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Finite-Element Approximation on Manifolds

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Abstract. A method of construction of the local approximations (in particurlar—generalization of finite-element ones, for example, plane finite-elements of Courant, Zlamal, Argyris etc.) in the case of functions defined on n-dimensional ($n \geq 1$) smooth manifold with boundary is proposed. A notion of nondegenerate simplicial subdivision of mentioned manifold is introduced, evaluations of approach and stability in Sobolev's spaces are discussed (last ones are optimal as to N-width of corresponding compact).

1. Introduction

The finite Courant's element, Zlamal's element, Argyris' element, Bell's one and a lot of other plane local approximations are well-known (see, for example, [1], [3]–[5]). The author discussed local approximations on smooth manifolds without boundary (see [2] and bibliography given there). The aim of presented work is construction of similar approximations for enough wide class of manifolds including manifolds with boundary.

Let $\Omega_j(x)$ are summable functions with compact support in n-dimensional space R^n where index j runs through some set J. We suppose the number of elements in $\{j | x \in \operatorname{supp} \Omega_j\}$ is finite and uniformly bounded as to $x \in R^n$. Let's assume the conditions

$$\sum_j x_j^\alpha \Omega_j(x) = x^\alpha, \quad |\alpha| \leq m,$$

are right; here $\{x_j\}$ is some net of knots in R^n, $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_i and m are nonnegative integer numbers, $x = (x^{(1)}, \ldots, x^{(n)})$, $x^{(i)} \in R^{(1)}$, $i = 1, \ldots, n$, $x^\alpha = (x^{(1)})^{\alpha_1} \ldots (x^{(n)})^{\alpha_n}$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

Linear combinations of functions Ω_j,

$$\widetilde{U}(x) = \sum_j c_j \Omega_j(x), \quad x \in R^n, \ c_j \in R^{(1)}, \tag{1.1}$$

are called the local (or finite-element) approximations (see [1],[3]–[5]). The same ones could be discussed in half-space $\overline{R}^n_+ \stackrel{\text{def}}{=} \{x | x \in R^n, x^{(n)} \geq 0\}$.

The approximations of Courant, Zlamal, Argyris etc. might be written in the form (1.1). The same constructions for functions defined on manifolds (for instance, on the surfaces) collide with great obstacles. In particular a surface is splitted in finite number of simple surfaces and its own approximation is disscussed on the each of them; however it is very difficult to stick the basic functions for adjoining simple surfaces with preservation of properties of approach, smoothness and stability.

We overcome these obstacles by usage of atlas with infinite (continual) quantity of maps (see [2]). For evaluation of the approach it is insufficient to use topological properties; therefore it is necessary to investigate certain metric characteristics of the discussed objects. Such way gives opportunity to get the approximations for enough wide class of manifolds with boundary.

2. Construction of the approximations

Let n-dimensional differentiated manifold $\mathcal{M}$ (maybe with boundary) is defined by atlas

$$A = \{(U_\zeta, \chi_\zeta)\}_{\zeta \in \mathcal{Z}},$$

where U_ζ is map support,

$$\chi_\zeta : U_\zeta \to \overline{R}^n_+,$$

$E_\zeta = \chi_\zeta(U_\zeta)$ is open set in the halfspace $\overline{R}^n_+$, $\cup \chi_\zeta U_\zeta = \mathcal{M}$, and mappings $\chi_\zeta \chi^{-1}_{\overline{\zeta}}$: $\chi_{\overline{\zeta}}(U_\zeta \cap U_{\overline{\zeta}}) \to \chi_\zeta(U_\zeta \cap U_{\overline{\zeta}})$ are diffeomorphisms for all ζ, $\overline{\zeta} \in \mathcal{Z}$.

In addition we suppose 1) $\mathcal{Z} = \mathcal{M}$, 2) $E_\zeta = K_\zeta \bigcap \overline{R}^n_+$, where K_ζ is open ball in R^n with center in point $\zeta' = \chi_\zeta(\zeta)$.

