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On two broad classes of heavy-tailed distributions

Since the introduction of Class $\mathcal{M}$ and Class $\mathcal{M}^*$, they have played important roles in insurance to describe tail equivalence of ruin probability and tail behavior of the deficit at ruin. And in insurance and finance most of heavy-tailed distributions with finite and positive expectations belong to Class $\mathcal{M}$. So it is important to study tail behaviors of Class $\mathcal{M}$ and Class $\mathcal{M}^*$. In this paper, we obtain some results on essential tail behaviors of these two classes.

1. Introduction

Let X be a non-negative random variable (r.v.) with distribution function(d.f.) F. X or F is called heavy-tailed if its moment generating function does not exist, i.e.,

$$g(t) := \int_0^\infty e^{tx} dF(x) = \infty, \quad \forall\ t > 0. \tag{1.1}$$

Otherwise, X or F is light-tailed, i.e., there exists some $t_0 > 0$ such that

$$g(t) < \infty, \quad \forall\ 0 \le t < t_0. \tag{1.2}$$

Write $\mathcal{K}$ as the class of heavy-tailed distributions (see [3]) and denote $\overline{F}(x) = 1 - F(x)$.

Several important classes of heavy-tailed distributions are enumerated here for our using, one can see [3] for detail:

$F \in \mathcal{S}$ iff for any $n \in \mathcal{N}$ (or equivalently for $n = 2$),

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{n\overline{F}(x)} = 1,$$

where F^{n*} denotes the n-fold convolution of F with the corresponding tail $\overline{F^{n*}} = 1 - F^{n*}$.

$F \in \mathcal{L}$ iff for any $l \in \mathcal{R}$ (or equivalently for some $l \neq 0$),

$$\lim_{x\to\infty} \frac{\overline{F}(x+l)}{\overline{F}(x)} = 1.$$

$F \in \mathcal{D}$ iff for any $0 < l < 1$ (or equivalently for some $0 < l < 1$)

$$\limsup_{x\to\infty} \frac{\overline{F}(lx)}{\overline{F}(x)} < \infty.$$

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Since the definitions of Class $\mathcal{L}$ and Class $\mathcal{D}$ do not involve convolution and give sensible explanations, they are convenient in applications and widely used in Finance and Insurance. And the two classes include almost all the heavy-tailed distributions which are usually used in applied probability, for instance, the class of regularly varying distributions $\mathcal{R}_{-\alpha}$ ($\alpha > 0$), which plays important roles in classical limit theory, the class of extend regularly varying distributions ERV$(-\alpha, -\beta)$, Class $\mathcal{C}$, etc.

$F \in \mathcal{R}_{-\alpha}$ ($\alpha > 0$) iff

$$\overline{F}(x) \sim x^{-\alpha} l(x), \quad x \to \infty,$$

where $l(x)$ is a slowly varying function as $x \to \infty$; $F \in$ERV$(-\alpha, -\beta)$ ($0 < \alpha \leq \beta < \infty$) iff

$$y^{-\beta} \leq \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\alpha}, \quad \forall\, y > 1;$$

and $F \in \mathcal{C}$ iff

$$\lim_{l\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(lx)}{\overline{F}(x)} = 1.$$

Class $\mathcal{C}$ has been used in different studies of applied probability such as queueing system and ruin theory; see, for example, [12], [5], [10], and references therein.

By the above definitions, one can easily get (see, for example, [3])

$$\mathcal{R}_{-\alpha} \subset ERV(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D} \subset \mathcal{K}; \quad \mathcal{S} \subset \mathcal{L} \subset \mathcal{K};$$
$$\mathcal{S} \not\subset \mathcal{D}; \quad \mathcal{D} \not\subset \mathcal{S}; \quad \mathcal{D} \cap \mathcal{L} \neq \Phi; \quad \mathcal{D} \cap \mathcal{L} \subset \mathcal{S}.$$

In the literature, as we know, most of the related problems in insurance and finance are considered in Class $\mathcal{D}$ or Class $\mathcal{L}$.

