Фуллерен C_{60} и другие многогранники с симметриями

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Мы изучаем трехмерные многогранники в пространствах постоянной кривизны, E^3, S^3 и H^3.

Определение

Выпуклый многогранник — это выпуклая комбинация (в соответствующей геометрии) конечного набора точек, называемых *вершинами многогранника*.
Ребро многогранника — это отрезок геодезической, соединяющей пару его вершин.
Грань — это геодезический многоугольник, лежащий на гиперплоскости, проходящей через соответствующие вершины.

Можем вычислять длины ребер, двугранные и плоские углы, площадь поверхности многогранника и его объем.

Определение

Два многогранника имеют один и тот же *комбинаторный тип*, если их реберные скелеты эквивалентны как топологические графы.

Метрические соотношения в гиперболическом треугольнике

Теорема синусов

$$\frac{\sin A}{\operatorname{sh} a} = \frac{\sin B}{\operatorname{sh} b} = \frac{\sin C}{\operatorname{sh} c}$$

Теорема косинусов I

$$\operatorname{ch} c = \operatorname{ch} a \ \operatorname{ch} b - \operatorname{sh} a \ \operatorname{sh} b \ \cos C$$

Теорема косинусов II

$$\cos C = -\cos A \ \cos B + \sin A \ \sin B \ \operatorname{ch} c$$

Площадь треугольника равна $S = \pi - A - B - C$.

Motivation: polyhedra and 3-manifolds

Calculating volumes of polyhedra is a classical problem, that has been well known since Euclid and remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is one of the main geometric invariants for a 3-dimensional manifold. Since Mostov rigidity theorem it is a topological invariant also.

Every 3-manifold can be presented by a fundamental polyhedron. That means we can pair-wise identify the faces of some polyhedron to obtain a 3-manifold. Thus the volume of 3-manifold is the volume of prescribed fundamental polyhedron.

Motivation: what we know about 3-manifolds in E^3, S^3, H^3?

- E^3 — there are just 10 Euclidean 3-forms (compact 3-manifolds without boundary) — 6 orientable and 4 non-orientable;
- S^3 — there is a complete list of spherical manifolds: cyclic case (lens spaces), dihedral case (prism manifolds), tetrahedral case, octahedral case and icosahedral case — all of them are prime, closed and orientable;
- H^3 — there is no such a list of hyperbolic 3-manifolds, we can just order them by volume or complexity or whatever... By the way there exist a manifold of minimal volume (Matveev–Fomenko–Weeks manifold). But still, there are some manifolds with the same volume or the same complexity. That is the reason why we need to know the exact volume.

Motivation: rational volume conjecture

The following problem is widely known and still open.

Rational volume conjecture. Let P be a spherical polyhedron whose dihedral angles are in $\pi \cdot \mathbb{Q}$. Then $Vol\,(P) \in \pi^2 \cdot \mathbb{Q}$.

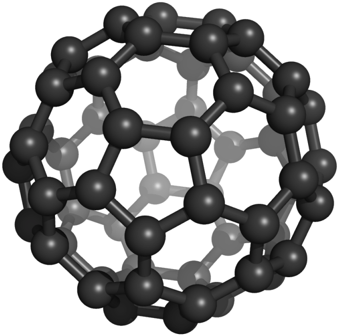
- Examples

1. Let L, be a spherical Lambert cube with dihedral angles $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}$. Then

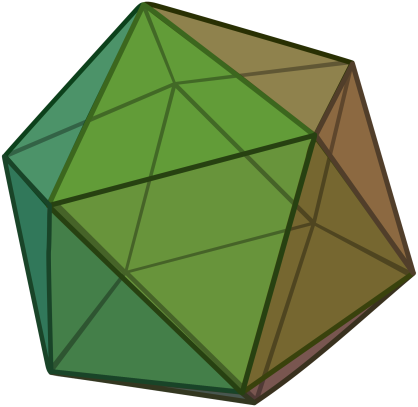
$$Vol\,L\left(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}\right) = \frac{31}{576}\,\pi^2.$$

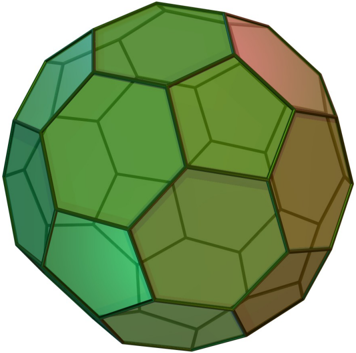
2. Let P be a Coxeter polyhedron in S^3 (that is all dihedral angles of P are $\frac{\pi}{n}$ for some $n \in \mathbb{N}$). Then the Coxeter group $\Delta(P)$ generated by reflections in faces of P is finite and

$$Vol\,(P) = \frac{Vol\,(S^3)}{|\Delta(P)|} = \frac{2\,\pi^2}{|\Delta(P)|} \in \pi^2 \cdot \mathbb{Q}.$$



Buckminsterfullerene or C_{60} is the smallest fullerene molecule containing pentagonal and hexagonal faces in which no two pentagons share an edge (which can be destabilizing, as in pentalene). It is also the most common in terms of natural occurrence, as it can often be found in soot.
The structure of C_{60} is a truncated icosahedron, which resembles an association football ball of the type made of 20 hexagons and 12 pentagons, with a carbon atom at the vertices of each polygon and a bond along each polygon edge.





Thank you for attention!