Example 1. It is discussed the sphere embedded in the space R^{n+1},

$$\mathcal{M} : (\xi, \xi) = r^2,$$

here symbol $(.,.)$ indicates the scalar product in R^{n+1}, $\xi = (\xi^{(1)}, \dots, \xi^{(n+1)})$ is vector of the space R^{n+1}.

Let L_ζ is tangent plane to sphere $\mathcal{M}$ in the point $\zeta \in \mathcal{M}$. Define mapping $\chi_\zeta : U_\zeta \to E_\zeta$ by relation $\chi_\zeta : \xi' = \xi - (\xi, \zeta)\zeta r^{-2}$, thus χ_ζ is orthogonal projection on the tangent plane L_ζ in which Cartesian coordinate system with origin in the point of tangency ζ is introduced. Here it is possible to assume (for sphere it is supposed further everywhere) $E_\zeta = \chi_\zeta(U_\zeta)$ is open ball with center in zero and with radius $\varepsilon > 0$, and ε is arbitrary fixed value in interval $(0, r)$.

Similar construction could be done for unrestricted n-dimensional smooth manifold $\mathcal{M}$ without boundary that was embedded in the space R^{n+1} locally smoothly and bijectively.

Example 2. Here we discuss the halfsphere embedded in the space R^{n+1},

$$\mathcal{M} : (\xi, \xi) = r^2, \quad \xi^{(n+1)} \geq 0.$$

The mapping $\chi_\zeta : U_\zeta \to E_\zeta$ on the tangent space L_ζ to halfsphere $\mathcal{M}$ in the point $\zeta \in \mathcal{M}$ is defined by formula

$$\chi_\zeta : \xi' = r^2\Big(\xi/(\zeta, \xi) - (\zeta - \zeta^{(n+1)} e_{n+1})/(r^2 - \zeta^{(n+1)})\Big),$$

where e_{n+1} is $(n+1)$-th ort of Cartesian coordinate system in R^{n+1}, and $\zeta = (\zeta^{(1)}, \dots, \zeta^{(n+1)})$.

Thus χ_ζ is central projection from zero of space R^{n+1} on the tangent plane L_ζ that have Cartesian coordinate system with origin in the point $\xi_{0,\zeta} = r^2(\zeta - \zeta^{(n+1)} e_{n+1})/(r^2 - \zeta^{(n+1)2})$. Here we assume that $E_\zeta = K_\zeta \bigcap R^n_+$, K_ζ is open ball with center in the point $\zeta' = \zeta - (\zeta - \zeta^{(n+1)} e_{n+1})/(1 - (\zeta^{(n+1)}/r)^2)$ and with radius $\varepsilon > 0$, and ε is arbitrary fixed (independent of ζ) number in interval $(0, r)$.

Such construction could be prepaired for arbitrary smooth manifold $\mathcal{M}$ with boundary that was embedded in the space R^{n+1} locally smoothly and bijectively.

According to (1.1) we define functions $\Omega_{j,\zeta}(\xi')$, $\xi' \in R^n_+$, with compact support in E_ζ for which the identity

$$\sum_j \xi_{j'}^{\alpha}\Omega_{j,\xi}(\xi') = \xi'^{\alpha}, \quad |\alpha| \leq m, \quad \xi' \in R^n,$$

is realized. The desired functions on the manifold $\mathcal{M}$ are defined by equalities

$$\omega_j(\zeta) = \Omega_{j,\zeta}(\zeta'), \quad \zeta' = \xi_\zeta(\zeta), \quad \zeta \in \mathcal{M}. \tag{2.1}$$

These functions are discussed as a basis of approximations given on $\mathcal{M}$.

The approximation is often constructed with usage of simplicial subdivision $\mathcal{T}$ of manifold $\mathcal{M}$. Naturally the subdivision is curvilinear but neither exact its construction nor approximate one is necessary. It's sufficient to take the set of vertices (0-skeleton) of the subdivision and to have collection of incidence tables. In the case of manifold with boundary the certain part of vertices would be on the boundary of the manifold, and adjacent to boundary simplexes (together with others) would be considered by the incidence tables.