2. The Class $\mathcal{M}$ and Class $\mathcal{M}^*$

In view that a risk random variable is usually non-negative and has a finite expectation

$$0 < \mu := EX = \int_0^\infty \overline{F}(x)dx < \infty, \tag{2.1}$$

in this paper we only discuss distributions with support on $[0, \infty)$ and satisfying condition (2.1). Denote by $\mathcal{E}_1$ the class of these distributions.

For $F \in \mathcal{E}_1$, write

$$V(x) = \int_0^x \overline{F}(u)du, \quad \overline{V}(x) = \int_x^\infty \overline{F}(u)du, \quad F_e(x) = \frac{1}{\mu} V(x), \quad r_e(x) = \frac{\overline{F}(x)}{\overline{V}(x)} := \frac{\epsilon(x)}{x}. \tag{2.2}$$

It is easy to see that $F_e(x)$ is also a distribution on $[0, \infty)$, which is called the integrated tail distribution of F, or its equilibrium distribution. In applied probability, it can be used to describe ruin probabilities of renewal risk model, tail distributions of ladder heights of a random walk, and limiting distributions of waiting time and busy period of queueing model, and so on. See [1], [3], [11] for details.

Let $F \in \mathcal{E}_1$, then for any $x > x_0 > 0$,

$$-\int_{x_0}^x \frac{\epsilon(t)}{t} dt = -\int_{x_0}^x \frac{\overline{F}(t)}{\overline{V}(t)} dt = \int_{x_0}^x \frac{d\overline{V}(t)}{\overline{V}(t)} = \log \frac{\overline{V}(x)}{\overline{V}(x_0)} = \log \frac{\overline{F}_e(x)}{\overline{F}_e(x_0)},$$

which is equivalent to

$$\overline{F}_e(x) = \overline{F}_e(x_0) \exp\left\{-\int_{x_0}^{x} \frac{\epsilon(t)}{t} dt\right\}. \tag{2.3}$$

Since the Class $(\mathcal{L} \cup \mathcal{D}) \cap \mathcal{E}_1$ does not contain all important heavy-tailed distribution in Insurance and Finance (please see Example 1 below), we introduced the following two classes of heavy-tailed distributions in [14] and [15]:

$F \in \mathcal{M}$ iff $F \in \mathcal{E}_1$ and

$$\lim_{x\to\infty} r_e(x) = \lim_{x\to\infty} \frac{\overline{F}(x)}{\overline{V}(x)} = 0; \tag{2.4}$$

$F \in \mathcal{M}^*$ iff $F \in \mathcal{E}_1$ and

$$\beta_F := \limsup_{x\to\infty} \epsilon(x) = \limsup_{x\to\infty} \frac{x\overline{F}(x)}{\overline{V}(x)} < \infty.$$

Clearly $\mathcal{M}^* \subset \mathcal{M}$. It has been proved in [14] and [15], that

$$\mathcal{D} \cap \mathcal{E}_1 \subset \mathcal{M}^*; \quad (\mathcal{L} \cup \mathcal{D}) \cap \mathcal{E}_1 \subset \mathcal{M}.$$

Moreover $F \in \mathcal{M} \Longleftrightarrow F_e \in \mathcal{L},\ F \in \mathcal{M}^* \Longleftrightarrow F_e \in \mathcal{C} \Longleftrightarrow F_e \in \mathcal{D}$ and

$$\mathcal{M}^* \subset \mathcal{M} \subset \mathcal{K}; \quad \mathcal{M}^* \setminus \mathcal{D} \neq \Phi; \quad \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{D}) \neq \Phi.$$

Therewithal, Class $\mathcal{M}$ not only extends $(\mathcal{L} \cup \mathcal{D}) \cap \mathcal{E}_1$, but also gives us a chance to deeply investigate common properties of Class $\mathcal{D}$ and Class $\mathcal{L}$. The following interesting example shows the necessity of introducing Classes $\mathcal{M}$:

Example 1. *Assume that the random variable X has a geometry distribution with parameter $p \in (0,1)$. Let $F(x) = P(X^r \le x)$ for some $r > 1$, then it is easy to show that $F \in \mathcal{M}$ but $F \notin \mathcal{D} \cup \mathcal{L}$.*

For our further discussion and use, a common method for constructing distribution $F \in \mathcal{M}$ is given below, where and throughout this paper, $h(x) \downarrow$ means that $h(x)$ is non-increasing.

