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DEGENERATE K -CONVOLUTED C -SEMIGROUPS AND DEGENERATE K -CONVOLUTED C -COSINE FUNCTIONS IN LOCALLY CONVEX SPACES

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The main purpose of this paper is to investigate degenerate K -convoluted C -semigroups and degenerate K -convoluted C -cosine functions in the setting of sequentially complete locally convex spaces. In our approach, degenerate operator families under consideration are defined locally or globally and their subgenerators are allowed to be multivalued linear operators.

Keywords: *degenerate K -convoluted C -semigroup, degenerate K -convoluted C -cosine function, multivalued linear operator, subgenerator, locally convex space.*

Introduction

The theory of abstract degenerate differential equations is still an active field of research of many mathematicians (cf. [1–5] for the basic source of information on the subject). In this paper, we introduce and systematically analyze degenerate K -convoluted C -semigroups and degenerate K -convoluted C -cosine functions subgenerated by multivalued linear operators (cf. [4; 6–9] for non-degenerate case), thus providing a new unification concept in the theory of abstract degenerate differential equations of first and second order. We are working in sequentially complete locally convex spaces, considering both local and global case. Special attention has been paid to explain, in a brief and concise manner, how we can improve our structural results from the second chapter of monograph [8] to degenerate operator families. We analyze the basic properties of subgenerators, extension and adjoint type theorems, real and complex characterization theorems, as well as generation of local degenerate K -convoluted C -semigroups (K -convoluted C -cosine functions). The study of degenerate K -convoluted C -groups and perturbation properties of the introduced classes is without the scope of this paper.

We use the standard notation throughout the paper. Unless specified otherwise, we assume that E is a Hausdorff sequentially complete locally convex space over the field of complex numbers, SCLCS for short. If X is also a SCLCS, then we denote by $L(E, X)$ the space consisting of all continuous linear mappings from E into X ; $L(E) \equiv L(E, E)$. By \otimes_E (\otimes , if there is no risk for confusion), we denote the fundamental system of seminorms which defines the topology of E . Let \mathcal{B} be the family of bounded subsets of E , and let $p_B(T) := \sup_{x \in B} p(Tx)$, $p \in \otimes_X$, $B \in \mathcal{B}$, $T \in L(E, X)$. Then $p_B(\cdot)$ is a seminorm on $L(E, X)$ and the system $(p_B)_{(p, B) \in \otimes_X \times \mathcal{B}}$ induces the Hausdorff locally convex topology on $L(E, X)$. Let us recall that the spaces $L(E)$ and E^* are sequentially

complete provided that E is barreled [10]. If E is a Banach space, then we denote by $\|x\|$ the norm of an element $x \in E$.

The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s) ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta > 0, t > 0$). If $0 < \tau \leq \infty$ and $(W(t))_{t \in [0, \tau)} \subseteq L(E)$ is strongly continuous, then we define $W^{[n]}(t)x := \int_0^t g_n(t-s)W(s)x ds$, $x \in E, t \in [0, \tau), n \in \mathbb{N}$; we use a similar notion for scalar-valued functions. If $K \in L_{\text{loc}}^1([0, \tau))$ and $n \in \mathbb{N}$, then we denote by $K^{*,n} \in L_{\text{loc}}^1([0, \tau))$ the n -th convolution power of $K(t)$.

1. Multivalued linear operators

In this section, we present some definitions and properties of multivalued linear operators that will be necessary for our further work. For more details, we refer the reader to the monographs [11] by R. Cross and [2] by A. Favini and A. Yagi.

Let X and Y be two SCLCSs. A multivalued map (multimap) $\mathcal{A} : X \rightarrow P(Y)$ is said to be a multivalued linear operator (MLO) iff the following holds:

- (i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X ;
- (ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x, y \in D(\mathcal{A})$, and $\lambda \mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}, x \in D(\mathcal{A})$.

If $X = Y$, then it is also said that \mathcal{A} is an MLO in X . An almost immediate consequence of definition is that, for every $x, y \in D(\mathcal{A})$ and for every $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda \mathcal{A}x + \eta \mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If \mathcal{A} is an MLO, then $\mathcal{A}0$ is a linear manifold in Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. The set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of \mathcal{A} and it is denoted by $N(\mathcal{A})$. The inverse \mathcal{A}^{-1} of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is easily seen that \mathcal{A}^{-1} is an MLO in X , as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if \mathcal{A}^{-1} is single-valued, then \mathcal{A} is said to be injective.

For any mapping $\mathcal{A} : X \rightarrow P(Y)$ we define $\check{\mathcal{A}} := \{(x, y) : x \in D(\mathcal{A}), y \in \mathcal{A}x\}$. Then \mathcal{A} is an MLO iff $\check{\mathcal{A}}$ is a linear relation in $X \times Y$, i.e., iff $\check{\mathcal{A}}$ is a linear subspace of $X \times Y$. Since no confusion seems likely, we will sometimes identify \mathcal{A} with its graph.

If $\mathcal{A}, \mathcal{B} : X \rightarrow P(Y)$ are two MLOs, then we define its sum $\mathcal{A} + \mathcal{B}$ by $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A} + \mathcal{B})$. It can be simply checked that $\mathcal{A} + \mathcal{B}$ is likewise an MLO.

Let $\mathcal{A} : X \rightarrow P(Y)$ and $\mathcal{B} : Y \rightarrow P(Z)$ be two MLOs, where Z is an SCLCS. The product of \mathcal{A} and \mathcal{B} is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. Then $\mathcal{B}\mathcal{A} : X \rightarrow P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \rightarrow P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. It is clear that $z\mathcal{A} : X \rightarrow P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z, \omega \in \mathbb{C}$.

The integer powers of an MLO $\mathcal{A} : X \rightarrow P(X)$ is defined recursively as follows: $\mathcal{A}^0 = I$; if \mathcal{A}^{n-1} is defined, set

$$D(\mathcal{A}^n) := \{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\},$$

and

$$\mathcal{A}^n x := (\mathcal{A}\mathcal{A}^{n-1})x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

It is well known that $(\mathcal{A}^n)^{-1} = (\mathcal{A}^{n-1})^{-1}\mathcal{A}^{-1} = (\mathcal{A}^{-1})^n =: \mathcal{A}^{-n}$, $n \in \mathbb{N}$ and $D(\mathcal{A}^n) = D((\lambda - \mathcal{A})^n)$, $n \in \mathbb{N}_0, \lambda \in \mathbb{C}$. Moreover, if \mathcal{A} is single-valued, then the above definitions are consistent with the usual definition of powers of \mathcal{A} .

If $\mathcal{A} : X \rightarrow P(Y)$ and $\mathcal{B} : X \rightarrow P(Y)$ are two MLOs, then we write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. Assume now that a linear single-valued operator $S : D(S) \subseteq X \rightarrow Y$ has domain $D(S) = D(\mathcal{A})$ and $S \subseteq \mathcal{A}$, where $\mathcal{A} : X \rightarrow P(Y)$ is an MLO. Then S is called a section of \mathcal{A} ; if this is the case, we have $\mathcal{A}x = Sx + \mathcal{A}0$, $x \in D(\mathcal{A})$ and $R(\mathcal{A}) = R(S) + \mathcal{A}0$.

We say that an MLO operator $\mathcal{A} : X \rightarrow P(Y)$ is closed if for any nets (x_τ) in $D(\mathcal{A})$ and (y_τ) in Y such that $y_\tau \in \mathcal{A}x_\tau$ for all $\tau \in I$ we have that $\lim_{\tau \rightarrow \infty} x_\tau = x$ and $\lim_{\tau \rightarrow \infty} y_\tau = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

If $\mathcal{A} : X \rightarrow P(Y)$ is an MLO, then we define the adjoint $\mathcal{A}^* : Y^* \rightarrow P(X^*)$ of \mathcal{A} by its graph

$$\mathcal{A}^* := \left\{ (y^*, x^*) \in Y^* \times X^* : \langle y^*, y \rangle = \langle x^*, x \rangle \text{ for all pairs } (x, y) \in \mathcal{A} \right\}.$$

It is simply verified that \mathcal{A}^* is a closed MLO, and that $\langle y^*, y \rangle = 0$ whenever $y^* \in D(\mathcal{A}^*)$ and $y \in \mathcal{A}0$. Furthermore, the equations [2, (1.2)–(1.6)] continue to hold for adjoints of MLOs acting on locally convex spaces.

Concerning the integration of functions with values in sequentially complete locally convex spaces, we follow the approach of C. Martinez and M. Sanz [12, pp. 99–102]. Denote by Ω a locally compact and separable metric space and by μ a locally finite Borel measure defined on Ω . Then the following fundamental lemma holds:

Lemma 1. *Suppose that $\mathcal{A} : X \rightarrow P(Y)$ is a closed MLO. Let $f : \Omega \rightarrow X$ and $g : \Omega \rightarrow Y$ be μ -integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f d\mu \in D(\mathcal{A})$ and $\int_\Omega g d\mu \in \mathcal{A} \int_\Omega f d\mu$.*

Let $C \in L(X)$ be injective and let $C\mathcal{A} \subseteq \mathcal{A}C$. The C -resolvent set of an MLO \mathcal{A} in X , $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which $R(C) \subseteq R(\lambda - \mathcal{A})$ and $(\lambda - \mathcal{A})^{-1}C$ is a single-valued bounded operator on X . The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C -resolvent of \mathcal{A} ($\lambda \in \rho_C(\mathcal{A})$). In this paper, we analyze the general situation in which the operator $C \in L(X)$ is not necessarily injective. Then the operator $(\lambda - \mathcal{A})^{-1}C$ appearing in our considerations is no longer single-valued and some unpleasant questions concerning the uniqueness of solutions of corresponding abstract differential inclusions of first and second order obviously appear. The resolvent set and spectrum of \mathcal{A} , $\rho(\mathcal{A})$ and $\sigma(\mathcal{A})$ shortly, are defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$ and $\sigma(\mathcal{A}) := \mathbb{C} \setminus \rho(\mathcal{A})$.

In the remaining part of paper, the underlying SCLCS will be denoted by E . Suppose that $\mu = dt$ is the Lebesgue's measure on $\Omega = [0, \infty)$ and $f : [0, \infty) \rightarrow E$ is a locally Lebesgue integrable function. As in the Banach space case, we denote the space consisting of such functions by $L_{\text{loc}}^1([0, \infty) : E)$. We are concerned with the existence of Laplace integral

$$(\mathcal{L}f)(\lambda) := \tilde{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) dt := \lim_{\tau \rightarrow \infty} \int_0^\tau e^{-\lambda t} f(t) dt,$$

for $\lambda \in \mathbb{C}$ and $f \in L_{\text{loc}}^1([0, \infty) : E)$. If $\tilde{f}(\lambda_0)$ exists for some $\lambda_0 \in \mathbb{C}$, then we define the abscissa of convergence of $\tilde{f}(\cdot)$ by

$$\text{abs}_E(f) := \inf \{ \text{Re } \lambda : \tilde{f}(\lambda) \text{ exists} \};$$

otherwise, $\text{abs}_E(f) := +\infty$. It is said that $f(\cdot)$ is Laplace transformable, or equivalently, that $f(\cdot)$ belongs to the class (P1)- E , iff $\text{abs}_E(f) < \infty$. We abbreviate $\text{abs}_E(f)$ to $\text{abs}(f)$ if there is no risk for confusion; if $E = \mathbb{C}$, then we also write (P1) in place of (P1)- E .

