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On multiparticle systems on a straight line with a triangular scattering reflection

A.Yu. Plakhov

1. We consider a Hamiltonian system in $\mathbf{R}^{2n}$ with the Hamiltonian

$$(1) \qquad H(q, p) = \sum \frac{p_i^2}{2} + \sum V(q_j - q_i).$$

We assume that the potential of the interaction V satisfies the following conditions:

$$(2) \quad V'(x) < 0, \quad x \in (0, \infty); \qquad V(x) \to +\infty \quad (x \to 0+); \quad \int^{\infty} |V(x)|\,dx < \infty.$$

It is known (see [1]) that they are sufficient for the existence of a reflection of the scattering σ: $(q^-, p^-) \to (q^+, p^+)$, where $q^{\pm}$, $p^{\pm} \in \mathbf{R}^n$.

We denote by M_n the set of potentials for which the scattering reflection in a system with the Hamiltonian (1) has triangular shape, that is, $p^+ = f(p^-)$, $q^+ = g(q^-, p^-)$. In the case $n = 2$ a scattering reflection always has triangular form: $q_1^+ = q_2^- + \delta(p_1^- - p_2^-)$, $q_2^+ = q_1^- - \delta(p_1^- - p_2^-)$, $p_1^+ = p_2^-$, $p_2^+ = p_1^-$. Thus, to every potential V we can assign the function δ (phase-shift) defined on $(0, \infty)$. A potential V belongs to M_n if and only if $p_i^+ = p_{n+1-i}^-$ for all (q^-, p^-) $(i = 1, ..., n)$.

It is known that for all n the potentials a^2x^{-2} and $a^2\sinh^{-2} bx$ belong to M_n (see [2]).

In [3] it is shown that $V(x) = x^{-2}$ if $V \in M_3$ and that $V(x)$ decreases like a power: $V(x) = x^{-k}(1 + o(1))\,(x \to +\infty), k > 1$.

***Theorem** 1. Let $V(x) = e^{-kx+a(x)} \in M_n, n \geqslant 3, k > 0, a \in c^2(0, \infty), a' \to 0, a'' \to 0$ (as $x \to +\infty$). Then $V(x) = b^2\sinh^{-2}(kx/2)$ for some $b > 0$.*

It can be shown that $M_n \supset M_{n+1}$ for $n \geqslant 2$, therefore, it suffices to consider the case $n = 3$.

The motion of a system of three particles is determined by six parameters: $q_1^-, q_2^-, q_3^-, p_1^-, p_2^-, p_3^-$. Fixing all paramaters except q_3^-, we consider the motion $\{q_i(t, q_3^-), p_i(t, q_3^-), i = 1, 2, 3\}$, as $q_3^- \to +\infty$. We put $t_1 = (3\delta_{12} - q_{12} - q_{13})(p_{12} + p_{13})^{-1}$, $t_2 = (3\delta_{13} - q_{13} - q_{23})(p_{13} + p_{23})^{-1}$, $d_1 = p_{12}t_1 + q_{12} - 2\delta_{12}$, $d_2 = p_{32}t_2 + q_{32} + \delta_{13} - \delta_{12}$, where $p_{ij} = p_i^- - p_j^-$, $q_{ij} = q_i^- - q_j^-$, and $\delta_{ij} = \delta(p_{ij})$. On each of the intervals $(-\infty, t_1)$, (t_1, t_2), and $(t_2, +\infty)$ we find the change of the momentum p_1.

On $(-\infty, t_1)$, we consider the motion $\{\bar{q}_j(t), \bar{p}_j(t), j = 1, 2\}$ of the first and second particles without taking into account the influence of the third. It turns out that

$$p_1(t_1) = \bar{p}_1(t_1) + O(V(2d_1(1-\varepsilon))) = p_2^- + p_{12}^{-1}V(d_1) + O(V(2d_1(1-\varepsilon)))$$

for $q_3^- \to +\infty$ for any $\varepsilon \in (0, 1)$. Next, since $\{q_i(-t), -p_i(-t), i = 1, 2, 3\}$ is a motion with the same potential and scattering data, as $t \to -\infty$, $(q_3^- + \delta_{13} + \delta_{23}, q_2^- + \delta_{12} - \delta_{23}, q_1^- - \delta_{12} - \delta_{13}, -p_3^-, -p_2^-, -p_1^-)$, we obtain immediately the analogous formula for $p_1(t_2)$. Hence,

$$(3) \quad p_1(t_2) - p_1(t_1) = -p_{12}^{-1}V(d_1) - p_{23}^{-1}V(d_2) + O(V(2d_1(1-\varepsilon)) + V(2d_2(1-\varepsilon))).$$

On the other hand, the change of the momentum p_1 on the interval (t_1, t_2) can be found by the formula

$$(4) \qquad p_1(t_2) - p_1(t_1) = \int_{t_1}^{t_2} [V'(q_2(t) - q_1(t)) + V'(q_3(t) - q_1(t))]\,dt.$$

