Journal of Siberian Federal University. Mathematics & Physics 2024, 17(2), 151–161

EDN: AAVCFR
УДК 539

# Optimal Control for an Elastic Frictional Contact Problem

**Ahlem Benraouda***
Department of mathematics
University of Science and Technology Houari Boumediene
Bab-Ezzouar, Algeria

Received 10.03.2023, received in revised form 15.06.2023, accepted 04.10.2023

**Abstract.** We consider a mathematical model which describes a frictional contact between an elastic body and a foundation. We prove the existence of a unique weak solution to the problem. Then, we study the continuous dependence of the solution with respect to the data. Finally, we address an optimal control problem for which we prove the existence of at least one solution.

**Keywords:** weak solution, Coulomb's friction, continuous dependence, lower semicontinuity, optimal control.

**Citation:** A. Benraouda, Optimal Control for an Elastic Frictional Contact Problem, J. Sib. Fed. Univ. Math. Phys., 2024, 17(2), 151–161. EDN: AAVCFR.

## Introduction

Contact problems abound in industry and everyday life. For this reason, the modelling, numerical analysis and computer simulations of such problems has been extensively studied in engineering and mathematical literature. See for instance [6, 9, 12–14].

Variational inequalities are a powerful mathematical tool to represent various nonlinear boundary value problems and mathematical models arising in Contact Mechanics. Their theory was developed based on arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. References in the field are [1, 3, 4, 7, 8, 10], for instance.

The optimal control theory in the study of mathematical models of contact is quite limited. The difficulties are generated by the strong nonlinearities which arise in the boundary conditions included in such models, also by some features like non-convexity and non-differentiability. Results on optimal control for various contact problems could be found in [2, 5, 11, 16].

In this paper, we consider a mathematical model which describes the contact between an elastic body and a foundation. We assume that the foundation is made of a rigid-plastic material of yield limit $\xi$. The body is acted upon by body forces of density $\boldsymbol{\varphi}_0$ and by tractions of density $\boldsymbol{\varphi}_2$, which act on a part of its boundary. The variational formulation of the model is in a form of an elliptic variational inequality in which the unknown is the displacement field and the data are the densities of applied forces $(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_2)$, the yield limit $\xi$ and the friction bound $F_b$.

The paper is structured as follows. In Section 1 we introduce some notation and preliminaries. In Section 2 we state the contact model, then we list the assumptions on the data and derive its variational formulation. Also, we state and prove the unique weak solvability of the problem, Theorem 2.1. Section 3 is dedicated to a convergence result, Theorem 3.1, which establishes the continuous dependence of the solution with respect to the densities of applied forces, the yield limit of the foundation and the friction bound. In Section 4 we state an optimal control problem and we prove its solvability, Theorem 4.2.

*abenraouda@usthb.dz

# 1. Preliminaries

In this section, we introduce the notation and some preliminaries materials we shall use. We use the notation $\mathbb{R}_+$ for the set of non-negative real numbers, $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 1, 2, 3)$ and the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$. The inner products and the corresponding norms on these spaces are defined by

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \end{aligned}$$

where the indices $i$ and $j$ run between 1 and $d$ and, unless stated otherwise, the summation convention over repeated indices is adopted.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega = \Gamma$ and let $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ be a partition of $\Gamma$ into three measurable disjoints parts such that $meas(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for the generic point in $\Omega \cup \Gamma$ and note that, in order to simplify the notation, we usually do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. Moreover, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$. Also, we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at $\Gamma$.

Everywhere in this paper, we use the standard notation for Sobolev and Lebesgue spaces of real-valued functions defined on $\Omega$ and $\Gamma$. In particular, we use the spaces

$$H = L^2(\Omega)^d,\ H_2 = L^2(\Gamma_2)^d,\ L^2(\Gamma_3)^d,\ L^2(\Gamma)^d \text{ and } H^1(\Omega)^d,$$

endowed with their canonical inner products and associated norms. Moreover, we recall that for a function $\boldsymbol{v} \in H^1(\Omega)^d$ we still write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v} \in L^2(\Gamma)^d$ of $\boldsymbol{v}$ on the boundary $\Gamma$. Let

$$\begin{aligned} V &= \{\, \boldsymbol{v} \in H^1(\Omega)^d :\ \boldsymbol{v} = \mathbf{0} \ \text{ a.e. on } \ \Gamma_1 \,\}, \\ Q &= \{\, \boldsymbol{\sigma} = (\sigma_{ij}) :\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\ 1 \leqslant i, j \leqslant d\}, \end{aligned}$$

which are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx.$$

The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here and below, $\boldsymbol{\varepsilon}$ and Div will represent the deformation and the divergence operators, respectively, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

where the quantity $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearised strain tensor associated with the displacement $\boldsymbol{u}$.