We need the following useful lemma.

Lemma 2. [3].

- (i) Suppose that $\mathcal{A} : E \rightarrow P(E)$ is a closed MLO₂, as well as $f \in (P1)\text{-}E$, $l \in (P1)\text{-}E$ and $(f(t), l(t)) \in \mathcal{A}$ for a.e. $t \geq 0$. Then $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ for $\operatorname{Re} \lambda > \max(\operatorname{abs}(f), \operatorname{abs}(l))$.
- (ii) Suppose that $\mathcal{A} : E \rightarrow P(E)$ is a closed MLO, as well as $f \in (P1)\text{-}E$, $l \in (P1)\text{-}E$ and $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ for $\operatorname{Re} \lambda > \max(\operatorname{abs}(f), \operatorname{abs}(l))$. Then $l(t) \in \mathcal{A}f(t)$ for any $t \geq 0$ which is a point of continuity of both functions $f(t)$ and $l(t)$.

The reader may consult [6; 8; 9; 13] for further information about vector-valued Laplace transform.

2. Convoluted C -semigroups and convoluted C -cosine functions in locally convex spaces

The notions of a (local) K -convoluted C -semigroup and a (local) K -convoluted C -cosine function in locally convex space (cf. [8; 14–18] for similar concepts) are introduced in the subsequent definitions. Observe that we do not require the injectiveness of operator $C \in L(E)$ here.

Definition 1. Let $0 \neq K \in L_{\text{loc}}^1([0, \infty))$. A strongly continuous operator family $(S_K(t))_{t \in [0, \tau)} \subseteq L(E)$ is called a (local, if $\tau < \infty$) K -convoluted C -semigroup iff the following holds:

- (i) $S_K(t)C = CS_K(t)$, $t \in [0, \tau)$;
- (ii) for all $x \in E$ and $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have

$$S_K(t)S_K(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] K(t+s-r)S_K(r)Cx \, dr. \quad (1)$$

Definition 2. Let $0 \neq K \in L_{\text{loc}}^1([0, \tau))$. A strongly continuous operator family $(C_K(t))_{t \in [0, \tau)} \subseteq L(E)$ is called a (local, if $\tau < \infty$) K -convoluted C -cosine function iff the following holds:

- (i) $C_K(t)C = CC_K(t)$, $t \in [0, \tau)$;
- (ii) for all $x \in E$ and $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$, we have

$$\begin{aligned} 2C_K(t)C_K(s)x &= \left(\int_t^{t+s} - \int_0^s \right) K(t+s-r)C_K(r)Cx \, dr \\ &\quad + \int_{t-s}^t K(r-t+s)C_K(r)Cx \, dr + \int_0^s K(r+t-s)C_K(r)Cx \, dr, \quad t \geq s; \\ 2C_K(t)C_K(s)x &= \left(\int_s^{t+s} - \int_0^t \right) K(t+s-r)C_K(r)Cx \, dr \\ &\quad + \int_{s-t}^s K(r+t-s)C_K(r)Cx \, dr + \int_0^t K(r-t+s)C_K(r)Cx \, dr, \quad t < s. \end{aligned} \quad (2)$$

By a (local) C -regularized semigroup, resp., (local) C -regularized cosine function, we mean any strongly continuous operator family $(S(t))_{t \in [0, \tau)} \subseteq L(E)$, resp., $(C(t))_{t \in [0, \tau)} \subseteq L(E)$, satisfying that $S(t)C = CS(t)$, $t \in [0, \tau)$ and $S(t+s)C = S(t)S(s)$ for all $t, s \in [0, \tau)$ with $t+s \in [0, \tau)$, resp., $C(t)C = CC(t)$, $t \in [0, \tau)$ and $2C(t)C(s) = C(t+s)C + C(|t-s|)C$ for all $t, s \in [0, \tau)$ with $t+s \in [0, \tau)$. A global C -regularized semigroup $(S(t))_{t \geq 0}$ is said to be entire iff, for every $x \in E$, the mapping $t \mapsto S(t)x$, $t \geq 0$ can be analytically extended to the whole complex plane.

A K -convoluted C -semigroup $(S_K(t))_{t \in [0, \tau)}$ is said to be locally equicontinuous iff, for every $t \in (0, \tau)$, the family $\{S_K(s) : s \in [0, t]\}$ is equicontinuous. In the case $\tau = \infty$, $(S_K(t))_{t \geq 0}$ is said to be exponentially equicontinuous (equicontinuous) iff there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t} S_K(t) : t \geq 0\}$ is equicontinuous; the infimum of such numbers is said to be the exponential type of $(S_K(t))_{t \geq 0}$. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geq 0$, then it is also said that $(S_K(t))_{t \in [0, \tau)}$ is an α -times integrated C -semigroup; 0-times C -integrated semigroup is nothing else but C -regularized semigroup. The above notion can be simply understood for the class of K -convoluted C -cosine functions.

For a K -convoluted C -semigroup $(S_K(t))_{t \in [0, \tau)}$, resp., K -convoluted C -cosine function $(C_K(t))_{t \in [0, \tau)}$, we define its (integral) generator $\hat{\mathcal{A}}$ by graph

$$\hat{\mathcal{A}} := \left\{ (x, y) \in E \times E : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)y ds, t \in [0, \tau) \right\}, \text{ resp.,}$$

$$\hat{\mathcal{A}} := \left\{ (x, y) \in E \times E : C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)y ds, t \in [0, \tau) \right\};$$

with $\Theta(t) \equiv 1$, we obtain the definition of integral generator of a C -regularized semigroup (C -regularized cosine function).

Denote by $(W(t))_{t \in [0, \tau)}$ any of the above considered operator families, and $a(t) \equiv 1$ ($a(t) \equiv t$) in the case of consideration semigroups (cosine functions). In what follows, we will refer to $(W(t))_{t \in [0, \tau)}$ as a (local) (a, Θ) -regularized C -resolvent family (cf. [13] for non-degenerate case). A fairly general class of degenerate (a, k) -regularized C -resolvent families has been recently considered in a series of recent research papers of the author, and our main aim here is to analyze the convoluted versions of semigroup property and d'Alambert functional equation for degenerate K -convoluted C -semigroups and degenerate K -convoluted C -cosine functions (in [19, Theorem 8], Z.-D. Mei, J.-G. Peng and J.-H. Gao have recently proved a composition property for the class of so-called K -convoluted α -order C -semigroups; in this paper, we will not follow this approach for the abstract time-fractional differential equations associated with the use of Caputo derivatives).

It is worth noting that the functional equality of $(W(t))_{t \in [0, \tau)}$ and its strong continuity together imply that $W(t)W(s) = W(s)W(t)$ for all $t, s \in [0, \tau)$ with $t+s \in [0, \tau]$; in general case, it is not true that $W(t)W(s) = W(s)W(t)$ for $\tau < t+s < 2\tau$ (cf. [20] and Example 1 below). In the present paper, we will accept the following notion of non-degeneracy: $(W(t))_{t \in [0, \tau)}$ is said to be non-degenerate iff the assumption $W(t)x = 0$ for all $t \in [0, \tau)$ implies $x = 0$ (cf. [17, p. 2] for more details on the subject).

It is clear that the integral generator $\hat{\mathcal{A}}$ of $(W(t))_{t \in [0, \tau)}$ is an MLO in E and that the local equicontinuity of $(W(t))_{t \in [0, \tau)}$ implies that $\hat{\mathcal{A}}$ is closed. Furthermore, we have that $\hat{\mathcal{A}} \subseteq C^{-1}\hat{\mathcal{A}}C$ in the MLO sense, and that $\hat{\mathcal{A}} = C^{-1}\hat{\mathcal{A}}C$ provided additionally that the operator C is injective.

In the following definition, we introduce the notion of an (a, k, C) -subgenerator of a strongly continuous operator family $(Z(t))_{t \in [0, \tau]} \subseteq L(E)$. This definition extends the corresponding one introduced by C.-C. Kuo [15; 16, Definition 2.4] in the setting of Banach spaces, where it has also been assumed that the operator $\mathcal{A} = A$ is linear and single-valued.

Definition 3. Suppose that $0 < \tau \leq \infty$, $C \in L(E)$, $a \in L^1_{\text{loc}}([0, \tau])$, $a \neq 0$, $k \in C([0, \tau])$ and $(Z(t))_{t \in [0, \tau]} \subseteq L(E)$ is a strongly continuous operator family. By an (a, k, C) -subgenerator of $(Z(t))_{t \in [0, \tau]}$ we mean any MLO \mathcal{A} in E satisfying the following two conditions:

- (i) $Z(t)x - k(t)Cx = \int_0^t a(t-s)Z(s)y ds$, whenever $t \in [0, \tau]$ and $y \in \mathcal{A}x$;
- (ii) for all $x \in E$ and $t \in [0, \tau]$, we have $\int_0^t a(t-s)Z(s)x ds \in D(\mathcal{A})$ and $Z(t)x - k(t)Cx \in \mathcal{A} \int_0^t a(t-s)Z(s)x ds$.

