Gradient methods for convex problems with stochastic inexact oracle

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Traditional Mathematical School „Control, Information and Optimization“
Moscow

Outline

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1. Introduction to convex optimization

2. Concept of (δ, L)-oracle

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4. Stochastic Intermediate Gradient Method

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4. Huge-scale problems, n or $\ln n$ operations per iteration is ok, coordinate descent schemes, sparsity, randomization. $N(\varepsilon) \approx O\left(\frac{n}{\varepsilon^2}\right)$.

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    2. Kullback–Leibler divergence: $V(x, z) = \sum_{i=1}^n x_i \ln \frac{x_i}{z_i}$.

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3. Uniformly convex functions:

$$f(y) \geq f(x) + \langle g(x), y - x \rangle + \frac{\kappa}{2}\|x - y\|^\rho, \quad \forall x, y \in Q, \forall g(x) \in \partial f(x),$$

where $\rho \geq 2$.

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3. $\Psi_k(x)$ – model of the function which approximates $f(x)$ in some sense.

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Below we consider mostly the smooth case.

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1. Convex case: $f(y_k) - f^* \leq \frac{L\|x_0 - x^*\|_2^2}{2k}$.
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$$f(y_k) - f^* \leq \frac{\beta_k}{A_k} d(x^*),$$

which can give us the rate of convergence.

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$f(x)$ is smooth.
Choose $d(x) = \frac{1}{2}\|x - x_0\|_2^2$, $\alpha_0 = \frac{L}{L-\mu}$, $A_k = \sum_{i=0}^{k} \alpha_i$, $\alpha_{k+1} = \frac{A_k\mu + L}{L-\mu}$, $\beta_k = L$.

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   Auxiliary problem $\min_{x\in Q}\{\langle g,x\rangle + d(x)\}$ can be solved explicitly:
   $$\hat{x}_i = \frac{exp(-g_i)}{\sum_{i=1}^n exp(-g_i)}, \quad i = 1, ..., n.$$
2. Composite optimization: $f(x) \to \varphi(x) := f(x) + h(x)$, where $h(x)$ is a simple convex function: the problem $\min_{x\in Q}\{\langle g,x\rangle + \alpha d(x) + \beta h(x)\}$ is easy solvable.
   Example: LASSO $\|x-a\|_2^2 + \lambda\|x\|_1 \to \min$.
   Non-smooth (but strongly convex) function $\Rightarrow$ lower bound $O\left(\frac{1}{k}\right)$.
   But $\|x-a\|_2^2$ is strongly convex and smooth $\Rightarrow$ we get method with $O\left(\exp\left(-k\cdot\mathrm{const}\right)\right)$.
3. Stochastic error, e.g. $\mathbb{E}_\xi f(x,\xi) \to \min_{x\in Q}$.
   On the step k we can get only
   $\nabla f(x,\xi_k): \quad \mathbb{E}_{\xi_k}\nabla f(x,\xi_k) = \nabla\mathbb{E}_{\xi_k} f(x,\xi_k)$ and
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Outline

(δ, L)-oracle

[Devolder, Glineur and Nesterov, 2011-2013]
For every $x \in Q$ there are $f_{\delta,L}(x) \in \mathbb{R}$ and $g_{\delta,L}(x) \in E^*$ such that

$$0 \le f(y) - f_{\delta,L}(x) - \langle g_{\delta,L}(x), y - x \rangle \le \frac{L}{2}\|x - y\|^2 + \delta, \quad \forall y \in Q.$$

