

CUT LOCUS IN THE RIEMANNIAN PROBLEM ON SO_3 IN AXISYMMETRIC CASE

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The parametrization of Riemannian geodesics on SO_3 is the classic L. Euler's result. But the global optimality of geodesics was not investigated yet. L. Bates and F. Fassio [1] have got the equation for conjugate time in the axisymmetric case and described conjugate locus depending on the ratio of eigenvalues of Riemannian metric.

We represent the Maxwell strata, cut locus and give the equation for the cut time.

When one eigenvalue of Riemannian metric moves to infinity, the parametrization of geodesics, conjugate time and locus, cut time and locus in the Riemannian problem converge to the sub-Riemannian ones that were considered by U. Boschain and F. Rossi [2].

Let $I_1 = I_2, I_3$ be the eigenvalues of the left invariant Riemannian metric, e_1, e_2, e_3 be corresponding basis in $\mathfrak{so}_3$, and p_1, p_2, p_3 be corresponding impulses, $\bar{p}_i = \frac{p_i}{|p|}$, $i = 1, 2, 3$.

Let $\eta = \frac{I_1}{I_3} - 1 > -1$.

Theorem. *Let $\tau_{cut}(\eta, \bar{p}_3)$ be the minimal positive root of the equation*

$$\cos \tau \cos(\tau \eta \bar{p}_3) - \bar{p}_3 \sin \tau \sin(\tau \eta \bar{p}_3) = 0$$

(1) *If $\eta \geq -\frac{1}{2}$, then the cut time is $\frac{2I_1\tau_{cut}(\eta, \bar{p}_3)}{|p|}$.*

(2) *If $\eta < -\frac{1}{2}$, then the cut time is*

$$\begin{cases} \frac{2\pi I_1}{|p|}, & \text{if } \frac{1}{2\eta} \leq |\bar{p}_3| < 1, \\ \frac{2I_1\tau_{cut}(\eta, \bar{p}_3)}{|p|}, & \text{if } |\bar{p}_3| < \frac{1}{2\eta}. \end{cases}$$

Theorem. (1) *If $\eta \geq -\frac{1}{2}$, then the cut locus is $\mathbb{R}P^2$ consisting of rotations by angles π in SO_3 .*

(2) *If $\eta < -\frac{1}{2}$, then the cut locus contains two components: $\mathbb{R}P^2$ and the segment*

$$J_\eta = \{\exp(\pm \varphi e_3) \mid \varphi \in [2\pi(1 + \eta), \pi]\}.$$

The proof of these theorems is based on considering the symmetry group of Hamiltonian vector field of Pontryagin maximum principle, finding its fixed points (Maxwell strata), considering some open sets bounded by Maxwell strata, which are diffeomorphic by exponential map. This method was presented by Yu. L. Sachkov for the Euler elastic problem [3].

Proposition. *The geodesics parametrization, conjugate time and locus, cut time and locus in sub-Riemannian problem on SO_3 are obtained from the Riemannian ones by $I_3 \rightarrow \infty$.*

Example. Cut locus in the sub-Riemannian problem has two components: $\mathbb{R}P^2$ and the circle without point

$$S^1 \setminus \{\text{id}\} = \{\exp(\varphi e_3) \mid \varphi \in (0, 2\pi)\}.$$

The stratum J_η of the cut locus for the Riemannian problem converges to this circle without point if $\eta \rightarrow -1$ (equivalent to $I_3 \rightarrow \infty$).

References

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3. *Yu. L. Sachkov.* Maxwell strata in the Euler elastic problem. // Journal of Dynamical and Control Systems. 2008. 14, 2. 169–234.