Let $\mathbf{0}_{H_2}$ denote the zero element of $H_2$ and $\mathbf{0}_V$ the zero element of $V$. For any element $\boldsymbol{v} \in V$ we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ its normal and tangential components on $\Gamma$ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Moreover, for a regular function $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ we denote by $\sigma_\nu$ and $\boldsymbol{\sigma}_\tau$ its normal and tangential components on $\Gamma$, that is $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$ and, we recall that the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}\, da \quad \forall\ \boldsymbol{v} \in H^1(\Omega)^d. \tag{1.1}$$

Also, recall that there exists a positive constant $c_{tr}$, depending on $\Omega$ and $\Gamma_1$, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \leqslant c_{tr} \|\boldsymbol{v}\|_V \quad \forall\ \boldsymbol{v} \in V. \tag{1.2}$$

Inequality (1.2) represents a consequence of the Sobolev trace theorem.

We end this section with the following result.

**Theorem 1.1.** *Let $X$ be a real Hilbert space and assume that $K$ is a nonempty closed convex subset of $X$, $A : X \to X$ is a strongly monotone Lipschitz continuous operator and $j : X \to \mathbb{R}$ is a convex lower semicontinuous function. Then, for each $f \in X$ there exists a unique solution to the variational inequality*

$$u \in K, \quad (Au, v-u)_X + j(v) - j(u) \geqslant (f, v-u)_X \quad \forall\, v \in K. \tag{1.3}$$

Theorem 1.1 will be used in Section 2 to prove the unique weak solvability of our mathematical model of contact. Its proof could be found in [14].

## 2. Problem statement and variational formulation

The physical setting of the problem is the following. We consider a body made of an elastic material which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary $\partial\Omega = \Gamma$, divided into three measurable disjoint parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is fixed on $\Gamma_1$, it is acted by given body forces of density $\boldsymbol{\varphi}_0$. Also, we assume that surface tractions of density $\boldsymbol{\varphi}_2$ act on $\Gamma_2$, and the body is in contact with an obstacle on $\Gamma_3$.

The classical formulation of the contact problem is as follows.

**Problem $\mathcal{P}$.** *Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ such that*

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$\operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{\varphi}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{\varphi}_2 \qquad \text{on} \quad \Gamma_2, \tag{2.4}$$

$$-\xi \leqslant \sigma_\nu \leqslant 0, \quad -\sigma_\nu = \begin{cases} 0 & \text{if} \quad u_\nu < 0 \\ \xi & \text{if} \quad u_\nu > 0 \end{cases} \qquad \text{on} \quad \Gamma_3, \tag{2.5}$$

$$\|\boldsymbol{\sigma}_\tau\| \leqslant F_b, \quad -\boldsymbol{\sigma}_\tau = F_b\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if} \quad \boldsymbol{u}_\tau \neq 0 \qquad \text{on} \quad \Gamma_3. \tag{2.6}$$

We now provide a description of the equations and boundary conditions in Problem $\mathcal{P}$. First, equation (2.1) represents the elastic constitutive law of the material. We assume that the nonlinear elasticity operator $\mathcal{E}$ satisfies the following conditions

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{E}} > 0 \text{ such that} \\
\qquad \|\mathcal{E}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{E}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\
\qquad (\mathcal{E}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{E}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{E}}\,\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{E}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{E}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{2.7}$$

Concrete examples of operators $\mathcal{E}$ which satisfy condition (2.7) can be found, for example, in [14, 17].

Equation (2.2) is the equation of equilibrium. Conditions (2.3), (2.4) represent the displacement and traction boundary conditions, respectively. We assume that the densities of body forces and tractions are such that

$$\boldsymbol{\varphi}_0 \in H, \tag{2.8}$$

$$\boldsymbol{\varphi}_2 \in H_2. \tag{2.9}$$

Next, (2.5) represent the contact condition in which $\sigma_\nu$ denotes the normal stress and $u_\nu$ is the normal displacement. Moreover, the function $\xi$ satisfies

$$\xi \in L^2(\Gamma_3), \qquad \xi(\boldsymbol{x}) \geqslant 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{2.10}$$

We now provide some comments on this condition. It is described by the multivalued relation between the normal displacement and the opposite of the normal stress. This condition was already used in [15], where a detailed description was provided, together with some mechanical interpretation. It models the contact with a foundation made of a rigid-plastic material. Indeed, this condition shows that the foundation behaves like a rigid body as far as the inequality $|\sigma_\nu| < \xi$ holds, where the function $\xi$ could be interpreted as the yield limit of the the foundation. It could allow penetration only when the equality $|\sigma_\nu| = \xi$ holds. In this case, the yield limit $\xi$ is reached and the foundation offers no additional resistance to penetration.