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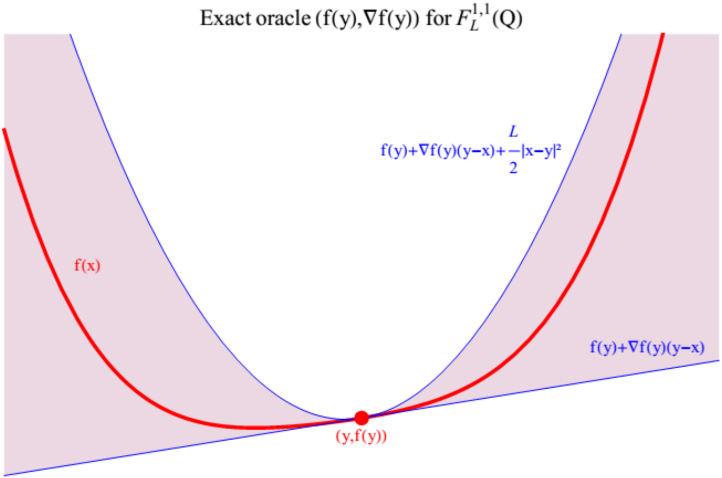
(δ, L)-oracle geometry



Picture from O.Devolder PhD Thesis, 2013

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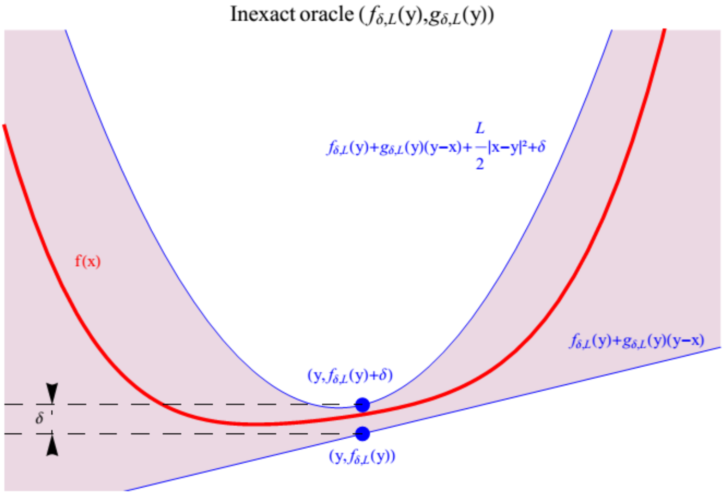

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(δ, L)-oracle examples

1. $f(x)$ – convex non-smooth function with Hölder continuous subgradient $\|g(x) - g(y)\|_* \leq L_\nu \|x - y\|^\nu, \quad \forall x, y \in Q$, for some $\nu \in [0, 1]$

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   If $\mathrm{diam}\, Q = D < +\infty$ then $(\tilde{f}_x - \Delta_1 - \Delta_2 D, \tilde{\nabla} f_x)$ is a $(2\Delta_1 + 2\Delta_2 D, M)$-oracle.

Convex case: PGM, DGM and FGM revisited

For $\mu = 0$ the only change we need to do in the schemes is $f(x) \to f_{\delta,L}(x)$, $\nabla f(x) \to g_{\delta,L}(x)$.

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These methods can be generalized to strongly convex case.

(δ, L, μ)-oracle

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As we have seen $f(x)$ – convex non-smooth function with Hölder continuous subgradient can be for any $\delta > 0$ equipped with (δ, L)-oracle with $L = L_\nu \left[\frac{L_\nu(1-\nu)}{2\delta(1+\nu)}\right]^{\frac{1-\nu}{1+\nu}}$.

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and optimal for every $\nu \in [0,1]$ rate

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$f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is convex, smooth with L_1 - Lipschitz continuous gradient, $f_2(x)$ is convex, non-smooth with M_2 bounded variation of subgradient.

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Fixing again number of iterations N and optimizing in δ we obtain

$$f(y_N) - f^* \leq \frac{2L_1R^2}{(N+1)^2} + \frac{2M_2R}{\sqrt{N+1}}.$$

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Theorem [Devolder, Glineur and Nesterov]

Consider a first-order method for the class of L-smooth functions with convergence rate $O\left(\frac{LR^2}{k^p}\right)$ when exact first-order information is used. Assume that the bounds on the performance of this method applied to a problem equipped with an inexact (δ, L)-oracle are given by inequality

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Let $\{\alpha_i\}_{i\geq 0}$, $\{B_i\}_{i\geq 0}$ be two sequences of coefficients satisfying

$$0 \leq \alpha_i \leq B_i, \quad \forall i \geq 0, \quad \alpha_k^2 \leq B_k \leq \sum_{i=0}^{k} \alpha_i, \quad \forall k \geq 1$$

We define also $A_k = \sum_{i=0}^{k} \alpha_i$ and $\tau_i = \frac{\alpha_{i+1}}{B_{i+1}}$.

