On the Moment (In)Determinacy of Probability Distributions

Jordan Stoyanov

Newcastle University, UK

e-mail: `stoyanovj@gmail.com`

St Petersburg Seminar in Probability

10 October 2014, 18:00

PLAN:

Discussion on recent works on probability distributions and their characterization as being unique (**M-determinate**) or non-unique (**M-indeterminate**) in terms of the moments.

1. Basics. Best known examples.
2. Briefly on Cramér's, Carleman's and Krein's conditions.
3. Hardy's condition.
4. Criteria based on the rate of growth of the moments.
5. Moment problem for Multivariate distributions.
6. Powers and products of r.v.s and their M-determinacy.
7. Open questions.

Standard Notations and Terminology

Basics: $\mathcal{M}$ = all $X \sim F$, f, finite moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$

$\{m_k\}$ is the moment sequence of F and of X

X, or F is either **M-determinate**, unique with these moments (**M-det**) or it is **M-indet**, there are many with the same moments.

supp(F): $[0, 1]$ (**Hausdorff**);
$\mathbb{R}^+ = [0, \infty)$ (**Stieltjes**);
$\mathbb{R}^1 = (-\infty, \infty)$ (**Hamburger**).

Best known M-indet distributions?

Log-normal distribution: $Z \sim \mathcal{N}(0,1), \;\; X = e^{Z} \sim LogN(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{x}\,\exp\left[-\frac{1}{2}(\ln x)^2\right],\; x > 0; \;\; f(x) = 0,\; x \le 0.$$

No m.g.f., heavy tail, $m_k = \mathbf{E}[X^k] = e^{k^2/2}, k \in \mathbb{N}$. X is M-indet.

Powers of Normal r.v.: $Z \sim \mathcal{N}(0,1)$; Z, Z^2 are M-det, Z^3 is M-indet.

Logarithmic skew normal: $Y_\lambda \sim LSN(\lambda), \;\; g_\lambda(y) = \frac{1}{y}\varphi(\ln y)\Phi(\lambda \ln y)$,
$y > 0$, or $Y_\lambda = e^{X_\lambda}$, $X_\lambda \sim SN(\lambda)$, $f_\lambda(x) = 2\varphi(x)\Phi(\lambda x)$, $x \in \mathbb{R}^1$ (Azzalini).
L&S JAP 2009: All $LSN(\lambda)$, $\lambda \in \mathbb{R}^1$ are M-indet. What if $\lambda \to \pm\infty$?

Powers of Exp r.v.: $\xi \sim Exp(1)$, density $e^{-x}, x > 0$; ξ is M-det. Then:
ξ^2, heavy tail, $m_k = (2k)!$, M-det; ξ^3, more heavy, $m_k = (3k)!$, M-indet.

Cramér: For a r.v. $X \sim F$ on $\mathbb{R}^1$, let the m.g.f. $M(t) = \mathbf{E}[e^{tX}]$ exist, i.e. $M(t)$ is well-defined for $t \in (-t_0, t_0)$, $t_0 > 0$ (**light tails**). Then:

- $X \in \mathcal{M}$ (finite all moments); • X, i.e. F, is M-det.

If no m.g.f., **heavy tails**, either M-det, or M-indet. How heavy?

Carleman: Depending on the support, $\mathbb{R}^1$ or $\mathbb{R}^+$,

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{1/2k}}, \qquad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_k)^{1/2k}}; \ \mathrm{C} = \infty \ \Rightarrow \ F \ \text{ is M-det.}$$

Krein: Assume density $f > 0$; $\mathrm{K}[f] < \infty \ \Rightarrow \ F$ is M-indet. Here

$$\mathrm{K}[f] \equiv \int_{-\infty}^{\infty} \frac{-\ln f(y)}{1+y^2} \mathrm{d}y, \qquad \mathrm{K}[f] \equiv \int_{a}^{\infty} \frac{-\ln f(y^2)}{1+y^2} \mathrm{d}y, \ a \geq 0.$$

Remark: Converses to Carleman and Krein (A. Pakes, G.D. Lin) and a discrete version of Krein (H. Pedersen).

How does Carleman's condition imply M-uniqueness?

Carleman (1926), ..., Koosis (1979) use quasi-analytic functions ...