Proposition 2.1. *Assume that $h : [0,\infty) \to (0,\infty)$ satisfies*

$$h(0) = 1; \quad 0 < h(x) \downarrow 0 \ (0 < x \to \infty), \quad \int_0^\infty h(u)du = \infty.$$

Denote

$$\overline{F}(x) = 1, \quad x \le 0; \qquad \overline{F}(x) = h(x) \exp\left\{-\int_0^x h(u)du\right\}, \quad x > 0.$$

Then $F(x) = 1 - \overline{F}(x)$ is a distribution function on $[0,\infty)$, $F \in \mathcal{M}$ and

$$\overline{V}(x) = \int_x^\infty \overline{F}(u)du = \exp\left\{-\int_0^x h(u)du\right\}, \quad x \ge 0. \tag{2.5}$$

Furthermore, we have

$$\overline{F}_e(x) = \overline{V}(x).$$

The conclusion is clear. In fact, $\overline{F}(x)$ is non-increasing, $\overline{F}(0) = 1$ and $\overline{F}(\infty) = 0$. Thus $F(x) = 1 - \overline{F}(x)$ is a distribution supported on $[0, \infty)$. Since $h(x)$ is monotone, $h(x)$ is continuous almost sure with respect to Lebesgue measure. Hence

$$\frac{d}{dx} \exp\left\{-\int_0^x h(u)du\right\} = -h(x)\exp\left\{-\int_0^x h(u)du\right\} = -\overline{F}(x) = \frac{d}{dx}\overline{V}(x), \quad a.s.$$

which yields (2.5) and

$$r_e(x) = \frac{\overline{F}(x)}{\overline{V}(x)} = h(x) \to 0, \quad x \to \infty,$$

i.e. $F \in \mathcal{M}$. Furthermore, by (2.5),

$$\int_0^\infty u dF(u) = \int_0^\infty \overline{F}(u)du = \overline{V}(0) = 1,$$

which means that $\overline{F}_e(x) = \overline{V}(x)$. #.

3. Some tail behaviors of heavy-tailed distributions

3.1. General discussion

It is well-known that for $F \in \mathcal{L}$ or $F \in \mathcal{D}$, we have (see, for example, [3])

$$\lim_{x\to\infty} e^{tx}\overline{F}(x) = \infty, \quad \forall\, t > 0. \tag{3.1}$$

This fact made some people believe that (3.1) is true for any distribution $F \in \mathcal{K}$ (see, for instance, p47 in [6] and section 2 of [4]). But this is wrong and we can find the following counterexample:

Example 1. *Let* $a_0 = 0, a_1 = 1, a_{n+1} > a_n$ *and* $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$. *Define*

$$P(X = a_n) = \exp\{-a_{n-1}\} - \exp\{-a_n\}, \quad n \in \mathcal{N}.$$

Clearly, we have $\lim\limits_{n\to\infty} a_n = \infty$ and

$$\overline{F}(x) = P(X > x) = P(X \geq a_n) = \exp\{-a_{n-1}\}, \quad a_{n-1} \leq x < a_n.$$

For any $t > 0,\ \delta > 0$, when n is large, $ta_n > 2a_{n-1},\ a_{n-1} < a_n - \delta < a_n$. Denoting $x_n = a_n - \delta$, we have

$$\limsup_{x\to\infty} e^{tx}\overline{F}(x) \geq \lim_{n\to\infty} e^{tx_n}\overline{F}(x_n) = \lim_{n\to\infty} e^{tx_n - a_{n-1}} \geq e^{-t\delta} \lim_{n\to\infty} e^{a_{n-1}} = \infty\ ,$$