Finally, (2.6) represents the contact with Coulomb's friction law where $F_b$ is a given friction bound. We assume that

$$F_b \in L^2(\Gamma_3), \qquad F_b(\boldsymbol{x}) \geqslant 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{2.11}$$

In this section, we derive the variational formulation of Problem $\mathcal{P}$ and, to this end, we assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (2.1)–(2.6). Let $\boldsymbol{v} \in V$. We use Green's formula (1.1), then we split the surface integral over $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ and use equalities (2.2), (2.4) to obtain that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q = (\boldsymbol{\varphi}_0, \boldsymbol{v} - \boldsymbol{u})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \boldsymbol{u})_{H_2} + \int_{\Gamma_1} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, da + \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, da.$$

Moreover, using this equality

$$\boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) = \sigma_\nu (v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \quad \text{a.e.} \quad \text{on} \quad \Gamma,$$

and the condition (2.3), we obtain that

$$\begin{aligned}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q = (\boldsymbol{\varphi}_0, \boldsymbol{v} - \boldsymbol{u})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \boldsymbol{u})_{H_2} \; + \\ + \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, da + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da.\end{aligned} \tag{2.12}$$

We use standard arguments and the hypothesis (2.10) to see that the contact condition (2.5) implies that

$$\int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, da \geqslant \int_{\Gamma_3} \xi (u_\nu^+ - v_\nu^+)\, da, \tag{2.13}$$

where $r^+$ denotes the positive part of $r$, i.e., $r^+ = \max\{r, 0\}$. In addition, it is easy to see that the condition (2.6) yields

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, da \geqslant \int_{\Gamma_3} F_b (\|\boldsymbol{u}_\tau\| - \|\boldsymbol{v}_\tau\|)\, da. \tag{2.14}$$

Next, we combine (2.12)–(2.14), then we use the constitutive law (2.1) to see that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_Q + \int_{\Gamma_3} \xi(v_\nu^+ - u_\nu^+)\, da + \int_{\Gamma_3} F_b(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, da \geqslant$$
$$\geqslant (\boldsymbol{\varphi}_0, \boldsymbol{v} - \boldsymbol{u})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \boldsymbol{u})_{H_2}. \tag{2.15}$$

Now, we introduce the operator $A : V \to V$ and the function $j : V \to \mathbb{R}$ defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{2.16}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} \xi v_\nu^+\, da + \int_{\Gamma_3} F_b \|\boldsymbol{v}_\tau\|\, da \qquad \forall\, \boldsymbol{v} \in V. \tag{2.17}$$

Using these definitions and inequality (2.15), we find the following variational formulation of Problem $\mathcal{P}$.

**Problem** $\mathcal{P}_V$. *Find a displacement field* $\boldsymbol{u} \in V$ *such that*

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geqslant$$
$$\geqslant (\boldsymbol{\varphi}_0, \boldsymbol{v} - \boldsymbol{u})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \boldsymbol{u})_{H_2} \quad \forall \boldsymbol{v} \in V. \tag{2.18}$$

We have the following existence and uniqueness result.

**Theorem 2.1.** *Assume that* (2.7)–(2.11) *hold. Then, Problem* $\mathcal{P}_V$ *has a unique solution* $\boldsymbol{u} \in V$.

*Proof.* We apply Theorem 1.1 with $K = X = V$. To this end, we use the definition (2.16) and assumption (2.7)(c) to see that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geqslant m_{\mathcal{E}}\, \|\boldsymbol{u} - \boldsymbol{v}\|_V^2 \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{2.19}$$

On the other hand, using assumption (2.7)(b), we obtain that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \leqslant L_{\mathcal{E}}\, \|\boldsymbol{u} - \boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{2.20}$$

We conclude from (2.19) and (2.20) that $A$ is a strongly monotone Lipschitz continuous operator on the space $V$.

Moreover, using (2.10)–(2.11) and (1.2), we see that the functional $j$ defined by (2.17) is a seminorm on $V$ and, in addition, it satisfies

$$j(\boldsymbol{v}) \leqslant c_{tr}(\|\xi\|_{L^2(\Gamma_3)} + \|F_b\|_{L^2(\Gamma_3)})\|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V.$$

It follows that $j$ is a continuous seminorm and, therefore, it is a convex and lower semicontinuous function on $V$. Finally, using the Riesz representation theorem, we define $\boldsymbol{f} \in V$ as follows

$$(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{\varphi}_0, \boldsymbol{v})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v})_{H_2} \qquad \forall\, \boldsymbol{v} \in V.$$

Theorem 2.1 now is a direct consequence of Theorem 1.1. □

# 3. A continuous dependence result

In this section, we study the dependence of the solution $\boldsymbol{u}$ of Problem $\mathcal{P}_V$ with respect to the data $\boldsymbol{\varphi}_0$, $\boldsymbol{\varphi}_2$, $\xi$ and $F_b$. To this end, we assume in what follows that (2.7)–(2.11) hold, and we consider a perturbation $\boldsymbol{\varphi}_{0\eta}$, $\boldsymbol{\varphi}_{2\eta}$, $\xi_\eta$ and $F_{b\eta}$ of $\boldsymbol{\varphi}_0$, $\boldsymbol{\varphi}_2$, $\xi$ and $F_b$, respectively, which satisfy (2.8)–(2.11). For each $\eta > 0$, we introduce the functional $j_\eta : V \to \mathbb{R}$ defined by

$$j_\eta(\boldsymbol{v}) = \int_{\Gamma_3} \xi_\eta v_\nu^+ \, da + \int_{\Gamma_3} F_{b\eta} \|\boldsymbol{v}_\tau\| \, da \qquad \forall\, \boldsymbol{v} \in V, \tag{3.1}$$

and, we consider the following variational problem.

