Differential Invariants of Feedback Transformations for Quasi-Harmonic Oscillation Equations

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The classification problem for a control-parameter-dependent second-order differential equations is considered. The algebra of the differential invariants with respect to Lie pseudo-group of feedback transformations is calculated. The equivalence problem for a control-parameter-dependent quasi-harmonic oscillation equation is solved. Some canonical forms of this equation are constructed.

Consider the problems of equivalence and classification for the differential equation:

$$\frac{d^2y}{dx^2} + f(y,u) = 0, \tag{1}$$

with respect to the feedback transformations [1]:

$$\varphi\colon (x,y,u) \longmapsto (X(x,y), Y(x,y), U(u)), \tag{2}$$

where the function$f(y,u)$ is smooth. Here u is a scalar control parameter. We will call an equation of form (1)*control-parameter-dependent quasi-harmonic oscillator equation* (QHO).

Definition. Operator

$$\nabla = M\frac{d}{dy} + N\frac{d}{du} \tag{3}$$

is called *G-invariant differentiation* if it commutes with every element of any prolongation of Lie algebra $\mathcal{G}$, where M and N are the functions on the jet space.

Theorem. *Differential operators*

$$\nabla_1 = \frac{z}{z_y}\frac{d}{dy}, \tag{4}$$

$$\nabla_2 = \frac{z}{z_u}\frac{d}{du} \tag{5}$$

are G-invariant differentiations.

Theorem. *Functions*

$$J_{21} = \frac{z_{yy}z}{z_y^2}, \quad J_{22} = \frac{z_{yu}z}{z_y z_u}$$

form a complete set of the basic second-order differential invariants, i.e.any other second-order differential invariants are the functions of J_{21} and J_{22}.

Theorem. *Quasi-harmonic oscillation equation differential invariants algebra is generated by second-order differential invariants J_{21}, J_{22} and invariant differentiations ∇_1 and ∇_2. This algebra separates regular orbits.*

Let us call an equation $\mathcal{E}_f$ *regular*, if

$$dJ_{21}(f) \wedge dJ_{22}(f) \neq 0.$$

Here $J(f)$ is the value of the differential invariant J on the function $f = f(y, u)$.

Theorem. *Suppose that the functions f and g are real-analytical. Two regular equations $\mathcal{E}_f$ and $\mathcal{E}_g$ are locally G-equivalent if and only if the functions Φ_{if} and Φ_{ig} identically equal ($i = 1, 2, 3$) and 3-jets of the functions f and g belong to the same connection component.*

References

1. *A. G. Kushner, V. V. Lychagin.* Petrov Invariants for 1-D Control Hamiltonian Systems. // Global and Stochastic Analysis. 2012. 2, 2. 241–264.