Method:

1. Choose $x_0 \in Q$.
2. $w_k = \pi_Q \left(x_k - \frac{1}{L} g_{\delta,L}(x_k)\right)$.
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Choose $\alpha_i = \left(\frac{i+p}{p}\right)^{p-1}, \quad B_i = \alpha_i^2$. Then rate of convergence for convex case:

$$f(y_k) - f^* \leq \Theta\left(\frac{L\|x_0 - x^*\|_2^2}{2k^p} \right) + \Theta(k^{p-1}\delta)$$

Outline

Concept of stochastic inexact oracle

[Devolder, Glineur and Nesterov, 2011-2013]
The function $f(x)$ is equipped with (δ, L)-oracle. For every $x \in Q$ there are $f_{\delta,L}(x) \in \mathbb{R}$ and $g_{\delta,L}(x) \in E^*$ such that

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We associate with x a random variable ξ whose probability distribution is supported $\Xi \subset \mathbb{R}^d$ and such that

$$\mathbb{E}_\xi F_{\delta,L}(x,\xi) = f_{\delta,L}(x)$$
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   The idea is to replace $\nabla f(x)$ by $G_{0,L}(x,\xi) = \frac{N}{M}\sum_{j=1}^M (x^T a_{\xi_j} - b_{\xi_j})a_{\xi_j}$, where $\{\xi_1,\dots,\xi_M\}$ is a subset of rows uniformly chosen from $\{1,\dots,N\}$.
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Choose $d(x) = \frac{1}{2}\|x - x_0\|_2^2$, $\beta_k > L$, $\gamma_k = \frac{1}{\beta_k}$.

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Can be generalized to non-Euclidean setup and composite optimization.

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- $\Psi_k(x) = \beta_k d(x) + \sum_{i=0}^{k} \alpha_i \left[F_{\delta,L}(x_i, \xi_i) + \langle G_{\delta,L}(x_i, \xi_i), x - x_i \rangle \right]$ – stochastic model of the objective function.

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$$f(y_k) - f^* \leq \frac{\beta_k}{A_k} d(x^*) + \frac{\bar{E}_k(x^*) + E_k}{A_k},$$

which can give us mean rate of convergence and probability of large deviations.

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3. For deterministic case Intermediate Gradient Method gives the rate
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Our goal is the method

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4. applicable to composite optimization problems.

Auxiliary objects

- Simple gradient mapping

$$y = \arg\min_{x \in Q} \{\beta_k d(x) + \alpha_k \left[F_{\delta,L}(x_k, \xi_k) + \langle G_{\delta,L}(x_k, \xi_k), x - x_k \rangle\right] + h(x)\}.$$

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- Minimum of smoothed model of the function

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We define also $A_k = \sum_{i=0}^{k} \alpha_i$ and $\tau_i = \frac{\alpha_{i+1}}{B_{i+1}}$.

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5. $$x_{k+1} = \tau_k z_k + (1-\tau_k)y_k$$
6. $$\hat{x}_{k+1} = \arg\min_{x\in Q}\{\beta_k V(x,z_k) + \alpha_{k+1}\langle G_{\delta,L}(x_{k+1},\xi_{k+1}), x - z_k\rangle + \alpha_{k+1}h(x)\}.$$
7. $$w_{k+1} = \tau_k\hat{x}_{k+1} + (1-\tau_k)y_k$$
8. $$y_{k+1} = \frac{A_{k+1}-B_{k+1}}{A_{k+1}}y_k + \frac{B_{k+1}}{A_{k+1}}w_{k+1}$$

Estimating functions

- $\Psi_k(x) = \beta_k d(x) + \sum_{i=0}^{k} \alpha_i \left[F_{\delta,L}(x_i, \xi_i) + \langle G_{\delta,L}(x_i, \xi_i), x - x_i\rangle + h(x)\right]$ – model of the objective function.