**Problem** $\mathcal{P}_V^\eta$. *Find a displacement field* $\boldsymbol{u}_\eta \in V$ *such that*

$$(A\boldsymbol{u}_\eta, \boldsymbol{v} - \boldsymbol{u}_\eta)_V + j_\eta(\boldsymbol{v}) - j_\eta(\boldsymbol{u}_\eta) \geqslant (\boldsymbol{\varphi}_{0\eta}, \boldsymbol{v} - \boldsymbol{u}_\eta)_H + (\boldsymbol{\varphi}_{2\eta}, \boldsymbol{v} - \boldsymbol{u}_\eta)_{H_2} \quad \forall\, \boldsymbol{v} \in V. \tag{3.2}$$

It follows from Theorem 2.1 that, for each $\eta > 0$, Problem $\mathcal{P}_V^\eta$ has a unique solution $\boldsymbol{u}_\eta \in V$. The behaviour of the solution $\boldsymbol{u}_\eta$ as $\eta \to 0$ is given in the following result.

**Theorem 3.1.** *Assume that* (2.7)–(2.11) *hold and, moreover, assume*

$$\boldsymbol{\varphi}_{0\eta} \rightharpoonup \boldsymbol{\varphi}_0 \quad \text{in} \quad H \quad \text{as} \quad \eta \to 0, \tag{3.3}$$

$$\boldsymbol{\varphi}_{2\eta} \rightharpoonup \boldsymbol{\varphi}_2 \quad \text{in} \quad H_2 \quad \text{as} \quad \eta \to 0, \tag{3.4}$$

$$\xi_\eta \to \xi \quad \text{in} \quad L^2(\Gamma_3) \quad \text{as} \quad \eta \to 0. \tag{3.5}$$

$$F_{b\eta} \to F_b \quad \text{in} \quad L^2(\Gamma_3) \quad \text{as} \quad \eta \to 0. \tag{3.6}$$

*Then, the following convergence holds*

$$\boldsymbol{u}_\eta \to \boldsymbol{u} \quad \text{in} \quad V \quad \text{as} \quad \eta \to 0. \tag{3.7}$$

The proof of Theorem 3.1 will be carried out in two steps. First, we provide the following weak convergence result.

**Lemma 3.2.** *The sequence* $\{\boldsymbol{u}_\eta\}$ *converges weakly in* $V$ *to* $\boldsymbol{u}$*, i.e.,*

$$\boldsymbol{u}_\eta \rightharpoonup \boldsymbol{u} \quad \text{in} \quad V \quad \text{as} \quad \eta \to 0. \tag{3.8}$$

*Proof.* Let $\eta > 0$. We take $\boldsymbol{v} = \mathbf{0}_V$ in (3.2) to obtain

$$(A\boldsymbol{u}_\eta - A\mathbf{0}_V, \boldsymbol{u}_\eta)_V + j_\eta(\boldsymbol{u}_\eta) \leqslant (\boldsymbol{\varphi}_{0\eta}, \boldsymbol{u}_\eta)_H + (\boldsymbol{\varphi}_{2\eta}, \boldsymbol{u}_\eta)_{H_2} - (A\mathbf{0}_V, \boldsymbol{u}_\eta)_V.$$

Next, using assumption (2.19), the positivity of the functional $j$ and the inequality (1.2), we deduce that

$$\begin{aligned} \|\boldsymbol{u}_\eta\|_V &\leqslant \frac{1}{m_{\mathcal{E}}}(\|\boldsymbol{\varphi}_{0\eta}\|_H + c_{tr}\|\boldsymbol{\varphi}_{2\eta}\|_{H_2} + \|A\mathbf{0}_V\|_V) \leqslant \\ &\leqslant \frac{\max(1, c_{tr})}{m_{\mathcal{E}}}(\|\boldsymbol{\varphi}_{0\eta}\|_H + \|\boldsymbol{\varphi}_{2\eta}\|_{H_2} + \|A\mathbf{0}_V\|_V). \end{aligned}$$

