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Infinite sets can be Ramsey in the Chebyshev metric

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Ramsey theory, which originated from a group of unrelated (at first glance) results from different areas of mathematics (Ramsey's theorem, which appeared due to the intrinsic development of mathematical logic, the number-theoretic Schur theorem, the geometric 'happy ending problem'), evolved into a separate discipline in the second half of the 20th century (see the classical book [1]). The present study is concerned with geometric Ramsey problems, which encompass the well-known currently open *problem of the chromatic number of the plane* (see the survey [2]) and can be traced back to the classical studies in [3] and [4] by Erdős with coauthors.

Let us give the formal definitions. Let $\mathbb{X} = (X, \rho_X)$, $\mathcal{Y} = (Y, \rho_Y)$ be metric spaces. A subset Y' of X is called a *copy* of $\mathcal{Y}$ if there exists a bijection $f\colon Y \to Y'$ that preserves distances. The *chromatic number* $\chi(\mathbb{X}, \mathcal{Y})$ *of* $\mathbb{X}$ *with forbidden space* $\mathcal{Y}$ is by definition the smallest k such that the elements of X can be coloured with k colours so that no monochromatic copy $Y' \subset X$ of $\mathcal{Y}$ arises. The *Euclidean Ramsey theory* treats the case where $\mathbb{X}$ is an n-dimensional Euclidean space $\mathbb{R}^n_2$. At present, there is no answer (nor even a generally accepted conjecture) in the problem of the classification of the finite *Ramsey* spaces (that is, spaces $\mathcal{Y}$ such that $\chi(\mathbb{R}^n_2, \mathcal{Y}) \to \infty$ as $n \to \infty$), even though this problem has been drawing the attention of many prominent mathematicians for several decades (see [5]–[11]). However, the situation turns out to be much simpler if the forbidden space $\mathcal{Y}$ is infinite: in [4] it was shown that in this case the exact equality $\chi(\mathbb{R}^n_2, \mathcal{Y}) = 2$ holds. The conjecture that this equality holds not only for infinite, but also for all *sufficiently large* spaces $\mathcal{Y}$ (in terms of the dimension n) is still open (see [12]).

Recently [13], [14] the authors of the present paper have extended problems in the Euclidean Ramsey theory to the metric spaces $\mathbb{R}^n_\infty = (\mathbb{R}^n, \ell_\infty)$, where ℓ_∞ is the standard Chebyshev metric defined by $\ell_\infty(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i|$ for all $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$. It particular, they showed that for any finite metric space $\mathcal{Y}$ there exists a constant $c = c(\mathcal{Y}) > 1$ such that $\chi(\mathbb{R}^n_\infty, \mathcal{Y}) > c^n$ for any $n \in \mathbb{N}$. However, the situation with infinite forbidden spaces $\mathcal{Y}$ remained unclear. Our paper fills this gap.

The subsets $\mathcal{B}(\boldsymbol{\lambda}) = \{0, \lambda_1, \lambda_1 + \lambda_2, \dots\}$ of the real line, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ is an arbitrary decreasing sequence of positive numbers, provide an example of most natural resolvable infinite metric spaces. It can be assumed without loss of generality that the series $\sum_i \lambda_i$ is convergent, because a simple 2-colouring of

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the space by nested colour-alternating cubes with sufficiently rapidly growing side lengths shows that $\chi(\mathbb{R}^n_\infty, \mathcal{Y}) = 2$ for all $\mathcal{Y}$ such that $\operatorname{diam}(\mathcal{Y}) = \infty$.

Theorem 1. *For any natural number n,*

(a) *if* $\lim_i \lambda_i/\lambda_{i+1} \leqslant 1 + 5^{-n}$, *then* $\chi(\mathbb{R}^n_\infty, \mathcal{B}(\boldsymbol{\lambda})) = 2$;

(b) *if* $\lambda_i \geqslant 32\lambda_{i+1}$ *for all* $i \in \mathbb{N}$, *then* $\chi(\mathbb{R}^n_\infty, \mathcal{B}(\boldsymbol{\lambda})) \geqslant \log_3 n$;

(c) *if* $\lambda_i \geqslant 2^n\lambda_{i+1}$ *for all* $i \in \mathbb{N}$, *then* $\chi(\mathbb{R}^n_\infty, \mathcal{B}(\boldsymbol{\lambda})) \geqslant n + 1$;

(d) *if* $|Y| \geqslant n^n$, *then* $\chi(\mathbb{R}^n_\infty, \mathcal{Y}) \leqslant n + 1$.

We note two corollaries of this result, which illustrate the fundamental difference between the Euclidean and Chebyshev Ramsey theories.

First, the estimate $\chi(\mathbb{R}^n_\infty, \mathcal{Y}) \leqslant n + 1$ is the best upper estimate for $\chi(\mathbb{R}^n_\infty, \mathcal{Y})$ that holds for all infinite metric spaces $\mathcal{Y}$. In particular, this estimate depends on the dimension n. Moreover, by contrast to the Euclidean case, one can prove its attainability not only for infinite spaces, but also for all sufficiently large spaces $\mathcal{Y}$.

Second, assertion (b) of Theorem 1 guarantees the existence of an infinite *Ramsey space* (in the Chebyshev metric), that is, a set $\mathcal{Y}$ such that $\chi(\mathbb{R}^n_\infty, \mathcal{Y}) \to \infty$ as $n \to \infty$. In fact, it suffices to put $\mathcal{Y} = \mathcal{B}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is an infinite decreasing geometric progression with ratio $1/32$ (or with any smaller ratio). Here the growth rate of $\chi(\mathbb{R}^n_\infty, \mathcal{Y})$ is at least logarithmic.

This result leaves open the following questions. How the quantity $\chi(\mathbb{R}^n_\infty, \mathcal{B}(\boldsymbol{\lambda}))$ behaves with respect to the dimension, where $\boldsymbol{\lambda}$ is an infinite decreasing geometric progression with ratio $1/32 < q < 1$? Does there exist an infinite metric space $\mathcal{Y}$ such that $\chi(\mathbb{R}^n_\infty, \mathcal{Y}) = n + 1$ for all natural numbers n?

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