In this paper, we focus our attention on (a, k, C) -subgenerators of K -convoluted C -semigroups and K -convoluted C -cosine functions; in the case that $(Z(t))_{t \in [0, \tau]} = (W(t))_{t \in [0, \tau]}$ and $k(t) \equiv \Theta(t)$, then we simply say that \mathcal{A} is a subgenerator of $(W(t))_{t \in [0, \tau]}$. By $\chi(W)$ we denote the set consisting of all subgenerators of $(W(t))_{t \in [0, \tau]}$. The local equicontinuity of $(W(t))_{t \in [0, \tau]}$ yields that for each subgenerator $\mathcal{A} \in \chi(W)$ we have $\overline{\mathcal{A}} \in \chi(W)$. The set $\chi(W)$ can have infinitely many elements; if $\mathcal{A} \in \chi(W)$, then $\mathcal{A} \subseteq \hat{\mathcal{A}}$ (cf. [17, Example 4.10, 4.11]; in these examples, the partially ordered set $(\chi_{\text{sv}}(W), \subseteq)$, where $\chi_{\text{sv}}(W)$ denotes the set consisting of all single-valued linear subgenerators of $(W(t))_{t \in [0, \tau]}$, does not have the greatest element) and, if $\chi(W)$ is finite, then it need not be a singleton [8]. In general, the set $\chi(W)$ can be empty and the integral generator of $(W(t))_{t \in [0, \tau]}$ need not be a subgenerator of $(W(t))_{t \in [0, \tau]}$ in the case that $\tau < \infty$:

Example 1. [20]. Let $0 < \tau < \infty$ and let $U : [\tau/2, \tau) \rightarrow L(E)$ be a strongly continuous function such that $U(\tau/2) = 0$ and $U(t)$ is injective for all $t \in (\tau/2, \tau)$. Define $T : [0, \tau) \rightarrow L(E \times E)$ by $T(t)(x \ y)^T := (0 \ y)^T$ for $t \in [0, \tau/2)$, $x, y \in E$ and $T(t)(x \ y)^T := (U(t)x \ y)^T$ for $t \in [\tau/2, \tau)$, $x, y \in E$. Set $C := T(0)$. Then C is not injective, $(T(t))_{t \in [0, \tau)} \subseteq L(E \times E)$ is a non-degenerate local C -regularized semigroup and it can be easily seen that the violation of condition

$$U(t) \int_{\tau/2}^s U(r)x dr = \int_{\tau/2}^t U(r)U(s)x dr, \quad x \in E, \quad t, s \in [\tau/2, \tau),$$

implies that the set $\chi(T)$ is empty (in the MLO sense) as well as that the integral generator $\hat{\mathcal{A}}$ of $(T(t))_{t \in [0, \tau)}$ is not a subgenerator of $(T(t))_{t \in [0, \tau)}$. We can similarly construct an example of a non-degenerate local C -regularized cosine function $(C(t))_{t \in [0, \tau)}$, with C being not injective, so that $(C(t))_{t \in [0, \tau)}$ does not have any subgenerator in the MLO sense [21]. Observe, finally, that the notion introduced in [17, Definition 4.3; 22, Definition 2.1; 8, Remark 2.1.8 (i)] cannot be used for proving the nonemptiness of set $\chi(W)$ in degenerate case.

Suppose that $K(t)$ is a kernel on $[0, \tau)$, in the case of consideration of convoluted operator families, C is injective and \mathcal{A} is an MLO. Then we can simply prove that there exists at most one locally equicontinuous (a, Θ) -regularized C -resolvent family $(W(t))_{t \in [0, \tau)}$ which have \mathcal{A} as a subgenerator.

If \mathcal{A} and \mathcal{B} are subgenerators of $(W(t))_{t \in [0, \tau)}$, then for any complex numbers α, β such that $\alpha + \beta = 1$ we have that $\alpha\mathcal{A} + \beta\mathcal{B}$ is a subgenerator of $(W(t))_{t \in [0, \tau)}$. Set $\mathcal{A} \wedge \mathcal{B} :=$

$(1/2)\mathcal{A} + (1/2)\mathcal{B}$. We define the operator $\mathcal{A} \vee_0 \mathcal{B}$ by $D(\mathcal{A} \vee_0 \mathcal{B}) := \text{span}[D(\mathcal{A}) \cup D(\mathcal{B})]$ and

$$\mathcal{A} \vee_0 \mathcal{B}(ax + by) := aAx + bBy, \quad x \in D(\mathcal{A}), \quad y \in D(\mathcal{B}), \quad a, b \in \mathbb{C};$$

$\mathcal{A} \vee \mathcal{B} := \overline{\mathcal{A} \vee_0 \mathcal{B}}$. Then $\mathcal{A} \vee_0 \mathcal{B}$ is a subgenerator of $(W(t))_{t \in [0, \tau]}$, and $\mathcal{A} \vee \mathcal{B}$ is a subgenerator of $(W(t))_{t \in [0, \tau]}$, provided that $(W(t))_{t \in [0, \tau]}$ is locally equicontinuous. In non-degenerate case, it is well known that the set $\chi(W)$, equipped with the operations \wedge and \vee , forms a complete Boolean lattice [22; 8, Remark 2.1.8(ii)-(iii)]. We will not discuss the properties of $(\chi(W), \wedge, \vee)$ in degenerate case.

If $\mathcal{A} \in \chi(W)$ and $y \in \mathcal{A}x$, then $(\int_0^t a(t-s)W(s)x ds, W(t)x - \Theta(t)Cx) \in \mathcal{A}$, $t \in [0, \tau)$, i.e., $(\int_0^t a(t-s)W(s)x ds, \int_0^t a(t-s)W(s)y ds) \in \mathcal{A}$, $t \in [0, \tau)$, which implies after differentiation that $(W(t)x, W(t)y) \in \overline{\mathcal{A}}$, $t \in [0, \tau)$. Hence, $W(t)\mathcal{A} \subseteq \mathcal{A}W(t)$, $t \in [0, \tau)$ for any closed subgenerator \mathcal{A} of $(W(t))_{t \in [0, \tau)}$. If this is the case, then $C^{-1}\mathcal{A}C$ also commutes with $W(t)$: Suppose that $(x, y) \in C^{-1}\mathcal{A}C$. Then $Cy \in \mathcal{A}Cx$, $CW(t)x = W(t)Cx \in D(\mathcal{A})$, $t \in [0, \tau)$ and $CW(t)y = W(t)Cy \in W(t)\mathcal{A}Cx \subseteq \mathcal{A}CW(t)x = \mathcal{A}W(t)Cx$, $t \in [0, \tau)$ so that $W(t)y \in C^{-1}\mathcal{A}CW(t)x$, $t \in [0, \tau)$ and $W(t)[C^{-1}\mathcal{A}C] \subseteq [C^{-1}\mathcal{A}C]W(t)$, $t \in [0, \tau)$.

Suppose again that \mathcal{A} is a closed subgenerator of $(W(t))_{t \in [0, \tau)}$ and $y \in \mathcal{A}x$. Then $(\int_0^t a(t-s)W(s)y ds, W(t)y - \Theta(t)Cy) = (W(t)x - \Theta(t)Cx, W(t)y - \Theta(t)Cy) \in \mathcal{A}$, $t \in [0, \tau)$, from which we may conclude that $\Theta(t)[Cy - \mathcal{A}Cx] \in W(t)y - \mathcal{A}W(t)x \in \mathcal{A}W(t)x - \mathcal{A}W(t)x = \mathcal{A}0$, $t \in [0, \tau)$. Taking any $t \in [0, \tau)$ with $\Theta(t) \neq 0$, we get that $Cy - \mathcal{A}Cx \in \mathcal{A}0$, $Cy \in \mathcal{A}Cx + \mathcal{A}0 = \mathcal{A}Cx$, and consequently, $C\mathcal{A} \subseteq \mathcal{A}C$, i.e., $\mathcal{A} \subseteq C^{-1}\mathcal{A}C$. Now we proceed by repeating some arguments from the proof of [8, Proposition 2.1.6 (i)]. Let $(x, y) \in \hat{\mathcal{A}}$. As above, we have $(\int_0^t a(t-s)W(s)x ds, \int_0^t a(t-s)W(s)y ds) \in \mathcal{A}$, $t \in [0, \tau)$ and $(W(t)x, W(t)y) \in \overline{\mathcal{A}} = \mathcal{A}$, $t \in [0, \tau)$. This implies

$$W(t)y \in \mathcal{A}W(t)x = \mathcal{A} \left[\Theta(t)Cx + \int_0^t a(t-s)W(s)y ds \right], \quad t \in [0, \tau),$$

and, since $\int_0^t a(t-s)W(s)y ds \in D(\mathcal{A})$ for $t \in [0, \tau)$, $Cx \in D(\mathcal{A})$ as well as $0 \in \mathcal{A}[\Theta(t)Cx + \int_0^t a(t-s)W(s)y ds - \int_0^t a(t-s)W(s)y ds] - \Theta(t)Cy$, $t \in [0, \tau)$. Hence, $Cy \in \mathcal{A}Cx$ and $\hat{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$. If, along with the closedness of \mathcal{A} we have that the operator C is injective, then we can simply verify that $C^{-1}\mathcal{A}C$ is likewise a closed subgenerator of $(W(t))_{t \in [0, \tau)}$, so that $\hat{\mathcal{A}} = C^{-1}\mathcal{A}C$ by previously proved inclusion $\hat{\mathcal{A}} \subseteq C^{-1}\mathcal{A}C$ and the fact that $\hat{\mathcal{A}}$ extends any subgenerator from $\chi(W)$.

Let \mathcal{A} and \mathcal{B} be two subgenerators of $(W(t))_{t \in [0, \tau)}$, let \mathcal{B} be closed, and let $y \in \mathcal{A}x$. Then we have $(\int_0^t a(t-s)W(s)y ds, W(t)y - \Theta(t)Cy) = (W(t)x - \Theta(t)Cx, W(t)y - \Theta(t)Cy) \in \mathcal{B}$, $t \in [0, \tau)$, which implies by Lemma 1 that $((a*W)(t)x - (a*\Theta)(t)Cx, (a*W)(t)y - (a*\Theta)(t)Cy) \in \mathcal{B}$, $t \in [0, \tau)$. Since $(a*W)(t)x \in D(\mathcal{B})$, $t \in [0, \tau)$, the above implies that $Cx \in D(\mathcal{B})$. Hence, $C(D(\mathcal{A})) \subseteq D(\mathcal{B})$.

In [17, Example 4.10], for each $\alpha > 0$ it has been constructed an example of a global degenerate α -times integrated C -semigroup with infinitely many single-valued bounded subgenerators. This example shows that the equivalence relation $A \subseteq B \Leftrightarrow D(A) \subseteq D(B)$, as well as any of the equalities $Ax = Bx$, $x \in D(A) \cap D(B)$ and $\rho(A) = \emptyset$, $A \neq \hat{A}$ does not hold for subgenerators in degenerate case (cf. [8, Proposition 2.1.6 (ii), (iii), (viii)], and [8, Proposition 2.1.16] for cosine operator functions case). Furthermore, the subgenerators from this example do not have the same eigenvalues, in general, so that the assertion of [8, Proposition 2.1.6 (v)] does not hold in degenerate case, as well; the same example shows that the equality $C^{-1}\mathcal{A}C = C^{-1}\mathcal{B}C$ (in the MLO sense) is not generally

true for subgenerators of degenerate integrated C -semigroups (cf. [8, Proposition 2.1.6 (ii)]), as well as that $C^{-1}\mathcal{A}C$ need not be a subgenerator of $(W(t))_{t \in [0, \tau]}$ and that $C^{-1}\mathcal{A}C$ can be a proper extension of $\hat{\mathcal{A}}$: Let $W(t) = T(t) := C \in L(E)$ for all $t \geq 0$. Then $(T(t))_{t \geq 0}$ is a global C -regularized semigroup with the integral generator $\hat{\mathcal{A}} = E \times N(C)$ and an MLO \mathcal{A} belongs to $\chi(T)$ iff $R(C) \times \{0\} \subseteq \mathcal{A}$ and $R(\mathcal{A}) \subseteq N(C)$. In particular, $\mathcal{A} = R(C) \times N(C)$ is a subgenerator of $(T(t))_{t \geq 0}$, $C^{-1}\mathcal{A}C = E \times N(C^2)$ which is, in general, a proper extension of $\hat{\mathcal{A}}$, and not a subgenerator of $(T(t))_{t \geq 0}$ provided that there exists an element $x \in E$ such that $C^2x = 0$ and $Cx \neq 0$.