We replace $q_1(t)$ on the right-hand side of (4) by $p_2^-t + q_2^- + \delta_{12}$, $q_2(t)$ and $q_3(t)$ by the coordinates at the moment of time t of particles interacting with the potential V with the scattering data, as $t \to -\infty$ $(q_1^- - \delta_{12}, q_3^-, p_1^-, p_3^-)$, and we show that the right-hand side of (4) changes by $o(V(h))$, where $h = p_{12}p_{13}^{-1}(\delta_{12} + \delta_{13} - q_{13}) + (q_{12} - 2\delta_{12} - \delta_{13}/2)$. Computing the expression resulting after the substitution, we obtain

$$(5) \qquad p_1(t_2) - p_1(t_1) = -p_{12}^{-1}V(d_1) - p_{23}^{-1}V(d_2) + V(h)\,\omega + o(V(h)),$$

where

(6) $$\omega = \frac{4k}{v}\int_0^{\infty}\left[\cosh\left(\frac{k}{2}q_v\left(\frac{x}{v}\right)\right) - e^{-\delta_{13}}\cosh\frac{kx}{2}\right]\cosh\frac{\lambda kx}{2}\,dx,$$

$\lambda = (p_{12} - p_{23})/p_{13}$, $v = p_{13}$, q_v (t) is the solution of the equation $\dot{q}^2/2 + 2V(q) = v^2/2$, $\dot{q}(0) = 0$.

We see that ω is independent of q_3^-. Comparing the right-hand sides of (3) and (5) and bearing in mind that $V(2d_1(1 - \varepsilon)) + V(2d_2(1 - \varepsilon)) = o(V(h))$ for some $\varepsilon \in (0, 1)$, we obtain $\omega = 0$. Hence, taking into account that $\lambda \in (-1, 1)$ and v are arbitrary, we deduce that the potential has the form $b^2\sinh^{-2}(kx/2)$.

2. Now we consider the case when the potential $V \in M_n$ is extended in a complex neighbourhood of 0 to an analytic function with an algebraic singularity at 0.

If $(-q(t), 0, q(t))$ is the motion in a system with potential V, then $(-q(t), q(t))$ is a motion in a 2-particle system with potential $V(x) + 2V(x/2)$. We denote by $\delta^W(p)$ the phase-shift corresponding to the potential inverse to W. The mapping $W \to \delta^W$ is linear, and $\delta^{W(\alpha y)}(p) = \delta^{W(y)}(\sqrt{(\alpha p)})$ for $\alpha > 0$ (see [4]).

Now $q_3^+ = -\delta^{W_1(y)}(2p)$, where $W_1(y)$ is the function inverse to $V(x) + 2V(x/2)$. On the other hand, $q_3^+ - q_1^- = -\delta(p_1^- - p_2^-) - \delta(p_1^- - p_3^-)$ for $V \in M_3$ (see [4]). Hence, $q_3^+ = -\delta^{W(y)}(p) - \delta^{W(y)}(2p) = -\delta^{W(y/4)+W(y)}(2p)$. Thus, $\delta^{W_1(y)} = \delta^{W(y/4)+W(y)}$. Using a result of Moser [4], we find that $W_1(y) - W(y/4) - W(y) = cy^{-1/2}$, provided that $V \in M_3$, $\int^{\infty} [V(x)]^{\theta}\,dx < \infty \quad (0 < \theta < 1)$.

Thus, we arrive at the following lemma.

Lemma. *Let $W(y)$ be the function inverse to $V(x)$ and suppose that V satisfies the conditions stated above. Then the functions $V(x/2) + V(x)/2$ and $W(2y) + W(y/2) + cy^{-1/2}$ are mutually inverse for some c.*

Using this lemma, we can prove the following theorem.

Theorem 2. *Suppose that $V(x] \in M_n$, $\int^{\infty} [V(x))]^{\theta}\,dx < \infty$, $\theta \in (0, 1)$, and that $V(x)$ coincides on $(0, \varepsilon)$ with one of the branches of an analytic function having an algebraic singularity at 0. Then either $V(x) = a^2x^{-2}$ or $V(x) = a^2\sinh^{-2} bx$.*

We sketch the proof of this theorem. First, using the lemma, we find the type of the singularity of the potential $V(x) = x^{-k}(1 + o(1))$ (as $x \to 0+$). It turns out that either a) $k = 2$ or b) $k = 1$. In the case a), we expand the function $[V(x)]^{-1/2} = x(a_0 + a_1x^{1/m} + a_2x^{2/m} + \ldots)$, $x \in (0, \varepsilon)$, in a power series. It can be shown that the coefficients a_n of the expansion are uniquely determined by the values a_0 and a_{2m}. Hence, $V(x) = a^2\sinh^{-2} bx$ for a suitable choice of a and b when $a_{2m} \neq 0$, and $V(x) = a^2x^{-2}$ when $a_{2m} = 0$. The case b) is treated similarly. It does not yield new potentials.

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Moscow State University

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