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and

$$\varphi(y_k) - \varphi^* \leq \frac{\beta_k}{A_k} d(x^*) + \frac{\bar{E}_k(x^*) + E_k}{A_k},$$

which can give us mean rate of convergence and probability of large deviations.

E_k and $\bar{E}_k(x)$

We managed to prove that

$$E_k = \sum_{i=0}^{k} B_i\delta + \sum_{i=0}^{k} \frac{B_i}{\beta_i - L}\left(\|G_{\delta,L}(x_i,\xi_i) - g_{\delta,L}(x_i)\|_*\right)^2 +$$

$$+\sum_{i=0}^{k} \alpha_i (f_{\delta,L}(x_i) - F_{\delta,L}(x_i,\xi_i)) +$$

$$+\sum_{i=1}^{k} (B_i - \alpha_i)\frac{\alpha_i}{B_i}\langle g_{\delta,L}(x_i) - G_{\delta,L}(x_i,\xi_i), z_{i-1} - y_{i-1}\rangle,$$

$$\bar{E}_k(x) = \sum_{i=0}^{k} \alpha_i \Big[F_{\delta,L}(x_i,\xi_i) - f_{\delta,L}(x_i) +$$

$$\langle G_{\delta,L}(x_i,\xi_i) - g_{\delta,L}(x_i), x - x_i\rangle\Big].$$

General rate of convergence

Theorem 1

Assume that the function f is endowed with stochastic inexact oracle with parameters δ, L, σ. Then the sequence y_k generated by the Stochastic Intermediate Gradient Method, when applied to the composite function φ, satisfies

$$
\begin{aligned}
\varphi(y_k) - \varphi^* \leq \frac{1}{A_k}\Bigg(&\beta_k d(x^*) + \sum_{i=0}^{k} B_i \delta + \\
&+ \sum_{i=0}^{k} \frac{B_i}{\beta_i - L} \|G_{\delta,L}(x_i,\xi_i) - g_{\delta,L}(x_i)\|_*^2 + \\
&+ \sum_{i=0}^{k} \alpha_i \langle G_{\delta,L}(x_i,\xi_i) - g_{\delta,L}(x_i), x^* - x_i\rangle + \\
&+ \sum_{i=1}^{k} (B_i - \alpha_i)\frac{\alpha_i}{B_i} \langle G_{\delta,L}(x_i,\xi_i) - g_{\delta,L}(x_i), y_{i-1} - z_{i-1}\rangle\Bigg).
\end{aligned}
$$

General mean rate of convergence

Taking expectation we get

Theorem 2

Assume that the function f is endowed with stochastic inexact oracle with parameters δ, L, σ. Then the sequence y_k generated by the Stochastic Intermediate Gradient Method, when applied to the composite function φ, satisfies

$$\mathbb{E}_{\xi_0,\dots,\xi_k}\varphi(y_k) - \varphi^* \leq \frac{\beta_k d(x^*)}{A_k} + \frac{\sum_{i=0}^{k} B_i \delta}{A_k} + $$
$$+ \frac{1}{A_k}\sum_{i=0}^{k}\frac{B_i}{\beta_i - L}\sigma^2.$$

Additional assumptions

1. $\xi_0, \dots, \xi_k$ are i.i.d random variables.

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$\xi_{[k]} = (\xi_0, \dots, \xi_k)$ – history of the random process after k iterations.

Auxiliary result 1

Lemma 1 [Juditsky, Lan, Nemirovski, Shapiro]

Let $\xi_0, \ldots, \xi_k$ be a sequence of realizations of the i.i.d. random variables $X_0, \ldots, X_k$ and let $\Delta_i = \Delta_i(\xi_{[i]})$ be a deterministic function of $\xi_{[i]}$ such that for all $i \geq 0$:

$$\mathbb{E}\left[\exp\left(\frac{\Delta_i^2}{\sigma^2}\right)|\xi_{[i-1]}\right] \leq \exp(1)$$

and $c_0, \ldots, c_k$ is a sequence of positive coefficients. Then we have for any $k \geq 0$ and any $\Omega \geq 0$:

$$\mathrm{P}\left(\sum_{i=0}^{k} c_i \Delta_i^2 \geq (1+\Omega)\sum_{i=0}^{k} c_i \sigma^2\right) \leq \exp(-\Omega).$$