We need the following useful extension of [15; 16, Lemma 2.1]:

Lemma 3. *Let $0 < \tau \leq \infty$, and let $(W(t))_{t \in [0, \tau]}$ be a strongly continuous operator family which commutes with C . Then the following is equivalent:*

- (i) $(W(t))_{t \in [0, \tau]}$ is an (a, Θ) -regularized C -resolvent family;
- (ii) for any complex non-zero polynomial $P(z)$ and $(W^{P, a_0}(t) \equiv (P * W)(t) + a_0W(t))_{t \in [0, \tau]}$ for any $a_0 \in \mathbb{C}$ is an $(a, P * \Theta + a_0\Theta)$ -regularized C -resolvent family;
- (iii) there exist a complex non-zero polynomial $P(z)$ and a number $a_0 \in \mathbb{C}$ such that $(W^{P, a_0}(t))_{t \in [0, \tau]}$ is an $(a, P * \Theta + a_0\Theta)$ -regularized C -resolvent family.

Proof. We will prove the lemma in the case that $(W(t))_{t \in [0, \tau]} = (S_K(t))_{t \in [0, \tau]}$ or $(W(t))_{t \in [0, \tau]} = (C_K(t))_{t \in [0, \tau]}$ for some $K \in L_{\text{loc}}^1([0, \tau])$, $K \neq 0$. The implication (i) \Rightarrow (ii) can be shown by induction on degree of $P(z)$. Consider first the semigroup case. If $\text{dg}(P) = 0$, then there exists a number $a_1 \in \mathbb{C}$ such that $P(z) \equiv a_1$ and, since $(S_\Theta(t) \equiv S_K^{[1]}(t))_{t \in [0, \tau]}$ is a Θ -convoluted C -semigroup by the proof of [15, Lemma 2.1], it suffices to show that, for every $x \in E$ and for every $t, s \in [0, \tau]$ with $t + s \in [0, \tau)$, we have:

$$\begin{aligned} S_\Theta(t)S_K(s)x + S_K(t)S_\Theta(s)x &= \\ &= \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \{ \Theta(t+s-r)S_K(r)Cx + K(t+s-r)S_\Theta(r)Cx \} dr. \end{aligned}$$

Let such x, t, s be fixed. Since $S_K(s)S_\Theta(t)x = S_K(t)S_\Theta(s)x$, the identities [15, (2.2) and (2.4)] yield

$$\begin{aligned} &\left[\int_0^{t+s} - \int_0^t - \int_0^s \right] K(t+s-r)S_\Theta(r)Cx dr = \\ &= S_K(t)S_\Theta(s)x + \Theta(s)S_\Theta(t)Cx = S_K(s)S_\Theta(t)x + \Theta(t)S_\Theta(s)Cx \end{aligned} \quad (3)$$

and

$$\begin{aligned} &\left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-r)S_K(r)Cx dr = \\ &= \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] K(t+s-r)S_\Theta(r)Cx dr - \Theta(s)S_\Theta(t)Cx - \Theta(t)S_\Theta(s)Cx = \\ &= S_K(s)S_\Theta(t)x + \Theta(t)S_\Theta(s)Cx - \Theta(s)S_\Theta(t)Cx - \Theta(t)S_\Theta(s)Cx = \\ &= S_K(s)S_\Theta(t)x - \Theta(s)S_\Theta(t)Cx. \end{aligned} \quad (4)$$

The claimed assertion follows by adding (3) and (4). Suppose now that (ii) holds for any complex non-zero polynomial $P(z)$ of degree strictly less than $n \in \mathbb{N}$, any number $a_0 \in \mathbb{C}$ and any K -convoluted C -semigroup $(S_K(t))_{t \in [0, \tau]}$. Let us prove that (ii) holds for arbitrary complex non-zero polynomial $P(z) = a_{n+1}g_{n+1}(z) + a_n g_n(z) + \dots + a_1 g_1(z)$ and

arbitrary number $a_0 \in \mathbb{C}$. Then we can always find complex numbers $A_0, A_1, B_1, \dots, B_n$ such that: $a_{n+1} = A_1 B_n$, $a_j = A_1 B_{j-1} + A_0 B_j$ for $1 \leq j \leq n$ and $a_0 = A_0 B_0$, so that $W^{P, a_0}(\cdot) = S_K^{P, a_0}(\cdot) = A_1(g_1 * S_{K_1})(\cdot) + A_0 S_{K_1}(\cdot) = S_{K_1}^{A_1, A_0}(\cdot)$, where $S_{K_1}(\cdot) = S_K^{P_1, B_0}(\cdot)$ with $P_1(z) = B_n g_n(z) + B_{n-1} g_{n-1}(z) + \dots + B_1 g_1(z)$. Hence, (ii) follows from its validity for constant polynomials and induction hypothesis. The implication (ii) \Rightarrow (iii) is trivial and the implication (ii) \Rightarrow (iii) holds on account of the proof of [15, Lemma 2.1], with $P(z) = 1$ and $a_0 = 0$. The proof for K -convoluted C -cosine functions is almost the same and here it is only worth pointing out how one can prove that the part (ii) holds for constant complex polynomials. Let $(C_K(t))_{t \in [0, \tau]}$ be a K -convoluted C -cosine function, let $P(z) \equiv a_1 \in \mathbb{C}$, and let $a_0 \in \mathbb{C}$. It is very simple to prove that (ii) holds provided $a_0 = 0$, so that $(C_\Theta(t) = C_K^{[1]}(t))_{t \in [0, \tau]}$ is a Θ -convoluted C -cosine function. In the remnant of proof, we assume that $t, s \in [0, \tau]$, $t + s < \tau$ and $t \geq s$ (the case $t < s$ can be considered analogously). It suffices to show that

$$\begin{aligned} & 2[C_\Theta(t)C_K(s)x + C_K(t)C_\Theta(s)x] = \\ & = \left(\int_t^{t+s} - \int_0^s \right) \{ \Theta(t+s-r)C_K(r)Cx + K(t+s-r)C_\Theta(r)Cx \} dr + \\ & \quad + \int_{t-s}^t \{ \Theta(r-t+s)C_K(r)Cx + K(r-t+s)C_\Theta(r)Cx \} dr + \\ & \quad + \int_0^s \{ \Theta(r+t-s)C_K(r)Cx + K(r+t-s)C_\Theta(r)Cx \} dr. \end{aligned} \quad (5)$$

Since $(C_\Theta(t))_{t \in [0, \tau]}$ is a Θ -convoluted C -cosine function, the final part of proof of [8, Theorem 2.1.13] shows that (see e.g. the equations [8, (36)–(41)])

$$\begin{aligned} & 2C_K(t)C_\Theta(s)x = 2\Theta(t)C_\Theta(s)Cx + \\ & \quad + \left(\int_t^{t+s} - \int_0^s \right) \Theta(t+s-r)C_K(r)Cx dr + \\ & \quad + \int_{t-s}^t \Theta(r-t+s)C_K(r)Cx dr - \int_0^s \Theta(r+t-s)C_K(r)Cx dr. \end{aligned} \quad (6)$$

Using the equality $2C_\Theta(t)C_K(s)x = 2(d/ds)C_\Theta(t)C_\Theta(s)x$ and the composition property for $(C_\Theta(t))_{t \in [0, \tau]}$, a straightforward computation shows that the value of term $2C_\Theta(t)C_K(s)x$, appearing on the left hand side of (5), is equal to $R_1 - R$, where R_1 , resp., R , is the term on the right hand side of (5), resp., (6). The proof of the lemma is thereby complete. \square

Inspecting the proofs of [15; 16, Theorem 2.2] yields the following (cf. also [13, Section 2.8], where the validity of equation (7) for all values $t, s \in [0, \tau]$ has been used for definition of an (a, Θ) -regularized C -resolvent family):

Theorem 1. *Let $0 < \tau \leq \infty$, and let $(W(t))_{t \in [0, \tau]}$ be a strongly continuous operator family which commutes with C .*

(i) *If $(W(t))_{t \in [0, \tau]}$ is an (a, Θ) -regularized C -resolvent family, then*

$$(a * W)(t)[W(s) - \Theta(s)C] = [W(t) - \Theta(t)C](a * W)(s) \text{ for } 0 \leq t, s, t+s < \tau. \quad (7)$$

- (ii) Suppose that $(W(t))_{t \in [0, \tau]}$ is locally equicontinuous and (7) holds. Then $(W(t))_{t \in [0, \tau]}$ is an (a, Θ) -regularized C -resolvent family.

By Theorem 1 (i), we have that the integral generator $\hat{\mathcal{A}}$ of a global (a, Θ) -regularized C -resolvent family $(W(t))_{t \geq 0}$ is always its subgenerator (as already seen, this statement is not true in local case).

The following theorem is a slight extension of [15; 16, Theorem 2.5].

Theorem 2. Let $0 < \tau \leq \infty$ and $C \in L(E)$. Suppose that $a(t) \equiv 1$ or $a(t) \equiv t$, \mathcal{A} is an MLO and $(Z(t))_{t \in [0, \tau]} \subseteq L(E)$ is a strongly continuous operator family which commutes with C . Let $(Z(t))_{t \in [0, \tau]}$ be locally equicontinuous. Then the following holds:

- (i) if \mathcal{A} is a subgenerator of $(Z(t))_{t \in [0, \tau]}$, then $(Z(t))_{t \in [0, \tau]}$ is an (a, Θ) -regularized C -resolvent family;
- (ii) suppose that (i) holds with $\mathcal{A} = A$ being single-valued and linear, as well as that C is injective. Then $(Z(t))_{t \in [0, \tau]}$ is non-degenerate.

Proof. Let $x \in E$ be fixed. Then (ii) of Definition 3 yields that

$$\int_0^s a(s-r)Z(r)x \, dr \in D(\mathcal{A})$$

and

$$Z(s)x - \Theta(s)Cx \in \mathcal{A} \int_0^s a(s-r)Z(r)x \, dr.$$

By (i) of Definition 3, we have

$$\begin{aligned} [Z(t) - \Theta(t)C] \int_0^s a(s-r)Z(r)x \, dr &= \int_0^t a(t-r)Z(r)[Z(s)x - \Theta(s)Cx] \, dr = \\ &= (a * Z)(t)[Z(s)x - \Theta(s)Cx], \end{aligned}$$

for $0 \leq t, s < \tau$. Now we can apply Theorem 1 (ii) in order to see that (i) holds true. The proof of (ii) is simple and therefore omitted. \square

Now we will construct an illustrative example of a global degenerate strongly continuous semigroup (i.e., I -regularized semigroup) which do not have any linear subgenerator:

Example 2. [2]. Suppose that \mathcal{A} is a non single-valued MLO which satisfies the Hille – Yosida condition [2, (H-Y), p. 28] on a Banach space E . By [2, Theorem 2.4], \mathcal{A} is the integral generator (a subgenerator) of a global strongly continuous semigroup $(T(t))_{t \geq 0}$ which vanishes on the closed subspace $\mathcal{A}0$ of E . An application of Theorem 2 (ii) shows that $(T(t))_{t \geq 0}$ has no linear, single-valued subgenerator.