Thus for any $t > 0,\ \int_0^\infty e^{tx}dF(x) = \infty$, which means that $F \in \mathcal{K}$. But it is easy to see that for any $0 < t < 1$, we have

$$\liminf_{x\to\infty} e^{tx}\overline{F}(x) \leq \lim_{n\to\infty} e^{ta_n}\overline{F}(a_n) = \lim_{n\to\infty} e^{(t-1)a_n} = 0\ ,$$

i.e. (3.1) fails. #

The example motivates us try to find some criterions under which (3.1) hold. At first, we give a proposition:

Proposition 3.1. *Assume that*

$$d_F(y) = \limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty, \quad \forall\, y > 0, \tag{3.2}$$

then for any

$$t > \nu_F := \inf_{y>0} \frac{\log d_F(y)}{y}, \tag{3.3}$$

we have

$$\lim_{x\to\infty} e^{tx}\overline{F}(x) = \infty.$$

P r o o f. For $t > \nu_F$, there is $y_0 > 0$ such that $t > \frac{\log d_F(y_0)}{y_0}$. Since $d_F(y_0) < \infty$, for any fixed $\varepsilon > 0$, there exists $0 < x_0 = x_0(\varepsilon) < \infty$ such that

$$\sup_{x\ge x_0} \frac{\overline{F}(x)}{\overline{F}(x+y_0)} < d_F(y_0) + \varepsilon.$$

Especially, for any $n \in \mathcal{N}$,

$$\frac{\overline{F}(x_0)}{\overline{F}(x_0+ny_0)} < (d_F(y_0)+\varepsilon)^n, \quad R(x_0+ny_0) \le n\log(d_F(y_0)+\varepsilon) + R(x_0),$$

where $R(x) := -\log \overline{F}(x)$. Since $R(x)$ is non-decreasing, for any $x \in [x_0+ny_0, x_0+(n+1)y_0)$, we get

$$\frac{R(x)}{x} \le \frac{(n+1)\ln(d_F(y_0)+\varepsilon) + R(x_0)}{x_0+ny_0}, \tag{3.4}$$

Hence $\limsup_{x\to\infty} \frac{R(x)}{x} \le \frac{1}{y_0}\ln(d_F(y_0)+\varepsilon)$. By the randomicity of ε, we get $\mu_F := \limsup_{x\to\infty} \frac{R(x)}{x} \le \frac{1}{y_0}\ln d_F(y_0)$. Thus $t > \frac{1}{y_0}\ln d_F(y_0) \ge \mu_F$, i.e. there exists $\delta > 0$ such that $\mu_F < t - 2\delta$. So for large x, $tx - R(x) \ge \delta x$, which shows

$$\lim_{x\to\infty} e^{tx}\overline{F}(x) = \lim_{x\to\infty} \exp\{tx - R(x)\} \ge \lim_{x\to\infty} \exp\{\delta x\} = \infty. \quad \#$$

Clearly, $\nu_F = 0$ for $F \in \mathcal{L}$. Thus

Corollary 1. (3.1) *holds for* $F \in \mathcal{L}$.

3.2. The case of Class $\mathcal{M}$

Noting that in Example 1, we have $\lim_{n\to\infty} \overline{F}(a_n)/\overline{V}(a_n) = 1$, which means $F \notin \mathcal{M}$. A problem will be put forward that "whether (3.1) holds for all $F \in \mathcal{M}$?"At first, we introduce two symbols which are useful in next discussion:

$$\Gamma_F := \left\{\gamma \,\middle|\, \int_0^\infty x^\gamma dF(x) < \infty \right\}, \quad \gamma_F := \sup\{\gamma \mid \gamma \in \Gamma_F\}. \tag{3.5}$$

From the definitions in section 2, we have that $\gamma_F \ge 1$ holds for $F \in \mathcal{M}$, and $1 \le \gamma_F < \infty$ for $F \in \mathcal{M}^*$.