In particular the simplexes of s-skeleton of simplicial subdivision could be enumerated for each $s = 0, 1, \ldots, n$. We discuss a incidence table T_v where the i-th row is listed numbers of vertices belonging to i-th simplex of n-skeleton. Let the set of vertices $\{\xi_j\}_{j=1,\ldots,N}$ in $\mathcal{M}$ is given. We are interested in a subset of the vertices with images $\xi'_j \stackrel{\text{def}}{=} \chi_\zeta(\xi_j), \xi_j \in U_\zeta$, which can be jointed by straight segments in E_ζ according to suitable fragment of mentioned incidence table T_v. The constructed straightline complex is added with all generated s-dimensional simplexes and is named by $\mathcal{T}'_\zeta$.

Let's suppose the point $\zeta' = \chi_\zeta(\zeta)$ is situated in one of the s-dimensional simplex T'^s_ζ of the complex $\mathcal{T}'_\zeta$.

This supposition means that discussed subdivision is enough fine.

Fixing mentioned simplex let's define s-dimensional curvilinear simplex T^s by formula

$$T^s = \{\zeta \mid \zeta \in \mathcal{M}, \ \zeta' = \chi_\zeta(\zeta), \ \zeta' \in T^{s\prime}_\zeta\}.$$

This formula is defined all simplexes of s-skeleton of the simlicial subdivision $\mathcal{T}$ for our manifold $\mathcal{M}$, $s = 1, 2, \ldots, n$. Thus simlicial subdivision $\mathcal{T}$ is defined completely. Thanks to it the analogs of Courant's, Zlamal's, Argyris' approximations, which were earlier discussed only on regions of Euclidean space, can be constructed on the manifold $\mathcal{M}$.

All functions $\Omega_{j,\zeta}(\xi')$ are often defined by so called pattern (see [3]) function Ω, depending on argument ξ' and on knots of net pattern,

$$\Omega_{j,\zeta}(\xi') = \Omega\Big(\xi', \{\xi'_{j'}(\zeta) - \xi'_j(\zeta)\}_{j' \in J(\zeta)}\Big),$$

where $J(\zeta) = \{j' | \xi_{j'} \in U_\zeta\}$. In this situation the function $\omega_j(\zeta)$ can be represented by next equality

$$\omega_j(\zeta) = \Omega\Big(\zeta', \{\xi'_{j'}(\zeta) - \xi'_j(\zeta)\}_{j' \in J(\zeta)}\Big).$$

The situation is the same one if there are several pattern functions.

Example 3. Let's construct interpolating basis of the local approximation (for $m = 1$) for simplicial subdivided sphere, $\mathcal{M} : (\xi, \xi) = r^2$, with usage of orthogonal projection onto tangent plane (see Example 1). We fix $j \in \{1, 2, \ldots, N\}$ and discuss the body $\mathcal{Z}_j$ of barycentric star of vertex ξ_j, $\mathcal{Z}_j = \cup_{T \in \mathcal{T}, \xi_j \in \overline{T}} \overline{T}$.

For $\zeta \notin \mathcal{Z}_j$ we define $\omega_j(\xi) \equiv 0$. Let $T \subset \mathcal{Z}_j$, and the knots ξ_j, $j = 1, 2, \ldots, n+1$, are vertices of T. Then for $\zeta \in T$ we find $\omega_j(\xi) = \Delta_j^T(\zeta)/\Delta_*^T(\zeta)$, where

$$\Delta_*^T(\zeta) = \begin{vmatrix} \xi_1^{(1)} - \xi_{n+1}^{(1)} & \ldots & \xi_n^{(1)} - \xi_{n+1}^{(1)} & \xi^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ \xi_1^{(n+1)} - \xi_{n+1}^{(n+1)} & \ldots & \xi_n^{(n+1)} - \xi_{n+1}^{(n+1)} & \xi^{(n+1)} \end{vmatrix},$$

$$\Delta_j^T(\zeta) = \begin{vmatrix} \xi_1^{(1)} & \ldots & \xi_{j-1}^{(1)} & \xi^{(1)} & \xi_{j+1}^{(1)} & \ldots & \xi_{n+1}^{(1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_1^{(n+1)} & \ldots & \xi_{j-1}^{(n+1)} & \xi^{(n+1)} & \xi_{j+1}^{(n+1)} & \ldots & \xi_{n+1}^{(n+1)} \end{vmatrix}.$$