It should also be observed that a global degenerate strongly continuous semigroup can have infinitely many subgenerators: Let $P \in L(E)$, $P^2 = P$ and $T(t) := P$, $t \geq 0$. Then it can be simply verified that, for every linear subspace V of $N(P)$, operator $\mathcal{A} = N(I - P) \times V$ is a subgenerator of $(T(t))_{t \geq 0}$.

The arguments used in the proofs of [15, Lemma 2.8] and [16, Lemma 2.9] enable one to deduce the following lemma:

Lemma 4. Let $0 < \tau \leq \infty$, $x \in E$, $0 \in \text{supp}(\Theta)$ and $(W(t))_{t \in [0, \tau]}$ be an (a, Θ) -regularized C -resolvent family. Then the existence of a number $\tau_0 \in (0, \tau)$ such that $W(t)x = 0$, $t \in [0, \tau_0]$ implies $CW(t)x = 0$, $t \in [0, \tau)$.

Keeping in mind Lemma 4, it is quite simple to prove the following extension of [15, Theorem 2.9] and [16, Theorem 2.10]:

Theorem 3. *Suppose that $0 < \tau \leq \infty$, $C \in L(E)$ is injective, $0 \in \text{supp}(\Theta)$ and $(W(t))_{t \in [0, \tau]}$ is an (a, Θ) -regularized C -resolvent family. Then $(W(t))_{t \in [0, \tau]}$ is non-degenerate iff the integral generator $\hat{\mathcal{A}}$ of $(W(t))_{t \in [0, \tau]}$ is its subgenerator.*

Before proceeding further, we would like to mention that the consideration from Example 2 shows that the existence of a subgenerator of a (local) C -regularized semigroup in the MLO sense does not imply its non-degeneracy, even in the case that $C = I$. The reader with a little experience will succeed in transferring the assertions of [15, Theorem 2.13] and [16, Theorem 2.14] to locally equicontinuous K -convoluted C -semigroups and K -convoluted C -cosine functions.

The following proposition extends the assertions of [8, Proposition 2.1.3] and [15; 16, Proposition 2.3] (for locally equicontinuous operator families, this proposition follows almost immediately from Theorem 3).

Proposition 1. *Let $0 < \tau \leq \infty$, and let $(W(t))_{t \in [0, \tau]}$ be an (a, Θ) -regularized C -resolvent family. Suppose that $H \in L^1_{\text{loc}}([0, \tau])$ and $H * K \neq 0$ in $L^1_{\text{loc}}([0, \tau])$. Set $W_H(t)x := \int_0^t H(t-s)W(s)x ds$, $x \in E$, $t \in [0, \tau)$. Then $(W_H(t))_{t \in [0, \tau)}$ is an $(a, H * \Theta)$ -regularized C -resolvent family.*

Proof. We will include all details of proof for semigroups, in purely convoluted case; the proof in all other cases can be given by applying the same trick. It is clear that $(W_H(t) = S_{K,H}(t))_{t \in [0, \tau)} \subseteq L(E)$ is a strongly continuous operator family which commutes with C . Therefore, it suffices to show that, for every $x \in E$ and $t, s \in [0, \tau)$ with $t + s \in [0, \tau)$, the following holds:

$$S_{H,K}(t)S_{H,K}(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] (H * K)(t + s - r)S_{H,K}(r)Cx dr,$$

i.e., that for each functional $x^* \in E^*$ we have:

$$\begin{aligned} & \int_0^t \int_0^s H(t-r)H(s-\sigma) \left[\int_0^{r+\sigma} - \int_0^r - \int_0^\sigma \right] K(r + \sigma - v) \langle x^*, S_K(v)Cx \rangle dv d\sigma dr = \\ & = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \int_0^r \int_0^{t+s-r} K(t + s - r - v)H(v)H(r - \sigma) \langle x^*, S_K(\sigma)Cx \rangle dv d\sigma dr. \end{aligned} \tag{8}$$

By Lemma 3, the above equality holds for all non-zero complex polynomials $H(\cdot)$ so that the final conclusion follows by applying Stone — Weierstrass theorem and the dominated convergence theorem in (8). \square

Remark 1.

- (i) Suppose that \mathcal{A} is a closed subgenerator of $(W(t))_{t \in [0, \tau)}$. Then we can simply prove with the help of Lemma 1 that \mathcal{A} is a subgenerator of $(W_H(t))_{t \in [0, \tau)}$.
- (ii) Suppose that \mathcal{A} (\mathcal{A}_H) is a closed subgenerator of $(W(t))_{t \in [0, \tau)}$ ($(W_H(t))_{t \in [0, \tau)}$). Then $\mathcal{A} \subseteq \mathcal{A}_H$, with the equality if $H(t)$ is a kernel on $[0, \tau)$.

Now we will extend the assertion of [8, Theorem 2.1.11] to degenerate operator families.

Theorem 4. *Suppose that \mathcal{A} is a closed MLO in E , $0 < \tau \leq \infty$, $K \in L_{\text{loc}}^1([0, \tau])$, $K \neq 0$ and $(C_K(t))_{t \in [0, \tau]}$ is a strongly continuous operator family which commutes with C . Set*

$$S_{\Theta}(t) = \begin{pmatrix} \int_0^t C_K(s) ds & \int_0^t (t-s)C_K(s) ds \\ C_K(t) - \Theta(t)C & \int_0^t C_K(s) ds \end{pmatrix}, \quad 0 \leq t < \tau,$$

and $\mathcal{C}(x y)^T := (Cx Cy)^T$, $x, y \in E$. Then we have:

- (i) *The following assertions are equivalent:*
 - (a) $(C_K(t))_{t \in [0, \tau]}$ is a K -convoluted C -cosine function on E .
 - (b) $(S_{\Theta}(t))_{t \in [0, \tau]}$ is a Θ -convoluted \mathcal{C} -semigroup $(S_{\Theta}(t))_{t \in [0, \tau]}$ on $E \times E$.

Suppose that the equivalence relation (a) \Leftrightarrow (b) in (i) holds. Then we have:

- (ii) \mathcal{A} is a subgenerator of $(C_K(t))_{t \in [0, \tau]}$ iff $\mathcal{B} := \begin{pmatrix} 0 & I \\ \mathcal{A} & 0 \end{pmatrix}$ is a subgenerator of $(S_{\Theta}(t))_{t \in [0, \tau]}$.
- (iii) Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be the integral generators of $(C_K(t))_{t \in [0, \tau]}$ and $(S_{\Theta}(t))_{t \in [0, \tau]}$, respectively. Then the inclusion $\begin{pmatrix} 0 & I \\ \hat{\mathcal{A}} & 0 \end{pmatrix} \subseteq \hat{\mathcal{B}}$ holds true. Furthermore, if $(C_K(t))_{t \in [0, \tau]}$ is non-degenerate, then $\begin{pmatrix} 0 & I \\ \hat{\mathcal{A}} & 0 \end{pmatrix} = \hat{\mathcal{B}}$.

Proof. Suppose that (a) holds. Then it is clear that $(S_{\Theta}(t))_{t \in [0, \tau]}$ is a strongly continuous operator family in $L(E \times E)$ which commutes with \mathcal{C} . Therefore, it suffices to show that the semigroup property holds for $(S_{\Theta}(t))_{t \in [0, \tau]}$, i.e., that the following holds for $0 \leq t, s, t+s < \tau$ and $x \in E$:

$$C_{\Theta}(t)C_{\Theta}(s)x + C_{\Theta}^{[1]}(t)[C_K(s) - \Theta(s)C]x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)C_{\Theta}(\sigma)Cx d\sigma, \quad (9)$$

$$C_{\Theta}(t)C_{\Theta}^{[1]}(s)x + C_{\Theta}^{[1]}(t)C_{\Theta}(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)C_{\Theta}^{[1]}(\sigma)Cx d\sigma, \quad (10)$$

$$\begin{aligned} & [C_K(t) - \Theta(t)C]C_{\Theta}(s)x + C_{\Theta}(t)[C_K(s) - \Theta(s)C]x = \\ & \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)\{C_K(\sigma)Cx - \Theta(s)C^2x\} d\sigma \end{aligned} \quad (11)$$

and

$$[C_K(t) - \Theta(t)C]C_{\Theta}^{[1]}(s)x + C_{\Theta}(t)C_{\Theta}(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)C_{\Theta}(\sigma)Cx d\sigma. \quad (12)$$

The proofs of equations (9)–(12) and implication (b) \Rightarrow (a) below will be given only in the case that $s \leq t$; the case $s > t$ can be considered similarly. First of all, we will prove (11). By [8, Lemma 2.1.12], we have that

$$\left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)\Theta(s) d\sigma = 0.$$

Therefore, we need to prove that

$$\begin{aligned} & [C_K(t) - \Theta(t)C]C_\Theta(s)x + C_\Theta(t)[C_K(s) - \Theta(s)C]x = \\ & = \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)C_K(\sigma)Cx \, d\sigma. \end{aligned} \quad (13)$$

In order to do that, observe that the partial integration and Newton — Leibniz formula imply:

$$\begin{aligned} & \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] K(t+s-\sigma)C_\Theta(\sigma)Cx \, d\sigma = \\ & = \Theta(s)C_\Theta(t)Cx + \Theta(t)C_\Theta(s)Cx + \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)C_\Theta(\sigma)Cx \, d\sigma, \end{aligned} \quad (14)$$

$$\int_{t-s}^t \{ \Theta(r-t+s)C_K(r)Cx + K(r-t+s)C_\Theta(r)Cx \} \, dr = \Theta(s)C_\Theta(t)Cx \quad (15)$$

and

$$\int_0^s \{ \Theta(r+t-s)C_K(r)Cx + K(r+t-s)C_\Theta(r)Cx \} \, dr = \Theta(t)C_\Theta(s)Cx. \quad (16)$$

Inserting (14)–(16) in (5) and dividing after that both sides of obtained equation with two, we immediately get (13). Consider now the equation (9). The both sides equal zero for $t = 0$ and it suffices therefore to show that their derivatives in variable t are equal, i.e., that:

$$C_K(t)C_\Theta(s)x + C_\Theta(t)[C_K(s) - \Theta(s)C]x = \frac{d}{dt} \left[\int_0^{t+s} - \int_0^t - \int_0^s \right] \Theta(t+s-\sigma)C_\Theta(\sigma)Cx \, d\sigma.$$

This follows from (11) and the usual limit procedure. The equation (12) is a consequence of (9) and Theorem 1 (i), so that it remains to be proved (9). Here we can clearly consider the function $\Theta(t)$ replaced in all places with the function $K(t)$: then (9) follows immediately from (11) by applying the partial intgration. Hence, (b) holds. The implication (b) \Rightarrow (a) can be proved as follows. By Lemma 3, we may assume without loss of generality that $K(t)$ is locally absolutely continuous on $[0, \tau)$. Since (11) holds, [8, Lemma 2.1.12] implies that the equation [8, (26)] holds true. Then the argumentation contained in the proof of [8, Theorem 2.1.13] (cf. [8, (27)–(35)]) shows that the composition property holds for $(C_K(t))_{t \in [0, \tau)}$. This completes the proof of (i). The proofs of (ii) and (iii) follow from simple computations and therefore omitted. \square

Remark 2.