Auxiliary result 2

Lemma 2 [Lan, Nemirovski, Shapiro]

Let $\xi_0, \ldots, \xi_k$ be a sequence of realizations of the i.i.d. random variables $X_0, \ldots, X_k$ and let Γ_i and η_i be a deterministic function of $\xi_{[i]}$ such that:

1. $\mathbb{E}\left[\Gamma_i | \xi_{[i-1]}\right] = 0.$
2. $|\Gamma_i| \leq c_i \eta_i$, where c_i is positive deterministic constant.
3. $\mathbb{E}\left[\exp\left(\frac{\eta_i^2}{\sigma^2}\right) | \xi_{[i-1]}\right] \leq \exp(1).$

Then for any $k \geq 0$ and any $\Omega \geq 0$:

$$P\left(\sum_{i=0}^{k} \Gamma_i \geq \sqrt{3\Omega}\sigma \sqrt{\sum_{i=0}^{k} c_i^2}\right) \leq \exp(-\Omega).$$

General result for large deviations

Theorem 3

If the assumptions 1, 2, 3 are satisfied, then for all $k \geq 0$ and all $\Omega \geq 0$, the sequence generated by the SIGM satisfies:

$$\mathrm{P}\left(\varphi(y_k) - \varphi^* \geq \frac{\beta_k d(x^*)}{A_k} + \frac{\sum_{i=0}^{k} B_i \delta}{A_k} + \right.$$

$$\left. + \frac{1+\Omega}{A_k} \sum_{i=0}^{k} \frac{B_i}{\beta_i - L} \sigma^2 + \frac{2D\sigma\sqrt{3\Omega}}{A_k} \sqrt{\sum_{i=0}^{k} \alpha_i^2}\right) \leq 3\exp(-\Omega).$$

Choice of the coefficients

Our goal is the rate of $\Theta\left(\frac{LR^2}{k^p} + \frac{\sigma R}{\sqrt{k}} + k^{p-1}\delta\right)$, $p \in [1, 2]$.

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Let $a = 2^{\frac{2p-1}{2}}$ and $b = 2^{\frac{5-2p}{4}} p^{\frac{1-2p}{2}}$, $R \geq \sqrt{2d(x^*)}$.
Then the sequences

$$
\alpha_i = \frac{1}{a}\left(\frac{i+p}{p}\right)^{p-1}, \quad \forall i \geq 0,
$$

$$
\beta_i = L + \frac{b\sigma}{R}(i+p+1)^{\frac{2p-1}{2}}, \quad \forall i \geq 0,
$$

$$
B_i = a\alpha_i^2 = \frac{1}{a}\left(\frac{i+p}{p}\right)^{2p-2}, \quad \forall i \geq 0.
$$

satisfy all the requirements.

Mean rate of convergence

Theorem 4

If the sequences $\{\alpha_i\}_{i\geq 0}$, $\{\beta_i\}_{i\geq 0}$, $\{B_i\}_{i\geq 0}$ are chosen from relations above and $p \in [1,2]$ then the sequence generated by the SIGM satisfies:

$$\mathbb{E}_{\xi_0,\ldots,\xi_k}\varphi(y_k) - \varphi^* \leq$$

$$\leq \frac{LR^2 p^p 2^{\frac{2p-3}{2}}}{(k+p)^p} + \frac{\sigma R 2^{\frac{3+2p}{4}}\sqrt{p}(k+p+2)^{p-\frac{1}{2}}}{(k+p)^p} +$$

$$+ 2^{2p-1}\left(\left(\frac{k+p}{p}\right)^{p-1} + 1\right)\delta =$$

$$= \Theta\left(\frac{LR^2}{k^p} + \frac{\sigma R}{\sqrt{k}} + k^{p-1}\delta\right).$$