The convergences (3.3) and (3.4) imply that the sequences $\{\boldsymbol{\varphi}_{0\eta}\}$ and $\{\boldsymbol{\varphi}_{2\eta}\}$ are bounded in $H$ and $H_2$, respectively. Therefore, we deduce that there exists $M > 0$, which does not depend on $\eta$, such that

$$\|\boldsymbol{u}_\eta\|_V \leqslant M. \tag{3.9}$$

Now, we combine (3.9) with a standard compactness argument to see that there exists $\widetilde{\boldsymbol{u}} \in V$ such that, passing to a subsequence, still denoted $\{\boldsymbol{u}_\eta\}$, we have

$$\boldsymbol{u}_\eta \rightharpoonup \widetilde{\boldsymbol{u}} \quad \text{in} \quad V \quad \text{as} \quad \eta \to 0. \tag{3.10}$$

We establish the equality

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u}. \tag{3.11}$$

Let $\eta > 0$. We take $\boldsymbol{v} = \widetilde{\boldsymbol{u}} \in V$ in (3.2) to obtain that

$$(A\boldsymbol{u}_\eta, \boldsymbol{u}_\eta - \widetilde{\boldsymbol{u}})_V \leqslant (\boldsymbol{\varphi}_{0\eta}, \boldsymbol{u}_\eta - \widetilde{\boldsymbol{u}})_H + (\boldsymbol{\varphi}_{2\eta}, \boldsymbol{u}_\eta - \widetilde{\boldsymbol{u}})_{H_2} + j_\eta(\widetilde{\boldsymbol{u}}) - j_\eta(\boldsymbol{u}_\eta).$$

Next, we pass to the upper limit as $\eta \to 0$ in this inequality and taking into account the convergences (3.3)–(3.6), (3.10) and the compactness of the trace operator, we deduce that

$$\limsup_{\eta \to 0} \, (A\boldsymbol{u}_\eta, \boldsymbol{u}_\eta - \widetilde{\boldsymbol{u}})_V \leqslant 0.$$

Therefore, assumptions (2.19)–(2.20) and the convergence (3.10) yield

$$\liminf_{\eta \to 0} \, (A\boldsymbol{u}_\eta, \boldsymbol{u}_\eta - \boldsymbol{v})_V \geqslant (A\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}} - \boldsymbol{v})_V \qquad \forall \boldsymbol{v} \in V. \tag{3.12}$$

On the other hand, we pass to the upper limit in (3.2) and we use again the convergences (3.3)–(3.6), (3.10) and the compactness of the trace operator to obtain that

$$\limsup_{\eta \to 0} \, (A\boldsymbol{u}_\eta, \boldsymbol{u}_\eta - \boldsymbol{v})_V \leqslant (\boldsymbol{\varphi}_0, \widetilde{\boldsymbol{u}} - \boldsymbol{v})_H + (\boldsymbol{\varphi}_2, \widetilde{\boldsymbol{u}} - \boldsymbol{v})_{H_2} + j(\boldsymbol{v}) - j(\widetilde{\boldsymbol{u}}) \qquad \forall \boldsymbol{v} \in V.$$

We combine now this inequality and (3.12) to see that

$$(A\widetilde{\boldsymbol{u}}, \boldsymbol{v} - \widetilde{\boldsymbol{u}})_V + j(\boldsymbol{v}) - j(\widetilde{\boldsymbol{u}}) \geqslant (\boldsymbol{\varphi}_0, \boldsymbol{v} - \widetilde{\boldsymbol{u}})_H + (\boldsymbol{\varphi}_2, \boldsymbol{v} - \widetilde{\boldsymbol{u}})_{H_2} \quad \forall \boldsymbol{v} \in V. \tag{3.13}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u}$ in (3.13) and $\boldsymbol{v} = \widetilde{\boldsymbol{u}}$ in (2.18), then, adding the resulting inequalities and using assumption (2.19), we obtain that the equality (3.11) holds.

A carefully examination of the proof of Lemma 3.2 shows that any weakly convergent subsequence of the sequence $\{\boldsymbol{u}_\eta\} \subset V$ converges weakly to $\boldsymbol{u} \in V$, where, $\boldsymbol{u}$ is the unique solution of (2.18). Moreover, the bound (3.9) shows that the sequence $\{\boldsymbol{u}_\eta\}$ is bounded in $V$ and, therefore, Lemma 3.2 is a consequence of a standard compactness argument. □

We proceed with the following strong convergence result.