- (i) Let \mathcal{A} be an MLO in E , and let \mathcal{B} be defined as in (i). Then \mathcal{A} is closed iff \mathcal{B} is closed. Furthermore, $\begin{pmatrix} C^{-1} & I \\ AC & 0 \end{pmatrix} \subseteq \mathcal{C}^{-1}\mathcal{B}\mathcal{C}$, with the equality in the case that the operator C is injective.

- (ii) Theorem 4 can be simply reformulated for C -regularized cosine functions on E and induced once integrated C -semigroups on the product space $E \times E$. In non-degenerate case, a similar assertion is also known for $(a * a, k)$ -regularized C -resolvent families on E and induced $(a, a * k)$ -regularized C -resolvent families on the product space $E \times E$ (cf. [8, Theorem 2.1.27 (xiv)]).

The assertions of [8, Proposition 2.3.3, Proposition 2.3.4] ([8, Proposition 2.3.8 (i), (ii)]) can be reword for abstract degenerate inclusions of first order (second order) by replacing the sequence $x_0 = x, Ax = x_1, \dots, A^k x = x_k$ in their formulations with an arbitrary sequence $(x_j)_{0 \leq j \leq k}$ satisfying $x_j \in \mathcal{A}x_{j-1}$ ($1 \leq j \leq k$). In order to fully transfer a great part of other results from [8, Section 2.3] to corresponding abstract degenerate inclusions of first and second order, the injectivity of operator C is almost inevitable to be assumed; we shall skip all related details on C -wellposedness of corresponding abstract degenerate inclusions of first and second order for the sake of brevity and better exposition.

The extension type theorems for non-degenerate integrated semigroups has been considered for the first time by W. Arendt, O. El-Mennaoui and V. Keyantuo [23], who have proved that a closed linear operator A generates a local $(2n)$ -times integrated semigroup on the interval $[0, 2\tau)$, provided that A generates a local n -times integrated semigroup on the interval $[0, \tau)$ ($n \in \mathbb{N}, 0 < \tau < \infty$). Immediately after that, I. Ciorănescu and G. Lumer [24] have extended their result to the class of local K -convoluted C -semigroups (see e.g. [8, Theorem 2.1.9] for a precise formulation). On the other hand, S. W. Wang and M. C. Gao [25] has considered automatic extensions of non-degenerate local C -regularized semigroups and non-degenerate local C -regularized cosine functions. It seems that the argumentation contained in [25] cannot be used for proving extension type theorems for degenerate C -regularized semigroups and degenerate C -regularized cosine functions which do not have subgenerators. The situation is much more simpler if degenerate operator families under our consideration have subgenerators. Keeping in mind Lemma 1 and elementary properties of multivalued linear operators, the following two theorems can be proved in almost the same way as in single-valued linear case (cf. V. Keyantuo, P. J. Miana, L. Sánchez-Lajusticia [26, Theorem 4.4], P. J. Miana, V. Poblete [27, Theorem 3.3], and [8, Theorem 2.1.9, Corollary 2.1.10, Theorem 2.1.14, Corollary 2.1.15] for some special cases of two last mentioned results):

Theorem 5. *Suppose that \mathcal{A} is a closed MLO. Let $n \in \mathbb{N}, 0 < \tau < \infty, 0 < \tau_0 < \tau, K \in L^1_{\text{loc}}([0, (n+1)\tau))$, $K \neq 0$, and let $(S_K(t))_{t \in [0, \tau)}$ be a local $K|_{[0, \tau)}$ -convoluted C -semigroup with a subgenerator \mathcal{A} . Define recursively the family of operators $(S_{K, n+1}(t))_{t \in [0, (n+1)\tau_0]}$ by $S_{K, n+1}(t)x := \int_0^t K(t-s)S_{K, n}(s)Cx ds$, $x \in E$, for $t \in [0, n\tau_0]$ and*

$$S_{K, n+1}(t)x := S_{K, n}(n\tau_0)S_K(t - n\tau_0)x + \int_0^{n\tau_0} K(t-s)S_{K, n}(s)Cx ds + \\ + \int_0^{t-n\tau_0} K^{*, n}(t-s)S_K(s)Cx ds$$

for $x \in E$ and $t \in (n\tau_0, (n+1)\tau_0]$. Then $(S_{K, n+1}(t))_{t \in [0, (n+1)\tau_0]}$ is a local $(K^{*, n+1})|_{[0, (n+1)\tau_0)}$ -convoluted C^{n+1} -semigroup with a subgenerator \mathcal{A} .

Theorem 6. *Suppose that \mathcal{A} is a closed MLO. Let $n \in \mathbb{N}, 0 < \tau < \infty, 0 < \tau_0 < \tau, K \in L^1_{\text{loc}}([0, (n+1)\tau))$, $K \neq 0$, and let $(C_K(t))_{t \in [0, \tau)}$ be a local $K|_{[0, \tau)}$ -convoluted*

C -cosine function with a subgenerator \mathcal{A} . Define recursively the family of operators $(C_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$ by $C_{K,n+1}(t)x := \int_0^t K(t-s)C_{K,n}(s)Cx ds$, $x \in E$, for $t \in [0, n\tau_0]$ and

$$\begin{aligned} C_{K,n+1}(t)x &:= 2C_{K,n}(n\tau_0)C_K(t-n\tau_0)x + \int_0^{n\tau_0} K(t-s)C_{K,n}(s)Cx ds + \\ &+ \int_0^{t-n\tau_0} K^{*,n}(t-s)C_K(s)Cx ds - \int_{2n\tau_0-t}^{n\tau_0} K(t+s-2n\tau_0)C_{K,n}(s)Cx ds - \\ &\quad - \int_0^{t-n\tau_0} K(s-t+2n\tau_0)C_K(s)x ds \end{aligned}$$

for $x \in E$ and $t \in (n\tau_0, (n+1)\tau_0]$. Then $(C_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$ is a local $(K^{*,n+1})_{|[0, (n+1)\tau_0]}$ -convoluted C^{n+1} -cosine function with a subgenerator \mathcal{A} .

Remark 3. Consider the situation of Theorem 5 (Theorem 6). Let $\hat{\mathcal{A}}$ be the integral generator of $(S_K(t))_{t \in [0, \tau]}$ $((C_K(t))_{t \in [0, \tau]}$). Then it is a very unpleasant question to precisely profile the integral generator $\hat{\mathcal{A}}_{n+1}$ of $(S_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$ $((C_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$) in general case. We will prove that the integral generator of $(S_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$ $((C_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$) is $\hat{\mathcal{A}}$, provided that the operator C is injective (observe that we do not require the condition $0 \in \text{supp}(K)$ here). The analysis is quite similar in both cases and we will consider only K -convoluted C -semigroups. Since \mathcal{A} is a subgenerator of $(S_{K,n+1}(t))_{t \in [0, (n+1)\tau_0]}$, and $\hat{\mathcal{A}}$ is the integral generator of $(S_K(t))_{t \in [0, \tau]}$, the foregoing arguments yield that $\hat{\mathcal{A}}_{n+1} = C^{-(n+1)}\mathcal{A}C^{n+1}$ and $\hat{\mathcal{A}} = C^{-1}\mathcal{A}C = C^{-1}\hat{\mathcal{A}}C$. Inductively, $C^{-l}\hat{\mathcal{A}}C^l = \hat{\mathcal{A}}$, $l \in \mathbb{N}$, so that

$$\hat{\mathcal{A}}_{n+1} = C^{-(n+1)}\mathcal{A}C^{n+1} = C^{-n}[C^{-1}\mathcal{A}C]C^n = C^{-n}\hat{\mathcal{A}}C^n = \hat{\mathcal{A}}.$$

The assertion of [8, Proposition 2.1.17] can be extended in the following way (a straightforward proof involving Lemma 1 is omitted):

Proposition 2. Let \mathcal{A} be a closed MLO, let $0 < \tau \leq \infty$, and let $K \in L_{\text{loc}}^1([0, \tau))$, $K \neq 0$. Suppose that $\pm\mathcal{A}$ are subgenerators of K -convoluted C -semigroups $(S_K^\pm(t))_{t \in [0, \tau]}$. Then \mathcal{A}^2 is a subgenerator of a K -convoluted C -cosine function $(C_K(t))_{t \in [0, \tau]}$, which is given by $C_K(t)x := \frac{1}{2}(S_K^+(t)x + S_K^-(t)x)$, $x \in E$, $t \in [0, \tau)$.

Concerning exponentially equicontinuous K -convoluted C -semigroups and exponentially equicontinuous K -convoluted C -cosine functions, we have the following fundamental theorem which can be simply proved with the help of Lemma 2 and Theorem 2 (i).

Theorem 7. Suppose that \mathcal{A} is a closed MLO in E , $C \in L(E)$, $\Theta(t)$ satisfies (P1), as well as that $(W(t))_{t \geq 0} \subseteq L(E)$ is a strongly continuous operator family which commutes with C on E . Let $\omega \geq \max(0, \text{abs}(\Theta))$ be such that the operator family $\{e^{-\omega t}W(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous. Then we have the equivalence of statements (i) and (ii), where:

- (i) $(W(t))_{t \geq 0}$ is an (a, Θ) -regularized C -resolvent family with a subgenerator \mathcal{A} .
- (ii) For every $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > \omega$ and $\tilde{\Theta}(\lambda) \neq 0$, we have $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$,

$$\int_0^\infty e^{-\lambda t}W(t)x dt \in \tilde{\Theta}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx, \quad x \in E, \quad (17)$$

and

$$\tilde{\Theta}(\lambda)Cx = \int_0^\infty e^{-\lambda t} [W(t)x - (a * W)(t)y] dt, \quad \text{whenever } (x, y) \in \mathcal{A}. \quad (18)$$

It is worth noting that [8, Proposition 2.2.6] continues to hold in our framework and that the validity of requirements from Theorem 7 does not imply, in general, that the operator $(I - \tilde{a}(\lambda)\mathcal{A})^{-1}C$ is single-valued (this holds provided that the operator C is injective). To verify this, we can consider the global C -regularized semigroup $(T(t) \equiv C)_{t \geq 0}$ and its integral generator $\hat{A} = E \times N(C)$; then $N(C) \subseteq (\lambda - \hat{A})^{-1}C0$ for all $\lambda > 0$.

