123, 1986, no. 1, pp. 33–45.
5. *C. Fefferman, C. R. Graham*, The ambient metric, Annals of mathematics studies, 178. Princeton University press, NJ, 2012, pp. x+113.
6. *J. M. Lee*, The Fefferman metric and pseudo-Hermitian invariants, Trans. Amer. Math. Soc., 296, 1986, no. 1, pp. 411–429.