**Lemma 3.3.** *The sequence $\{\boldsymbol{u}_\eta\}$ converges strongly in $V$ to $\boldsymbol{u}$, i.e.,*

$$\boldsymbol{u}_\eta \to \boldsymbol{u} \quad \text{in } V \quad \text{as } \eta \to 0. \tag{3.14}$$

*Proof.* Let $\eta > 0$. We take $\boldsymbol{v} = \boldsymbol{u}$ in (3.2) to obtain that

$$(A\boldsymbol{u}_\eta, \boldsymbol{u}_\eta - \boldsymbol{u})_V \leqslant (\boldsymbol{\varphi}_{0\eta}, \boldsymbol{u}_\eta - \boldsymbol{u})_H + (\boldsymbol{\varphi}_{2\eta}, \boldsymbol{u}_\eta - \boldsymbol{u})_{H_2} + j_\eta(\boldsymbol{u}) - j_\eta(\boldsymbol{u}_\eta).$$

Next, we use this inequality and assumption (2.19) to see that

$$\begin{aligned} m_{\mathcal{E}} \, \|\boldsymbol{u}_\eta - \boldsymbol{u}\|_V^2 &\leqslant (A\boldsymbol{u}_\eta - A\boldsymbol{u}, \boldsymbol{u}_\eta - \boldsymbol{u})_V = \\ &= (A\boldsymbol{u}_\eta, \boldsymbol{u}_\eta - \boldsymbol{u})_V - (A\boldsymbol{u}, \boldsymbol{u}_\eta - \boldsymbol{u})_V \leqslant \\ &\leqslant (\boldsymbol{\varphi}_{0\eta}, \boldsymbol{u}_\eta - \boldsymbol{u})_H + (\boldsymbol{\varphi}_{2\eta}, \boldsymbol{u}_\eta - \boldsymbol{u})_{H_2} + j_\eta(\boldsymbol{u}) - j_\eta(\boldsymbol{u}_\eta) - (A\boldsymbol{u}, \boldsymbol{u}_\eta - \boldsymbol{u})_V. \end{aligned}$$

We now pass to the limit as $\eta \to 0$ and we use (3.3)–(3.6), (3.8) and the compactness of the trace operator. As a result we deduce that

$$\|\boldsymbol{u}_\eta - \boldsymbol{u}\|_V \to 0 \quad \text{as } \eta \to 0,$$

which concludes the proof. □

We are now in position to present the proof of Theorem 3.1.

*Proof.* The convergence (3.7) is a consequence of Lemma 3.2. □

The convergence result (3.7) is important from mechanical point of view, since it shows that the weak solution of the elastic contact problem (2.1)–(2.6) depends continuously on the densities of applied forces, the yield limit and the friction bound.

# 4. The optimal control problem

In this section, we formulate an optimal control problem associate to Problem $\mathcal{P}_V$. To this end, we assume that conditions (2.7)–(2.11) hold and, in order to control the solution of Problem $\mathcal{P}_V$ by the density of surface tractions $\boldsymbol{\varphi}_2$, we assume that $\boldsymbol{\varphi}_0$, $\xi$ and $F_b$ are given and satisfy (2.8), (2.10), (2.11), respectively. Let $\boldsymbol{\phi} \in V$ and $\delta, \gamma > 0$ be two positive constants and let us define the cost functional $\mathcal{L} : H_2 \times V \to \mathbb{R}$ by

$$\mathcal{L}(\boldsymbol{\varphi}_2, \boldsymbol{u}) = \delta \|\boldsymbol{u} - \boldsymbol{\phi}\|_V + \gamma \|\boldsymbol{\varphi}_2\|_{H_2} \qquad \forall (\boldsymbol{\varphi}_2, \boldsymbol{u}) \in H_2 \times V. \tag{4.1}$$

Using standard arguments it is easy to see that $\mathcal{L}$ is a convex lower semicontinuous functional on $H_2 \times V$ and, therefore, it is weakly lower semicontinuous. Also, we define the following admissible set

$$\mathcal{V}_{ad} = \{ (\boldsymbol{\varphi}_2, \boldsymbol{u}) \in H_2 \times V, \ \text{ such that } (2.18) \text{ holds } \}. \tag{4.2}$$

We formulate now the following optimal control problem.

**Problem $\mathcal{O}$.** *Find $(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*) \in \mathcal{V}_{ad}$ such that*

$$\mathcal{L}(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{\varphi}_2, \boldsymbol{u}) \in \mathcal{V}_{ad}} \mathcal{L}(\boldsymbol{\varphi}_2, \boldsymbol{u}).$$

An element $(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*)$ is called an *optimal pair* and the corresponding surface traction force $\boldsymbol{\varphi}_2^*$ is called an *optimal control*. The mechanical interpretation of Problem $\mathcal{O}$ is the following : we are looking for a given surface traction force $\boldsymbol{\varphi}_2 \in H_2$ such that the displacement $\boldsymbol{u} \in V$ given by (2.18) is as close as possible to the "desired displacement" $\boldsymbol{\phi}$. Furthermore, this choice has to fulfil a minimum expenditure condition which is taken into account by the second term in the definition (4.1).

Our result in this section is the following.

**Theorem 4.1.** *Assume that* (2.7)–(2.8) *and* (2.10)–(2.11) *hold. Then, there exists at least one solution $(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*) \in \mathcal{V}_{ad}$ of Problem $\mathcal{O}$.*

The proof of Theorem 4.1 will be carried out in two steps, that we present in what follows. We start by considering the following functional $J : H_2 \to \mathbb{R}$ defined by

$$J(\boldsymbol{\varphi}_2) = \delta \|\boldsymbol{u}(\boldsymbol{\varphi}_2) - \boldsymbol{\phi}\|_V + \gamma \|\boldsymbol{\varphi}_2\|_{H_2} \qquad \forall \boldsymbol{\varphi}_2 \in H_2, \tag{4.3}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{\varphi}_2)$ is the solution of (2.18). Next, we consider the following optimization problem.