Large deviations bound

Theorem 5

If the sequences $\{\alpha_i\}_{i\geq 0}$, $\{\beta_i\}_{i\geq 0}$, $\{B_i\}_{i\geq 0}$ are chosen from relations above and $p \in [1,2]$ then the sequence generated by the SIGM satisfies:

$$\mathrm{P}\Bigg(\varphi(y_k) - \varphi^* >$$

$$> \frac{LR^2 p^p 2^{\frac{2p-3}{2}}}{(k+p)^p} + \frac{(1+\Omega)\sigma R 2^{\frac{3+2p}{4}}\sqrt{p}(k+p+2)^{p-\frac{1}{2}}}{(k+p)^p} +$$

$$+ 2^{2p-1}\left(\left(\frac{k+p}{p}\right)^{p-1} + 1\right)\delta + \frac{2D\sigma\sqrt{6p\Omega}}{\sqrt{k+p}}\Bigg) \leq$$

$$\leq 3\exp(-3\Omega).$$

SIGM: mean complexity

For SIGM we have ($\sqrt{V(x_0, x^*)} \le R$).

$$\mathbb{E}\varphi(y_k) - \varphi^* \le \frac{c_1 L V(x_0, x^*)}{k^p} + \frac{c_2 \sigma R}{\sqrt{k}} + c_3 k^{p-1} \delta.$$

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If we choose $k \geq \max\left\{ \left(\frac{2c_1 L V(x_0,x^*)}{\varepsilon}\right)^{\frac{1}{p}}, \frac{4c_2^2\sigma^2 R^2}{\varepsilon^2} \right\}$,

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The goal is to obtain for μ–strongly convex function

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$$N(\varepsilon) = \tilde{c}_1 \left(\frac{L}{\mu}\right)^{\frac{1}{p}} \ln \frac{\mu R_0^2}{\varepsilon} + \tilde{c}_2 \frac{\sigma^2}{\mu\varepsilon}$$

oracle calls.

SIGM: strongly convex case

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Let us use restart technique.

SIGM: strongly convex case

Denote $\Delta_k = \varphi(y_{N_k}) - \varphi^*$. Let us perform N_0 iterations with $R^2 = \frac{\varkappa}{\mu}\Delta_0 \stackrel{\text{def}}{=} R_0^2$ and $m = m_0$.

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$$\mathbb{E}\Delta_1 \leq \frac{c_1 L \varkappa \Delta_0}{\mu N_0^p} + \frac{c_2 \sigma \sqrt{\frac{\varkappa}{\mu}\Delta_0}}{\sqrt{m_0 N_0}} + c_3 N_0^{p-1}\delta$$

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Let us choose $N_0 = \left(\frac{4c_1 L\varkappa}{\mu}\right)^{\frac{1}{p}}$ and $m_0 \geq \max\left\{1, \frac{16c_2^2\sigma^2\varkappa}{\mu N_0\Delta_0}\right\}$.

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$$\frac{c_1 L \varkappa \Delta_0}{\mu N_0^p} = \frac{\Delta_0}{4}, \quad \frac{c_2 \sigma \sqrt{\frac{\varkappa}{\mu}\Delta_0}}{\sqrt{m_0 N_0}} \leq \frac{\Delta_0}{4},$$

SIGM: strongly convex case

Denote $\Delta_k = \varphi(y_{N_k}) - \varphi^*$. Let us perform N_0 iterations with $R^2 = \frac{\varkappa}{\mu}\Delta_0 \stackrel{\text{def}}{=} R_0^2$ and $m = m_0$. Then

$$\mathbb{E}\Delta_1 \leq \frac{c_1 L \varkappa \Delta_0}{\mu N_0^p} + \frac{c_2 \sigma \sqrt{\frac{\varkappa}{\mu}\Delta_0}}{\sqrt{m_0 N_0}} + c_3 N_0^{p-1} \delta$$

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Also we have

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Discussion

We have obtained the method with mean rate of convergence of $\Theta\left(\frac{LR^2}{k^p} + \frac{\sigma R}{\sqrt{k}} + k^{p-1}\delta\right)$, where we can chose $p \in [1, 2]$ in advance. It has the following advantages.

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Thank you for your attention!