

THE AVERAGE NUMBER OF CONNECTED COMPONENTS OF AN ALGEBRAIC HYPERSURFACE

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How many zeros of a random polynomial are real? M. Kac [2] tackled this question for a Gaussian ensemble of univariate polynomials (1943). Addressing the case of a real algebraic hypersurface in $\mathbb{R}P^n$, we discuss asymptotic estimates for the number (and relative position) of connected components. This addresses a random version of Hilbert’s Sixteenth Problem.

The outcome depends on the definition of “random”. We consider Gaussian ensembles that are invariant under an orthogonal change of coordinates. Following E. Kostlan [3] we parameterize this family of ensembles in terms of a generalized Fourier series of eigenfunctions of the spherical Laplacian. With some regularity assumptions on the choice of weights assigned to each eigenspace, we calculate the order of growth (as the degree d goes to infinity) of the average number of connected components. The order of growth turns out to be the same as the n th power of the average number of zeros on a one-dimensional sample slice:

$$\mathbb{E}b_0(X) = \Theta \left(\left[\mathbb{E}b_0(X \cap \mathbb{R}P^1) \right]^n \right), \quad \text{as } d \rightarrow \infty.$$

This relates the multivariate case to the classical problem of Kac.

The proof uses random matrix theory to prove an upper bound and harmonic analysis to prove a lower bound. (This is joint work with Yan V. Fyodorov and Antonio Lerario [1, 4].)

References

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3. *E. Kostlan.* On the expected number of real roots of a system of random polynomial equations. // Proceedings of the conference Foundations of computational mathematics (Hong Kong, 2000), World Sci. Publishing, River Edge, NJ, 2002. 149–188.
4. *A. Lerario, E. Lundberg.* Statistics on Hilbert’s sixteenth problem. // arXiv:1212.3823.