Keeping in mind Theorem 7 and [13, Theorem 1.2.2], it is very simple to prove the following complex characterization theorem:

Theorem 8. *Suppose that \mathcal{A} is a closed MLO in E , $C \in L(E)$, $\Theta(t)$ satisfies (P1), $\omega_0 > \max(0, \text{abs}(\Theta))$ and, for every $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > \omega_0$ and $\tilde{\Theta}(\lambda) \neq 0$, we have $R(C) \subseteq R(I - \tilde{a}(\lambda)\mathcal{A})$. If there exists a function $\Upsilon : \{\lambda \in \mathbb{C} : \text{Re } \lambda > \omega_0\} \rightarrow L(E)$ which satisfies:*

- (a) $\Upsilon(\lambda)x \in \tilde{\Theta}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$ for $\text{Re } \lambda > \omega_0$, $\tilde{\Theta}(\lambda) \neq 0$, $x \in E$;
- (b) the mapping $\lambda \mapsto \Upsilon(\lambda)x$, $\text{Re } \lambda > \omega_0$ is analytic for every fixed $x \in E$;
- (c) there exists $r \geq -1$ such that the family $\{\lambda^{-r}\Upsilon(\lambda) : \text{Re } \lambda > \omega_0\} \subseteq L(E)$ is equicontinuous;
- (d) $\Upsilon(\lambda)x - \tilde{a}(\lambda)\Upsilon(\lambda)y = \tilde{\Theta}(\lambda)Cx$ for $\text{Re } \lambda > \omega_0$, $(x, y) \in \mathcal{A}$;
- (e) $\Upsilon(\lambda)Cx = C\Upsilon(\lambda)x$ for $\text{Re } \lambda > \omega_0$, $x \in E$;

then, for every $\alpha > 1$, \mathcal{A} is a subgenerator of a global $(a, \Theta * g_{\alpha+r})$ -regularized C -resolvent family $(W_\alpha(t))_{t \geq 0}$ which satisfies that the family $\{e^{-\omega_0 t}W_\alpha(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous.

The real representation theorem for generation of degenerate K -convoluted C -semigroups and degenerate K -convoluted C -cosine functions in locally convex spaces reads as follows.

Theorem 9.

- (i) *Assume $\omega \in \mathbb{R}$, $\Theta(t)$ satisfies (P1), and \mathcal{A} is a closed subgenerator of a global $(a, \Theta^{[1]})$ -regularized C -resolvent family $(W_1(t))_{t \geq 0}$ such that $W_1(0) = 0$ and the family*

$$\left\{ h^{-1}e^{-\omega t} \min(e^{\omega h}, 1) (W_1(t+h) - W_1(t)) : t \geq 0, h > 0 \right\} \quad (19)$$

is equicontinuous. Then there exists $b \geq \max(0, \omega, \text{abs}(\Theta))$ such that the mapping $\lambda \mapsto \Upsilon(\lambda) \cdot := \lambda \int_0^\infty e^{-\lambda t} W_1(t) \cdot dt$, $\lambda > b$, is infinitely differentiable in $L(E)$, $\Upsilon(\lambda)x \in \tilde{\Theta}(\lambda)(I - \tilde{a}(\lambda)\mathcal{A})^{-1}Cx$, $\lambda > b$, $\tilde{\Theta}(\lambda) \neq 0$, $x \in E$, the family

$$\left\{ k!^{-1}(\lambda - \omega)^{k+1} \frac{d^k}{d\lambda^k} \Upsilon(\lambda) : k \in \mathbb{N}_0, \lambda > b, \tilde{\Theta}(\lambda) \neq 0 \right\} \quad (20)$$

is equicontinuous, and (d)–(e) of Theorem 8 hold.

- (ii) *Suppose $\omega \in \mathbb{R}$, $\Theta(t)$ satisfies (P1), $b \geq \max(0, \omega, \text{abs}(\Theta))$, \mathcal{A} is a closed MLO, the function $\Upsilon : D(H) \equiv \{\lambda > b : \tilde{\Theta}(\lambda) \neq 0\} \rightarrow L(E)$ satisfies that (a) and (d)–(e) of Theorem 8 hold for $\lambda \in D(H)$, as well as that the mapping $\lambda \mapsto \Upsilon(\lambda)x$,*

$\lambda \in D(H)$, is infinitely differentiable for every fixed $x \in E$ and that, for every $p \in \circledast$, there exist $c_p > 0$ and $r_p \in \circledast$ such that:

$$p\left(k!^{-1}(\lambda - \omega)^{k+1} \frac{d^k}{d\lambda^k} \Upsilon(\lambda)x\right) \leq c_p r_p(x), \quad x \in E, \quad \lambda \in D(H), \quad k \in \mathbb{N}_0. \quad (21)$$

Then, for every $r \in (0, 1]$, \mathcal{A} is a subgenerator of a global $(a, \Theta * g_r)$ -regularized C -resolvent family $(W_r(t))_{t \geq 0}$ satisfying that $W_r(0) = 0$ as well as that, for every $p \in \circledast$,

$$p(W_r(t+h)x - W_r(t)x) \leq \frac{2c_p r_p(x)}{r\Gamma(r)} \max(e^{\omega(t+h)}, 1) h^r, \quad t \geq 0, \quad h > 0, \quad x \in E, \quad (22)$$

and that, for every $p \in \circledast$ and $B \in \mathcal{B}$, the mapping $t \mapsto p_B(R_r(t))$, $t \geq 0$, is locally Hölder continuous with exponent r .

- (iii) Suppose that the requirements of (ii) hold with \mathcal{A} being densely defined. Then \mathcal{A} is a subgenerator of a global (a, Θ) -regularized C -resolvent family $(W(t))_{t \geq 0}$ satisfying that the family $\{e^{-\omega t}W(t) : t \geq 0\} \subseteq L(E)$ is equicontinuous.

The statements clarified in Theorems 5, 6, Remark 3, Proposition 2, Theorem 7 and Theorem 9 can be simply reformulated for the classes of local C -regularized semigroups and local C -regularized cosine functions. Involving into consideration the conditions (d)–(e) from Theorem 8, one can simply prove the analogons of [8, Proposition 2.4.2. Corollary 2.4.3, Theorem 2.4.5, Theorem 2.4.8, Corollary 2.4.9] for degenerate analytic (K -convoluted) C -semigroups in locally convex spaces (primarily from the time and space limitations, we have not been able to precisely formulate the assertions on analytic and differential properties of degenerate operator families). The adjoint type theorems [8, Theorem 2.2.7 (i), (iii), Theorem 2.2.8] continue to hold for (a, Θ) -regularized C -regularized families subgenerated by closed multivalued linear operators ($C \in L(E)$ non-injective), and it is not necessary to assume that the operator \mathcal{A} is densely defined or that $R(C)$ is dense in X in the case of consideration of [8, Theorem 2.2.7 (i)].

Now we will reconsider our structural results from [8, Section 2.7] for degenerate K -convoluted C -semigroups subgenerated by multivalued linear operators. The subsequent theorem follows, more or less, by repeating the arguments given in the proof of [8, Theorem 2.7.1].

Theorem 10.

- (i) Suppose that E is Banach space, $M > 0$, $\beta \geq 0$, $|K(t)| \leq Me^{\beta t}$, $t \geq 0$, $(S_K(t))_{t \in [0, \tau)}$ is a (local) K -convoluted semigroup with a closed subgenerator \mathcal{A} and, for every $\epsilon > 0$, there exist $\epsilon_0 \in (0, \tau\epsilon)$ and $T_\epsilon > 0$ such that $1/|\tilde{K}(\lambda)| \leq T_\epsilon e^{\epsilon_0|\lambda|}$, $\operatorname{Re} \lambda > \beta$, $\tilde{K}(\lambda) \neq 0$. Then, for every $\epsilon > 0$, there exist $\bar{C}_\epsilon > 0$ and $\bar{K}_\epsilon > 0$ such that, for every λ which belongs to the following set

$$\Omega_\epsilon^1 := \{\lambda \in \mathbb{C} : \tilde{K}(\lambda) \neq 0, \operatorname{Re} \lambda > \beta, \operatorname{Re} \lambda \geq \epsilon|\lambda| + \bar{C}_\epsilon\},$$

there exists an operator $F(\lambda) \in L(E)$ so that $F(\lambda)\mathcal{A} \subseteq \mathcal{A}F(\lambda)$, $\lambda \in \Omega_\epsilon^1$, $F(\lambda)x \in (\lambda - \mathcal{A})^{-1}x$, $\lambda \in \Omega_\epsilon^1$, $x \in E$, $F(\lambda)x - x = F(\lambda)y$, whenever $\lambda \in \Omega_\epsilon^1$ and $(x, y) \in \mathcal{A}$,

$$\|F(\lambda)\| \leq \bar{K}_\epsilon e^{\epsilon_0|\lambda|}, \quad \lambda \in \Omega_\epsilon^1, \quad \tilde{K}(\lambda) \neq 0,$$

and that the mapping $\lambda \mapsto F(\lambda) \in L(E)$, $\lambda \in \Omega_\epsilon^1$ is analytic.

- (ii) Suppose that E is Banach space, $\alpha > 0$, $M > 0$, $\beta \geq 0$, $\Phi : \mathbb{C} \rightarrow [0, \infty)$, $|K(t)| \leq Me^{\beta t}$, $t \geq 0$, $(S_K(t))_{t \in [0, \tau]}$ is a local K -convoluted semigroup with a closed subgenerator \mathcal{A} and $1/|\tilde{K}(\lambda)| \leq e^{\Phi(\alpha\lambda)}$, $\operatorname{Re} \lambda > \beta$, $\tilde{K}(\lambda) \neq 0$. Then, for every $t \in (0, \tau)$, there exist $\beta(t) > 0$ and $M(t) > 0$ such that, for every λ which belongs to the following set

$$\Lambda_{t, \alpha, \beta(t)} := \left\{ \lambda \in \mathbb{C} : \tilde{K}(\lambda) \neq 0, \operatorname{Re} \lambda \geq \frac{\Phi(\alpha\lambda)}{t} + \beta(t) \right\},$$

there exists an operator $F(\lambda) \in L(E)$ so that $F(\lambda)\mathcal{A} \subseteq \mathcal{A}F(\lambda)$, $\lambda \in \Lambda_{t, \alpha, \beta(t)}$, $F(\lambda)x \in (\lambda - \mathcal{A})^{-1}x$, $\lambda \in \Lambda_{t, \alpha, \beta(t)}$, $x \in E$, $F(\lambda)x - x = F(\lambda)y$, whenever $\lambda \in \Lambda_{t, \alpha, \beta(t)}$ and $(x, y) \in \mathcal{A}$,

$$\|F(\lambda)\| \leq M(t)e^{\Phi(\alpha\lambda)}, \quad \lambda \in \Lambda_{t, \alpha, \beta(t)}, \quad \tilde{K}(\lambda) \neq 0$$

and that the mapping $\lambda \mapsto F(\lambda) \in L(E)$, $\lambda \in \Lambda_{t, \alpha, \beta(t)}$ is analytic. Furthermore, the existence of a sequence (t_n) in $[0, \tau)$ satisfying $\lim_{n \rightarrow \infty} t_n = \tau$ and $\sup_{n \in \mathbb{N}} \ln \|S_K(t_n)\| < \infty$ implies that there exist $\beta' > 0$ and $M' > 0$ such that the above holds with the region $\Lambda_{t, \alpha, \beta(t)}$ replaced by $\Lambda_{\tau, \alpha, \beta'}$ and the number $M(t)$ replaced by M' .

