

MA–TRUDINGER–WANG TENSOR, FROM PDE REGULARITY TO GEOMETRIC INFORMATION

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The Ma–Trudinger–Wang (MTW) tensor was introduced in [4] to guarantee a regularity theory for the fully non linear Monge–Ampère Equation. In particular this PDE is satisfied by the solution of an optimal transport problem [5]. This tensor also leads to a regularity theory (TCP theory) for optimal transportation problem on a Riemannian manifold [3]. For example we can set in dimension 2 the following theorem :

Theorem. *The TCP condition (continuity of optimal transport map) holds if and only if (M, g) satisfies (MTW) (positivity of MTW tensor) and all its injectivity domains are convex.*

Cédric Villani conjecture that the convexity of injectivity domains is in fact a consequence of (MTW). He makes a step in this direction [1]. Together with Alessio Figalli and Ludovic Rifford [2] we improved this result and prove the following "Boney M theorem":

Theorem. *Let (M, g) be a nonfocal Riemannian manifold satisfying (MTW). Then all injectivity domains of M are convex.*

During this talk we will introduce the optimal transportation problem and the Ma–Trudinger–Wang tensor. We then review some applications of MTW tensor in order to make it less mysterious and prove that it contains many geometric informations. In particular we will explain why the MTW tensor can be seen as a curvature one. We will conclude with the convexity of injectivity domains for a non focal manifold.

References

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