Idea: Zolotarev metric approch. Work by Klebanov et al. ~ 1982.
$\mathcal{D}$ = all d.f.s (on the real line), metric, $d(F,G)$ between $F, G \in \mathcal{D}$. Then:
(a) $d(F,G)$ is symmetric; (b) $d(F,G)$ satisfies the triangle inequality;
(c) $d(F,G) = 0 \iff F = G$.

Assume $d(F,G)$ is the Lévy metric, or the Kolmogorov (uniform) metric.

Result: Let F and G have finite all moments and the first $2n$ coincide:
$m_k(F) = m_k(G) = m_k,\ k = 1, 2, \ldots, 2n$. Denote $C_n = \sum_{k=1}^{n} (m_{2k})^{-1/2k}$.

$$d(F,G) \le K_2 \frac{\log(1 + C_{n-1})}{(C_{n-1})^{1/4}} \quad (\text{here } K_2 = K_2(m_1, m_2)).$$

Corollary: If $C = \sum_{k=1}^{\infty} (m_{2k})^{-1/2k} = \infty$ (Carleman, (H)), then $C_n \to \infty$,
as $n \to \infty$, and, see (c), $d(F,G) \to 0 \Rightarrow F = G$.

Stieltjes Class: Given $X \sim F$, $F \in \mathcal{M}$, f, M-indet.

$$\mathbf{S} = \mathbf{S}(f,h) = \{f_\varepsilon(x) = f(x)[1+\varepsilon\, h(x)],\ x \in \mathbb{R}^1,\ \varepsilon \in [-1,1]\}.$$

Here h is a **perturbation function:** $|h(x)| \le 1$ for all $x \in \mathbb{R}^1$ and the product $f(x)h(x)$, $x \in \mathbb{R}^1$, has **vanishing moments**,

$$\int x^k f(x)h(x)\,\mathrm{d}x = 0,\ k = 0,1,2,\ldots$$

$L^2[f] =$ Hilbert space, 'weight' f, $h \perp$ all polynomials $\mathcal{P}$.

If h is proper, for any $\varepsilon \in [-1,1]$, f_ε is density. If $X_\varepsilon \sim F_\varepsilon$, f_ε, and

$$\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[X^k],\ k = 1,2,\ldots;\ \varepsilon \in [-1,1];\ X_0 = X.$$

Remark: If F is M-det $\Rightarrow$ $h = 0$ and $\mathbf{S}$ consists only of F, $F' = f$.

Index of dissimilarity in S: $\mathrm{D}(f,h) = \int |h(x)| f(x)\mathrm{d}x$, in $[0,1]$.

"New" Criterion: **G. Hardy (1917/1918).** *The Math. Messenger*

Statement (Hardy): r.v. $X > 0$, $X \sim F$. Suppose $\sqrt{X}$ has m.g.f.:

$\mathbf{E}[e^{t\sqrt{X}}] < \infty$ for $t \in [0, t_0)$, $t_0 > 0$ (H)=Hardy's condition; $\frac{1}{2}$-Cramér.

Then $X \in \mathcal{M}$ and X is M-det: all moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$ are finite and F is the only d.f. with the moment sequence $\{m_k\}$.

Proofs: (a) Original, Titchmarch's book. (b) Our S&L TPA (2012).
Condition (H) $\Longleftrightarrow m_k(X) \le c^k\,(2k)! \Rightarrow$ $\mathrm{C}[\{m_k\}] = \infty \Rightarrow$ X is M-det.

Notice: The condition is on $\sqrt{X}$ but the conclusion is for X.

Corollary: If a r.v. $X > 0$ has a m.g.f., then its square X^2 is M-det.

Result: In (H), $\frac{1}{2}$ is the best possible constant for X to be M-det.
For each $\rho \in (0, \frac{1}{2})$ there is a r.v. Y with $\mathbf{E}[e^{tY^\rho}] < \infty$ s.t. Y is M-indet.

Comment: Hardy's condition is sufficient but not necessary for M-det.

Rate of growth of the moments and (in)determinacy

Stieltjes: r.v. $X \in \mathbb{R}_+$, moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$ Define:

$\Delta_k = \dfrac{m_{k+1}}{m_k}$. It increases in k and let $\Delta_k = \mathcal{O}((k+1)^\gamma)$ as $k \to \infty$.

The number $\gamma =$ **rate of growth of the moments** of X.

Statement 1: If $\gamma \leq 2$, then X is M-det.

Statement 2: $\gamma = 2$ is the best possible constant for which X is M-det.
Equiv: If $\Delta_k = \mathcal{O}((k+1)^{2+\delta})$, $\delta > 0$, there is a r.v. Y which is M-indet.