Theorem 11. Suppose that $K(t)$ satisfies (P1), \mathcal{A} is a closed MLO, $C \in L(E)$, $r_0 \geq \max(0, \operatorname{abs}(K))$, $\Phi : [r_0, \infty) \rightarrow [0, \infty)$ is a continuously differentiable, strictly increasing mapping, $\lim_{t \rightarrow \infty} \Phi(t) = +\infty$, $\Phi'(\cdot)$ is bounded on $[r_0, \infty)$ and there exist $\alpha > 0$, $\gamma > 0$ and $\beta > r_0$ such that, for every λ which belongs to the following set

$$\Psi_{\alpha, \beta, \gamma} := \left\{ \lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \frac{\Phi(\alpha |\operatorname{Im} \lambda|)}{\gamma} + \beta \right\}$$

there exists an operator $F(\lambda) \in L(E)$ so that $F(\lambda)\mathcal{A} \subseteq \mathcal{A}F(\lambda)$, $\lambda \in \Psi_{\alpha, \beta, \gamma}$, $F(\lambda)x \in (\lambda - \mathcal{A})^{-1}Cx$, $\lambda \in \Psi_{\alpha, \beta, \gamma}$, $x \in E$, $F(\lambda)C = CF(\lambda)$, $\lambda \in \Psi_{\alpha, \beta, \gamma}$, $F(\lambda)x - Cx = F(\lambda)y$, whenever $\lambda \in \Psi_{\alpha, \beta, \gamma}$ and $(x, y) \in \mathcal{A}$, and that the mapping $\lambda \mapsto F(\lambda)x$ is analytic on $\Omega_{\alpha, \beta, \gamma}$ and continuous on $\Gamma_{\alpha, \beta, \gamma}$, where $\Gamma_{\alpha, \beta, \gamma}$ denotes the upwards oriented boundary of $\Psi_{\alpha, \beta, \gamma}$ and $\Omega_{\alpha, \beta, \gamma}$ the open region which lies to the right of $\Gamma_{\alpha, \beta, \gamma}$. Let the following conditions hold.

- (i) There exists $\sigma > 0$ such that the operator family $\{e^{\Phi(\sigma|\lambda)}F(\lambda) : \lambda \in \overline{\Omega_{\alpha, \beta, \gamma}}\} \subseteq L(E)$ is equicontinuous.
- (ii) There exists a function $m : [0, \infty) \rightarrow (0, \infty)$ such that $m(s) = 1$, $s \in [0, 1]$ and that, for every $s > 1$, there exists an $r_s > r_0$ so that $\frac{\Phi(t)}{\Phi(st)} \geq m(s)$, $t \geq r_s$.
- (iii) $\lim_{t \rightarrow \infty} te^{-\Phi(\sigma t)} = 0$.
- (iv) $(\exists a \geq 0)(\exists r'_a > r_0)(\forall t > r'_a) \frac{\ln t}{\Phi(t)} \geq a$.

Then \mathcal{A} subgenerates a local K -convoluted C -semigroup on $[0, a + m(\frac{\alpha}{\sigma\gamma})]$.

We can similarly formulate analogons of [8, Theorem 2.7.2(iii)-(iv)] for local integrated (C) -semigroups and [8, Theorem 2.7.3] for local K -convoluted C -cosine functions. The proof of [14, Theorem 3.15] essentially shows that the existence of numbers $r \geq 0$ and $\theta \in (0, \pi/2)$ and an injective operator $C \in L(E)$ such that $-\mathcal{A}$ is a closed subgenerator of an exponentially equicontinuous, analytic r -times integrated C -semigroup $(S_r(t))_{t \geq 0}$ of angle θ (defined as in non-degenerate case, see e.g. [14]) implies

that there exists an operator $C_1 \in L(E)$ such that \mathcal{A} is a subgenerator of an entire C_1 -regularized semigroup in E . Using the usual matrix reduction, we can apply this result in the analysis of problem

$$\frac{d}{dt}(Cu') - Bu' + Au(t) = F(t), \quad 0 < t \leq T; \quad u(0) = u_0, \quad Cu'(0) = Cu_1,$$

provided that the conditions from [2, Section 6.1] hold. Some instructive examples of exponentially bounded integrated semigroups generated by multivalued linear operators can be found in [2, Section 5.8] (multivalued matricial operators on product spaces can also serve one for construction of exponentially bounded degenerate integrated semigroups; see e.g. [13, Example 3.2.24]).

We close the paper by providing some other illustrative examples and applications of our abstract theoretical results.

Example 3. Let $E := l_2(\mathbb{C})$ be the Hilbert space consisted of all square-summable complex sequences equipped with the norm $\|x\| := \|\langle x_1, x_2, \dots, x_n, \dots \rangle\| := (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$, $x \in E$. Let $(b_n)_{n \in \mathbb{N}}$ be any real sequence with the property that $0 \leq b_n \leq 1$, $n \in \mathbb{N}$ and let $a_n = n + (n^{-2}e^{-2n} - n^2)^{1/2}$, $n \in \mathbb{N}$. Define, for every $\langle x_1, x_2, \dots, x_n, \dots \rangle \in E$ and $t \in [0, 1)$,

$$T(t)\langle x_1, x_2, \dots, x_n, \dots \rangle := \langle b_1 e^{ta_1} x_1, b_2 e^{ta_2} x_2, \dots, b_n e^{ta_n} x_n, \dots \rangle,$$

as well as $C := T(0)$, $I := \{j \in \mathbb{N} : b_j \neq 0\}$ and $D \in L(E)$ by $D\langle x_1, x_2, \dots, x_n, \dots \rangle := \langle y_1, y_2, \dots, y_n, \dots \rangle$, where $y_j = a_j x_j$ for $j \in I$ and $y_j = 0$ for $j \notin I$. Then $(T(t))_{t \in [0, 1)}$ is a local C -regularized semigroup with the integral generator

$$\hat{\mathcal{A}} := \{(\langle x_1, x_2, \dots, x_n, \dots \rangle, \langle y_1, y_2, \dots, y_n, \dots \rangle) : y_j = a_j x_j \text{ for all } j \in I\}$$

and any linear subspace \mathcal{A} of $E \times E$ satisfying $E \times R(D) \subseteq \mathcal{A} \subseteq \hat{\mathcal{A}}$ is a subgenerator of $(T(t))_{t \in [0, 1)}$.

Example 4. Put $E := \{f \in C^\infty([0, \infty)) : \lim_{x \rightarrow +\infty} f^{(k)}(x) = 0 \text{ for all } k \in \mathbb{N}_0\}$ and $\|f\|_k := \sum_{j=0}^k \sup_{x \geq 0} |f^{(j)}(x)|$, $f \in E$, $k \in \mathbb{N}_0$. This calibration induces a Fréchet topology on E . Let $J = [a, b] \subseteq (0, \infty)$, and let $m_b \in C^\infty([0, \infty))$ satisfy $0 \leq m_b(x) \leq 1$, $x \geq 0$, $m_b(x) = 1$, $x \notin J$ and $m_b(x) = 0$, $x \in [a + \epsilon, b - \epsilon]$ for some $\epsilon > 0$. Consider the multiplication operators $A : D(A) \rightarrow E$ and $B \in L(E)$, where $D(A) = \{f(x) \in E : (-1 - x + ie^x)f(x) \in E\}$, $Af(x) := (-1 - x + ie^x)f(x)$ and $Bf(x) := m_b(x)f(x)$ ($x \geq 0$, $f \in E$). Set $\mathcal{A} := B^{-1}A$. Then it is not difficult to prove that, for every $s > 1$, $d > 0$ and $\omega > 0$, the operator family $\{e^{-d|\lambda|^{1/s}}(\lambda - \mathcal{A})^{-1} : \operatorname{Re} \lambda > \omega\} \subseteq L(E)$ is equicontinuous. Now we can apply Theorem 8 in order to see that \mathcal{A} is the integral generator of an exponentially equicontinuous $\mathcal{L}^{-1}(e^{-d|\lambda|^{1/s}})$ -convoluted semigroup on E . On the other hand, there do not exist numbers $n \in \mathbb{N}$ and $\tau > 0$ such that \mathcal{A} is the integral generator of a local n -times integrated semigroup $(S_n(t))_{t \in [0, \tau)}$ on E . Speaking-matter-of-factly, $(S_n(t))_{t \in [0, \tau)}$ must be given by the following formula

$$(S_n(t)f)(x) := \left[\frac{e^{t(-1-x+ie^x)}}{(-1-x+ie^x)^n} - \frac{t^{n-1}}{(n-1)!} \frac{1}{-1-x+ie^x} - \dots - \frac{t}{(-1-x+ie^x)^{n-1}} - \frac{1}{(-1-x+ie^x)^n} \right] f(x),$$

for any $f \in E$, $x \geq 0$ and $t \in [0, \tau)$. This immediately implies that for each $t \in (0, 1)$ there exists $f_t \in E$ such that $\|S_n(t)f_t\|_{n+1} = +\infty$, which is a contradiction.

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ВЫРОЖДЕННЫЕ K -СВЁРТОЧНЫЕ C -ПОЛУГРУППЫ И ВЫРОЖДЕННЫЕ K -СВЁРТОЧНЫЕ C -КОСИНУС-ФУНКЦИИ В ЛОКАЛЬНО ВЫПУКЛЫХ ПРОСТРАНСТВАХ

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Главная цель этой работы — исследовать вырожденные K -свёрточные C -полугруппы и вырожденные K -свёрточные C -косинус-функции в секвенциально полных локально выпуклых пространствах. Используется подход, при котором вырожденные операторные семейства определены локально или глобально, а их субгенераторы являются, вообще говоря, многозначными линейными операторами.

Ключевые слова: *вырожденная K -свёрточная C -полугруппа, вырожденная K -свёрточная C -косинус-функция, многозначный линейный оператор, субгенератор, локально выпуклое пространство.*

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