**Problem $\mathcal{O}_1$.** *Find $\boldsymbol{\varphi}_2^* \in H_2$ such that*

$$J(\boldsymbol{\varphi}_2^*) = \min_{\boldsymbol{\varphi}_2 \in H_2} J(\boldsymbol{\varphi}_2). \tag{4.4}$$

We have the following existence result.

**Lemma 4.2.** *There exists at least one solution $\boldsymbol{\varphi}_2^* \in H_2$ of Problem $\mathcal{O}_1$.*

*Proof.* Let

$$\theta = \inf_{\boldsymbol{\varphi}_2 \in H_2} J(\boldsymbol{\varphi}_2) \in \mathbb{R}, \tag{4.5}$$

and let $\{\boldsymbol{\varphi}_{2n}\} \subset H_2$ such that

$$\lim_{n \to \infty} J(\boldsymbol{\varphi}_{2n}) = \theta. \tag{4.6}$$

We prove that the sequence $\{\boldsymbol{\varphi}_{2n}\}$ is bounded in $H_2$. Arguing by contradiction, assume that $\{\boldsymbol{\varphi}_{2n}\}$ is not bounded in $H_2$. Then, we pass to a subsequence, still denoted $\{\boldsymbol{\varphi}_{2n}\}$, to see that

$$\|\boldsymbol{\varphi}_{2n}\|_{H_2} \to +\infty \quad \text{in} \quad H_2 \quad \text{as} \quad n \to +\infty. \tag{4.7}$$

Using the definition (4.3) and the positivity of the parameters $\delta$ and $\gamma$ to see that

$$J(\boldsymbol{\varphi}_{2n}) = \delta\,\|\boldsymbol{u}(\boldsymbol{\varphi}_{2n}) - \boldsymbol{\phi}\|_V + \gamma\,\|\boldsymbol{\varphi}_{2n}\|_{H_2} \geqslant \gamma\,\|\boldsymbol{\varphi}_{2n}\|_{H_2},$$

then, passing to the limit as $n \to +\infty$ and using (4.7) we deduce that

$$\lim_{n\to+\infty} J(\boldsymbol{\varphi}_{2n}) = +\infty.$$

We combine this equality with (4.6) to see that $\theta = +\infty$ which is a contradiction with (4.5) and, therefore, we conclude that the sequence $\{\boldsymbol{\varphi}_{2n}\}$ is bounded in $H_2$. Thus, a standard compactness argument implies that there exists $\boldsymbol{\varphi}_2^* \in H_2$ such that, passing to a subsequence, still denoted $\{\boldsymbol{\varphi}_{2n}\}$, we have

$$\boldsymbol{\varphi}_{2n} \rightharpoonup \boldsymbol{\varphi}_2^* \quad \text{in} \quad H_2 \quad \text{as} \quad n \to +\infty. \tag{4.8}$$

In addition, using the convergence (4.8) and the continuous dependence result given by Theore 3.1, we have that

$$\boldsymbol{u}(\boldsymbol{\varphi}_{2n}) \to \boldsymbol{u}(\boldsymbol{\varphi}_2^*) \quad \text{in} \quad V \quad \text{as} \quad n \to +\infty. \tag{4.9}$$

We now use (4.8) and (4.9) to see that

$$\begin{aligned}
&\lim_{n\to+\infty} \|\boldsymbol{u}(\boldsymbol{\varphi}_{2n}) - \boldsymbol{\phi}\|_V = \|\boldsymbol{u}(\boldsymbol{\varphi}_2^*) - \boldsymbol{\phi}\|_V,\\
&\liminf_{n\to+\infty} \|\boldsymbol{\varphi}_{2n}\|_{H_2} \geqslant \|\boldsymbol{\varphi}_2^*\|_{H_2},
\end{aligned}$$

which imply that

$$\liminf_{n\to+\infty} J(\boldsymbol{\varphi}_{2n}) \geqslant J(\boldsymbol{\varphi}_2^*). \tag{4.10}$$

It follows from (4.6) and (4.10) that

$$\theta \geqslant J(\boldsymbol{\varphi}_2^*). \tag{4.11}$$

On the other hand, (4.5) implies that

$$\theta \leqslant J(\boldsymbol{\varphi}_2^*). \tag{4.12}$$

Finally, we combine (4.11) and (4.12) to see that (4.4) holds, which concludes the proof. □

We proceed with the following existence result.