Statement 3: If $\gamma > 2$, we add one condition, Lin's condition, and show that X is M-indet.

Hamburger: Similar statments hold for r.v.s on $\mathbb{R}^1$, with $m_{2(k+1)}/m_{2k}$ ...

Use Hardy's and Rate of Growth: new proofs of known results:

Exp Example: $\xi \sim Exp(1)$, density e^{-x}, $x > 0$, m.g.f.

Result: ξ^r is M-det for $0 \le r \le 2$ and M-indet for $r > 2$. By Krein-Lin.

Now, if $X = \xi^2$, $\sqrt{X} = \xi$ is Cramér $\Rightarrow$ $X = (\sqrt{X})^2$ is M-det, by Hardy.

Similarly, $X^{r/2}$ is Cramér for $r \in (0, 2]$ $\Rightarrow$ X^r is M-det for $r \in (0, 2]$.

Write $\mathbf{E}[\xi^r]$ via $\Gamma(\cdot)$, rate growth $\gamma \le 2$ for $r \in (0, 2] \Rightarrow \xi^r$ is M-det

For $X = \xi^3$, $m_k = \mathbf{E}[X^k] = (3k)!$, fast $\nearrow$. Rate $\gamma > 2$. Density of ξ^3 is $g(x) = \frac{1}{3} x^{-2/3} e^{-x^{1/3}}$; it satisfies Lin's condition $\Rightarrow \xi^3$ is M-indet.

Stieltjes class: use $g(x)$ and perturbation $h(x) = \sin(\sqrt{3}\, x^{1/3} - \pi/3)$
$\mathbf{S}(g, h) = \{g_\varepsilon(x) = g(x)[1 + \varepsilon h(x)],\ x > 0,\ \varepsilon \in [-1, 1]\}$.
Similarly for any $r > 2$ by using properties of $\Gamma(\cdot)$.

Normal Example: $Z \sim \mathcal{N}(0,1)$, Z^2, Z^3, Z^4, $|Z|^r$.

Z is Cramér $\Rightarrow$ Z is M-det. $|Z|$ is Cramér $\Rightarrow$ Z^2 is M-det, by Hardy.
However, $Z^2 = \chi_1^2$ (light tail) is also Cramér $\Rightarrow$ Z^4 is M-det, by Hardy.

Comment: To apply twice Cramér, and twice Hardy, is the shortest way to prove that power 4 of the normal r.v. Z, Z^4 is M-det.

General Result: $|Z|^r$ is M-det for $0 \le r \le 4$, and M-indet for $r > 4$.

1. Proof by CB, AP 1988. Explicit families, Stieltjes classes.
2. Use Krein for $r > 4$ and Krein and Lin condition for $0 \le r \le 4$.
3. Use rate of growth of moments conditions.

Case: Z^n, $n = 3, 5, \ldots$, on $\mathbb{R}^1$, $\mathrm{C} < \infty$. By Krein $\Rightarrow$ all are M-indet.
Same conclusion by using our rate growth result.

Strange Fact: $X = Z^3$ is M-indet, however $|X| = |Z|^3$ is M-det. Why?
Hint: X on $\mathbb{R}^1$ and $|X|$ on $\mathbb{R}_+$ have different rate of growth of moments.

Multidimensional Moment Problem: Work Going ... [full of traps]

Picture Today: Not too much done for multivariate distributions ...

analytic: Petersen (1982), Berg-Thill (1991), Schmüdgen-Putinar (2008)

probability/statistics: K&S (2011–2013) + a few references therein.

Random vector $X = (X_1, ..., X_n) \in \mathbb{R}^n$ with arbitrary distribution F.

Finite are all multi-indexed moments

$$m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}], \ k_j \geq 0, \ k_1 + \ldots + k_n = k, \ k = 1, 2, ...$$

Same kind of questions and terminology as in dim. 1.

Tools: Cramér, n-dim. m.g.f.; Carleman, next slide; but ... **no Krein**.