In particular for $n = 1$ the manifold $\mathcal{M}$ is circumference, and defining $\xi_j = (r \cos \varphi, r \sin \varphi)$, $\xi_{j+N} = \xi_j$, for functions $\widetilde{\omega}_j(\varphi) \stackrel{\text{def}}{=} \omega_j(r \cos \varphi, r \sin \varphi)$, we have

$$\widetilde{\omega}_j(\varphi) = \begin{cases} \dfrac{\sin(\varphi_{j\pm1} - \varphi)}{2\sin((\varphi_{j\pm1} - \varphi_j)/2)\cos(\varphi - (\varphi_{j\pm1} + \varphi_j)/2)}, & \text{if } \varphi \in [\varphi_j, \varphi_{j\pm1}), \\ 0, & \text{if } \varphi \notin (\varphi_{j-1}, \varphi_{j+1}). \end{cases}$$

Example 4. The usage of the central projection (see Example 2) leads us to next formulas for the basis of local approximation ($m = 1$) on halfsphere mentioned above: for ζ, $\xi_j \in \overline{T}$ we have

$$\omega_j^0(\zeta) = r^{-2}(\xi_j, \zeta)\Delta_j^T(\zeta)/\Delta^T,$$

where

$$\Delta^T = \begin{vmatrix} \xi_1^{(1)} & \ldots & \xi_n^{(1)} \\ \ldots & \ldots & \ldots \\ \xi_1^{(n+1)} & \ldots & \xi_n^{(n+1)} \end{vmatrix},$$

and the determinant $\Delta_j^T(\zeta)$ was pointed in the Example 3. In the case of at least one of the points ζ, ξ_j isn't in simplex $\overline{T}$, the equality $\omega_j^0(\zeta) = 0$ holds.

3. Approach and stability

Further it is necessary to use the family of refining subdivisions of manifold under the certain conditions of nondegeneracy. Let's fix some quantity $\varepsilon \in (0, 0.1)$ and introduce a class $\mathcal{S}_A(\varepsilon)$ of simplicial subdivisions $\mathcal{T}$, such that angles α between edges (between one-dimensional simplexes) with common vertex in each straightline simplex $T_\zeta^{s'}$, $\zeta \in \mathcal{M}$, have property $\sin(\alpha) > \varepsilon$.

We discuss linear combination of the basic functions (2.1)

$$\widetilde{u}(\zeta) = \sum_{j=1}^{N} v_j \omega_j(\zeta). \tag{3.1}$$

Theorem 1. *Let the manifold $\mathcal{M}$ of class C^{m+1} has been embedded smoothly in the space R^{n+1}. Let's discuss a family of its simplicial subdivisions of class $\mathcal{S}_A(\varepsilon)$.*

Then for arbitrary function $u \in W_p^{m+1}(\mathcal{M})$ there are quantities v_j such that inequality

$$\|u - \widetilde{u}\|_{W_p^s(\mathcal{M})} \le K h^{m+1-s+n(1/p-1/q)} \|u\|_{W_q^{m+1}(\mathcal{M})}$$

is right; here $1 \leq q \leq p \leq +\infty$,

$$h = \max_{T \in \mathcal{T}} \max_{\xi_i, \xi_j \in \overline{T}} \|\xi_i - \xi_j\|_{R^{n+1}},$$

K is positive constant depending on ε, p, q, m, s, n, but independing on h and u.

If functions $\Omega_{j,\xi}(\xi')$ satisfy to interpolation relation $\Omega_{j,\xi}(\xi') = \delta_{j,j'}$, then for function $u(\zeta)$ from the class $C(\mathcal{M})$ expression

$$\tilde{u}(\zeta) = \sum_{j=1}^{N} u(\xi_j)\omega_j(\zeta)$$

interpolates the function u in knots ξ_j, $j = 1, 2, \dots, N$.