**Lemma 4.3.** *There exists at least one solution $(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*) \in \mathcal{V}_{ad}$ of Problem $\mathcal{O}$.*

*Proof.* We note that

$$(\boldsymbol{\varphi}_2, \boldsymbol{u}) \in \mathcal{V}_{ad} \iff \boldsymbol{\varphi}_2 \in H_2 \text{ and } \boldsymbol{u} = \boldsymbol{u}(\boldsymbol{\varphi}_2) \text{ is the solution of (2.18)}. \tag{4.13}$$

The definitions (4.1) and (4.3) imply that

$$J(\boldsymbol{\varphi}_2) = \mathcal{L}(\boldsymbol{\varphi}_2, \boldsymbol{u}(\boldsymbol{\varphi}_2)) \qquad \forall\, \boldsymbol{\varphi}_2 \in H_2.$$

Let $\boldsymbol{\varphi}_2^* \in H_2$ be a solution of Problem $\mathcal{O}_1$ and $\boldsymbol{u}^* = \boldsymbol{u}(\boldsymbol{\varphi}_2^*)$ be the solution of (2.18) with the data $\boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_2^*$. Then, by using (4.13) we deduce that

$$(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*) \in \mathcal{V}_{ad}. \tag{4.14}$$

Moreover, we have that

$$\mathcal{L}(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*) = J(\boldsymbol{\varphi}_2^*) \leqslant J(\boldsymbol{\varphi}_2) = \mathcal{L}(\boldsymbol{\varphi}_2, \boldsymbol{u})$$

for all $(\boldsymbol{\varphi}_2, \boldsymbol{u}) \in \mathcal{V}_{ad}$. Combining this inequality with (4.14), we deduce that $(\boldsymbol{\varphi}_2^*, \boldsymbol{u}^*)$ is a solution of Problem $\mathcal{O}$, which concludes the proof. □

We are now in position to present the proof of Theorem 4.1.

*Proof.* Theorem 4.1 is a direct consequence of Lemma 4.3. □

# References


[1] C.Baiocchi, A.Capelo, Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems, John Wiley, Chichester, 1984.

[2] V.Barbu, Optimal Control of Variational Inequalities, Pitman Advanced Publishing, Boston, 1984.

[3] H.Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier (Grenoble)*, **18**(1968), 115–175.

[4] A.Capatina, Variational Inequalities and Frictional Contact Problems, *Advances in Mechanics and Mathematics*, Vol. 31, Springer, New York, 2014.

[5] A.Friedman, Optimal control for variational inequalities, *SIAM J. Control Optim.* **24**(1986), no. 3, 439–451.

[6] W.Han, M.Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, Studies in Advanced Mathematics, Vol. 30, Americal Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.

[7] I.Hlaváček, J Haslinger, J.Nečas, J.Lovíšek, Solution of Variational Inequalities in Mechanics, Springer-Verlag, New York, 1988.

[8] N.Kikuchi, J.T.Oden, Theory of variational inequalities with applications to problems of flow through porous media, *Int. J. Engng. Sci.*, **18**(1980), 1173–1284.

[9] T.A.Laursen, Computational Contact and Impact Mechanics, Springer, Berlin, 2002.

[10] J.-L.Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Gauthiers-Villars, Paris, 1969.

[11] R.Mignot, J.P.Puel, Optimal control in some variational inequalities, *SIAM J. Control Optim.*, **22**(1984), 466–476.

[12] S.Migórski, A.Ochal, M.Sofonea, Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Advances in Mechanics and Mathematics, Vol. 26, Springer, New York, 2013.

[13] M.Shillor, M.Sofonea, J.J.Telega, Models and Analysis of Quasistatic Contact, Lect. Notes Phys., Vol. 655, Springer, Berlin, 2004.

[14] M.Sofonea, A.Matei, Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Note Series **398**, Cambridge University Press, 2012.

[15] M.Sofonea, S.Migórski, Variational Hemivariational Inequalities with Applications, Chapman and Hall/CRC, New York, 2017.

[16] A.Touzaline, Optimal control of a frictional contact problem, *Acta Mathematicae Applicatae Sinica, English Series* **31**(2015), 991–1000. DOI: 10.1007/s10255-015-0519-8

[17] E.Zeidler, Nonlinear Functional Analysis and its Applications. IV: Applications to Mathematical Physics, New York, Springer-Verlag,

## Оптимальное управление для задачи упругого фрикционного контакта

**Ахлем Бенрауда**
Университет науки и технологий Хуари Бумедьена
Баб-Эззуар, Алжир

**Аннотация.** Рассматривается математическая модель, описывающая фрикционный контакт упругого тела с фундаментом. Доказано существование единственного слабого решения задачи. Изучаеся непрерывная зависимость решения от данных. Наконец, мы рассматриваем задачу оптимального управления, для которой доказываем существование хотя бы одного решения.

**Ключевые слова:** слабое решение, кулоновское трение, непрерывная зависимость, полунепрерывность снизу, оптимальное управление.