Carleman Condition in Dimension n

We need the numbers M_{2k} and M_k, for F on $\mathbb{R}^n$ and $\mathbb{R}^n_+$:

$$M_{2k} = m_{2k,0,\ldots,0} + m_{0,2k,0\ldots,0} + \ldots + m_{0,0,\ldots,0,2k} \quad \text{(Hamburger)},$$

$$M_k = m_{k,0,\ldots,0} + m_{0,k,0\ldots,0} + \ldots + m_{0,0,\ldots,0,k} \quad \text{(Stieltjes)}.$$

Now the n-Carleman quantity is defined, respectively, as follows:

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(M_{2k})^{1/2k}} \quad \text{and} \quad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(M_k)^{1/2k}}.$$

$\mathrm{C} = \infty \ \Rightarrow$ the vector X, or equiv. its n-dimensional d.f. F, is M-det.

If n-Carleman holds for X, then 1-Carleman holds for each X_j. Converse not in general true. There are c.e.s; related to Müntz theorem.

Known Result: Given $X \sim F$ in $\mathbb{R}^n$, marginals $F_1, \ldots, F_n$.

(a) If each of $F_1, \ldots, F_n$ is M-det, then the n-dim. d.f. F is M-det.

(b) If $X_1, \ldots, X_n$ are independent, and F is M-det, then each F_j is M-det.

Comments:

- In (a) we do not say in which way F_j are M-det.
- In (b), $F = F_1 \cdots F_n$, this is used for the converse.
- There are M-det n-dim. d.f.s with M-indet marginals. [Illustrate!]

The last claim is strange, counter-intuitive, but true. Quite analytic.

New Result: Again, $X \sim F$ in $\mathbb{R}^n$, marginals $F_1, \ldots, F_n$ with densities $f_1, \ldots, f_n$ which are positive and smooth. Assume that for each j, Krein condition $(= \infty)$ and Lin condition hold for f_j, $j = 1, \ldots, n$.

Then for any indep/dep structure of X, the n-dim. d.f. F is M-det.

Recent Result: SL, TPA (2012).

Given is a random vector $X \sim F$ with arbitrary distribution in $\mathbb{R}^n$ and finite all multi-indexed moments $m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}], \dots$ Consider the lenght of X: $\|X\| = \sqrt{\|X\|^2} = \sqrt{X_1^2 + \dots + X_n^2}$.

Suppose: 1-dim. non-neg. r.v. $\|X\|$ is Cramér: $\mathbf{E}[e^{c\|X\|}] < \infty, \ c > 0$.

Then the n-dim. Hamburger moment problem for F has a unique solution. Or, we say, the random vector $X \in \mathbb{R}^n$ is M-det, also that F is the only n-dim. d.f. with the set of multi-indexed moments $\{m_{k_1,\dots,k_n}\}$.

Proof: We follow two steps.

Step 1: Cramér for $\|X\| \ \Rightarrow \ \|X\|^2$ is M-det, by Hardy (Stieltjes case).

Step 2: Amazing statement by Putinar-Schmüdgen: If $\|X\|^2$ is M-det (1-dim. Stieltjes), then F is M-det (n-dim. Hamburger).

Products and Powers of Random Variables: ξ and $\perp\!\!\!\perp \xi_1, \dots, \xi_n$

When are $Y_n = \xi_1 \cdots \xi_n$ and $X_n = \xi^n$ M-det, and when M-indet? Same?

Stieltjes case: $\xi > 0$, moments of X_n dominate those of Y_n; 'expect': M-det of $X_n \Rightarrow$ M-det of Y_n, M-indet of $Y_n \Rightarrow$ M-indet of X_n.

Generalized gamma-distributions: $GG(a, b, c)$, $a, b, c > 0$. Density $f(x) = Kx^{a-1}e^{-bx^c}, x > 0$. Here: Exp, gamma, half-normal, χ^2, half-Bessel.

Result 1: $\xi \sim Exp(1)$. Then Y_2 is M-det, $Y_n = \xi_1 \cdots \xi_n$, $n = 3, 4, \dots,$ are all M-indet. Recall ξ^2 is M-det, ξ^n for $n = 3, 4, \dots,$ are all M-indet.

Result 2: $Z \sim \mathcal{N}$, $|Z|$, half-normal. Product of 2, 3 or 4 $\perp\!\!\!\perp$ half-normals is M-det, while product of 5 or more $\perp\!\!\!\perp$ half-normals is M-indet.

Result 3: Half-logistic, $2e^{-x}/(1 + e^{-x})^2$, $x > 0$. Product of 3 or more half-logistic r.v.s is M-indet.

Result 4: Product of 3 or more χ^2 r.v.s is M-indet.