Theorem 2. *Under suppositions of Theorem 2 the inequality of stability*

$$\frac{C_1}{2} \sum_{T \in \mathcal{T}} \sum_{\xi_j \in \overline{T}} |v_j|^2 mT \leq \left\| \sum_{j=1}^{N} v_j \omega_j \right\|^2_{L_2(\mathcal{M})} \leq 2C_2 \sum_{T \in \mathcal{T}} \sum_{\xi_j \in \overline{T}} |v_j|^2 mT$$

holds. Here mT is mesure of simplex T, and constants C_1 and C_2 depend on ε, but they independ on fineness of discussed simplicial subdivision.

4. Realization

On manifold $\mathcal{M}$ let's define atlas $A = \{(U_\zeta, \chi_\zeta)\}$ and discuss such fine simplicial subdivision $\mathcal{T}$ of the manifold $\mathcal{M}$ that for arbitrary $T \in \mathcal{T}$ and $\zeta \in T$ relation

$$T \subset U_\zeta \tag{4.1}$$

is correct.

Then $T' \stackrel{\text{def}}{=} \chi_\zeta(T) \subset E_\zeta$, and as the function $\Omega_{j,\zeta}(\xi')$ is defined for $\xi' \in T'$, it is possible to find $\omega_j(\zeta)$ for $\zeta \in T$ by (2.1). Notice that we don't need in exact or approximate methods of searching of curvilinear simplexes T from $\mathcal{T}$.

For construction of the functions $\omega_j(\zeta)$ it is sufficient to have

a) representation of manifold with atlas $\{(U_\zeta, \chi_\zeta)\}_{\zeta \in \mathcal{M}}$,

b) incidence tables of simplicial subdivision of the manifold $\mathcal{M}$ satisfying conditions (4.1),

c) description of vertices of mentioned subdivision (for example, their coordinates in the case of /local/ embedding in $(m+1)$-dimensional space R^{m+1}),

d) formulas for functions $\Omega_{j,\zeta}(\xi')$, $\xi' \in T' \subset \mathcal{T}'_\zeta$ (or formula of pattern function Ω).

For example, let two-dimensional surface $\mathcal{M}$ is done in the form $G(x, y, z) = 0$, where coincident gradient field of function G isn't degenerate on $\mathcal{M}$. Main difficulty of realization is construction of simplicial subdivision (triangulation).

Let's suppose there is rough triangulation (division of the surface by a few large curviliniar triangles). For further constructions it needs only 1) table of incidence T_v and 2) coordinates of vertices V_c. The fineness of the subdivision can be fulfiled subdividing of each triangle by connecting of middle points of its sides.

For realization of the process in the case of arbitrary surface it is sufficient to find points which is near-by to mentioned middle points of curvilinear sides. Everybody attains it by inflating of spheres with centers in the middles of corresponding chords until they touch of surface $\mathcal{M}$ (points of contact are found approximatly). Mentioned points of contacts are searching approximations. Such approximations don't destroy disired topological structure of triangulation and don't have influence on the exactness of interesting approximation. Notice the representations of curvilinear elements of subdivision aren't used in explicit form, and for solution of question about belonging of a point either curvilinear triangle it is sufficient to use the map χ_ζ for definition of belonging its image to corresponding straightlinear triangle of subdivision $\mathcal{T}'_\zeta$.

The proposed algorithm (for $n = 2$ and $m = 1$) had been realized with computers for approximation of functions defined on the sphere, on the torus and on the sphere with two handles.

Notice also the same algorithm can be used in the case of $n > 2$.

References

1. Ciarlet, P. *The finite element method for elliptic problems.* Oxford, 1978.
2. Demjanovich, Yu. K. *Local approximation on manifold and minimal splines.* Sanct-Petersburg University Press, 1994 (in Russian).
3. Demjanovich, Yu. K. *On the approximations of local function spaces.* Vestnik Leningrad University **1** (1977), pp. 35–41 (in Russian).
4. Demjanovich, Yu. K. *Stability of net method for elliptic problems.* Dokl. AN USSR **165**(1) (1965), pp. 20–23 (in Russian).
5. Michlin, S. G. *Approximation auf dem Kubischen Gitter.* Berlin, 1976.