Metric Geometry of Carnot–Carathéodory Spaces and Its Applications

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We describe new fine properties of Carnot–Carathéodory spaces under minimal assumptions on smoothness of the basis vector fields. As a corollary, we discover new geometric properties of weighted Carnot-Caratheodory spaces. All these results are new even for a "smooth" case. They play crucial role in the development of the differentiability theory on sub-Riemannian structures (see, e.g., works by S. Vodopyanov [1, 2]), in the investigation of non-equiregular Carnot–Carathéodory spaces (see, e.g., work by S. Selivanova [3]) and imply many basic results of the theory of non-holonomic spaces (see, e.g., work by S. Basalaev and S. Vodopyanov [4]).

Definition see, e.g., [5, 2, 4, 6, 7]**.** Fix a connected Riemannian C^∞-manifold $\mathbb{M}$ of topological dimension N. The manifold $\mathbb{M}$ is called the *Carnot–Carathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that every point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of C^1-smooth vector fields $X_1, \ldots, X_N$ enjoying the following two properties.

(1) At every point $v \in U$ we have a subspace

$$H_i\mathbb{M}(v) = H_i(v) = \operatorname{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\} \subset T_v\mathbb{M}$$

of the dimension $\dim H_i$ independent of v, $i = 1, \ldots, M$.

(2) The inclusion $[H_i, H_j] \subset H_{i+j}$, $i + j \leq M$, holds.

Moreover, if the third condition holds then the Carnot–Carathéodory space is called the *Carnot manifold*:

(3) $H_{j+1} = \operatorname{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$, where $k = \left[\frac{j+1}{2}\right]$, $H_0 = \{0\}$, $j = 1, \ldots, M-1$.

The subbundle $H\mathbb{M}$ is called *horizontal*.

The number M is called the *depth* of the manifold $\mathbb{M}$.

The main result is the following

Theorem [6, 7]**.** *Let $\mathbb{M}$ be a Carnot–Carathéodory space with $C^{1,\alpha}$-smooth basis vector fields, $\alpha \geq 0$ (if $\alpha = 0$ then the fields belong to the class C^1). Then for each point of $\mathbb{M}$, there exists a sufficiently small neighborhood $\mathcal{U} \Subset \mathbb{M}$ possessing the following property: for $u, v \in \mathcal{U}$, $w = \gamma(1)$ and $\widehat{w} = \widehat{\gamma}(1)$, where $\gamma, \widehat{\gamma}\colon [0,1] \to \mathbb{M}$ are absolutely continuous (in the classical sense) curves contained in* $\operatorname{Box}(u, \varepsilon)$ *such that*

$$\dot{\gamma}(t) = \sum_{i=1}^{N} b_i(t) X_i(\gamma(t)), \quad \gamma(0) = v, \quad \textit{and} \quad \dot{\widehat{\gamma}}(t) = \sum_{i=1}^{N} b_i(t) \widehat{X}_i^u(\gamma(t)), \quad \widehat{\gamma}(0) = v,$$

and each measurable function $b_i(t)$ meets the property

$$\int_0^1 |b_i(t)|\, dt < S\varepsilon^{\deg X_i}, \tag{1}$$

$S < \infty$, $i = 1, \dots, N$, we have

$$\max\{d_\infty(w, \widehat{w}), d^u_\infty(w, \widehat{w})\} = \begin{cases} O(1) \cdot \varepsilon^{1+\frac{\alpha}{M}} \text{ if } \alpha > 0, \\ o(1) \cdot \varepsilon \text{ if } \alpha = 0, \end{cases} \tag{2}$$

with $O(1)$ and $o(1)$ to be uniform in $u \in \mathcal{U}$ and all collections of functions $\{b_i(t)\}_{i=1}^N$ with the property (1).

Remark [7]; see also [8]**.** For weighted Carnot–Carathéodory spaces, the estimate in (2) is $O(1) \cdot \varepsilon^{1+\frac{\alpha}{l_M}}$ for $\alpha > 0$, where l_M is the maximal weight [3, 7, 8].

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