Hamburger case: r.v. ξ on $\mathbb{R}^1$ and $\perp\!\!\!\perp$ copies $\xi_1, \ldots, \xi_n$

Result 1: $Z \sim \mathcal{N}$. Then, $Z_1 Z_2$ is M-det, while $Y_n = Z_1 \cdots Z_n$ for $n = 3, 4, \ldots,$ are all M-indet. Recall, Z^2 is M-det, Z^3 is M-indet, Z^4 is M-det, any next power, $5, 6, \ldots,$ is M-indet. Compare Z^4 and $Z_1 \cdots Z_4$.

Result 2: The product of two $\perp\!\!\!\perp$ Laplace r.v. $(\frac{1}{2} e^{-|x|})$ is M-indet. (Above: product of two *Exp* is M-det.) Square of Laplace r.v. is M-det.

Result 3: Logistic: $1/(2 + e^x + e^{-x})$. Products of 3 or more is M-indet.

Result 4: Product of Laplace r.v. and logistic r.v. is M-indet.

Result 5: Symmetric X on $\mathbb{R}^1$ and $Y > 0$, $T = XY$. If X or Y is M-indet, T is M-indet. If X and Y have rates of growth of moments a and b, and $a + 2b \leq 2$, T is M-det. If $a + 2b > 2$ + cond get M-indet T.

Result 6: N = r.v. in $\mathbb{N}_0$, $\tilde{X} = \xi^N$, $\tilde{Y} = \xi_1 \cdots \xi_N$. All r.v.s $\perp\!\!\!\perp$. We have conditions for M-det and M-indet. E.g., if $N \sim$ Poisson, $Z \sim \mathcal{N}$, both $\tilde{X}$ and $\tilde{Y}$ are M-indet.

Q1: How to construct discr. distr. with moments $\{(3k)!, k = 1, 2, \ldots\}$?

Q2: How to define Krein's condition in dimension 2 or more?

$(\mathsf{K}[f] = \int \frac{-\ln f(x)}{1+x^2} dx < \infty$ or $-\ln f(x^2) \ldots)$

Q3: $X \sim F, F' = f$, finite moments, inf.div., M-indet. Stieltjes class

$\mathbf{S}(f, h) = \{f_\varepsilon = f(1 + \varepsilon h),\ \varepsilon \in [-1, 1]\}$, any perturbation h.

Conjecture: f_ε with $\varepsilon \neq 0$ is not inf.div.

Q4: Continuous r.v. $X \Rightarrow$ discrete (rounded) r.v. $\lfloor X \rfloor$ or $\lfloor X + \frac{1}{2} \rfloor$.

Conjecture: X and $\lfloor X \rfloor$ share the same M-det/indet property.

Q5: $X \sim F$ on $\mathbb{R}^+$, $\mathbf{E}[X^k] = e^{k^2/2}, k \in \mathbb{N}$ ($LogN(0, 1)$ moments!)

Conjecture: If F is unimodal, F is unique and $F = LogN(0, 1)$.

Q6: **Conjecture:** If X is a positive r.v., then for any n, the power X^n and the product $X_1 \cdots X_n$ share the same determinacy property. If X is on $\mathbb{R}^1$, then X^{2n+1} and $X_1 \cdots X_{2n+1}$ share the same determinacy.

References: Books (ST, A, BC)

Hardy, GH (1917/1918): *The Mathematical Messenger* **46/47**

Berg, C (1988): *Ann. Probab.* **16** 910–913

Pedersen, HL (1988): *J. Approx. Theory* **95** 90–100

Lin, GD (1997): *Statist. Probab. Letters* **35** 85–90

De Jeu, M (2003): *Ann. Probab.* **31** 1205–1227

Putinar, M, Schmüdgen, K (2008): *Indiana Univ. Math. J.* **57** 2931–68

Kleiber, C, Stoyanov, J (2011/2013): *J. Multivar. Analysis* **113** 7–18

Stoyanov, J, Lin, GD (2012): *Theory Probab. Appl.* **57**, no. 4, 811–820

Lin, GD, Stoyanov, J (2013): *JOTP* doi 10.1007/s10959-014-0546-z

Stoyanov, J, Lin, GD, DasGupta, A (2014) *JSPI* **154** 166–177

Lin, GD, Stoyanov, J (2014): arXiv: 1406.1654v1 [mathPR] 6 Jun 2014

Stoyanov, J (2013): **Counterexamples in Probability. 3rd edn.** Dover Publications, New York. (1st ed, 2nd ed: Wiley 1987, 1997)