We discover an interesting property for $F \in \mathcal{M}$:

Theorem 1. *For* $F \in \mathcal{M}$ *with* $\gamma_F > 1$, (3.1) *holds; but for* F *with* $\gamma_F = 1$, (3.1) *doesn't always hold.*

A counterexample for $F\in\mathcal{M}$ with $\gamma_F=1$ (even an example $F\in\mathcal{M}^*$) is as follows:

Example 2. *The example is constructed by the method in Proposition 2.1. Let*

$$a_1=1,\quad a_{n+1}=a_n+\frac{1}{n}e^{a_n},\quad n\in\mathcal{N},$$

then $a_n>0$ is strictly increasing and $a_n\to\infty$. Assume

$$h(x)=1,\quad 0\le x<a_1;\quad h(x)=e^{-a_n},\quad a_n\le x<a_{n+1},\quad n\in\mathcal{N}.$$

Clearly $\int_0^\infty h(u)du=1+\sum\limits_{n=1}^{\infty}\frac{1}{n}=\infty$ and $h(x)$ satisfies the conditions in Proposition 2.1. If we define non-negative distribution F as that in Proposition 2.1, then $F\in\mathcal{M}$ and $\frac{\overline{F}(x)}{\overline{V}(x)}=r_e(x)=h(x)$. For $0<t<1$, we have

$$\liminf_{x\to\infty}e^{tx}\overline{F}(x)\le\liminf_{x\to\infty}e^{tx}h(x)\le\lim_{n\to\infty}e^{ta_n}h(a_n)=\lim_{n\to\infty}e^{(t-1)a_n}=0,$$

which shows that (3.1) doesn't hold. On the other hand,

$$\limsup_{x\to\infty}\frac{x\overline{F}(x)}{\overline{V}(x)}=\limsup_{x\to\infty}xh(x)\le\lim_{n\to\infty}a_{n+1}h(a_n)=\lim_{n\to\infty}(a_ne^{-a_n}+\frac{1}{n})=0,$$

which means that $F\in\mathcal{M}^*$. By a conclusion in [13], $\gamma_F=1$. #

This example shows that the condition "$\gamma_F>1$"in Theorem 1 can't be cancelled.

Remark. Theorem 1 shows that distributions F with $\gamma_F=1$ behave strangely. This has never been noticed before.

In order to prove the conclusion in Theorem 1 concerning $F\in\mathcal{M}$ with $\gamma_F>1$, we first give the following lemma.

Lemma 3.1. *If $\gamma_F>1$, then for any $0<p<1-\frac{1}{\gamma_F}$, we have*

$$\lim_{x\to\infty}\frac{\overline{F}^p(x)}{\overline{V}(x)}=\lim_{x\to\infty}\frac{\overline{F}^p(x)}{\int_x^\infty\overline{F}(u)du}=\infty.\tag{3.6}$$

P r o o f. For any $0<p<1-\frac{1}{\gamma_F}$, there exists some $r<\gamma_F$ such that $r(1-p)>1$. Choose $\delta=1-p$ and $\rho:=2^{1-r\delta}<1$. It is well-known that for distribution F supported on $[0,\infty)$, its rth moment exists if and only if $\sum\limits_{k=1}^{\infty}2^{kr}\overline{F}(2^k)<\infty$. Then for large k, we have $2^{kr}\overline{F}(2^k)<1$, which shows that $2\overline{F}^{\frac{\delta}{k}}(2^k)<2^{1-r\delta}=\rho$. Hence for large n and $2^{n-1}\le x<2^n$, we have

$$\begin{aligned}\frac{\overline{V}(x)}{\overline{F}^p(x)}&=\frac{\int_x^\infty\overline{F}(u)du}{\overline{F}^{1-\delta}(x)}\le\int_x^\infty\overline{F}^\delta(u)du\le\sum_{k=n-1}^{\infty}\int_{2^k}^{2^{k+1}}\overline{F}^\delta(u)du\\&\le\sum_{k=n-1}^{\infty}2^k\overline{F}^\delta(2^k)=\sum_{k=n-1}^{\infty}\left\{2\overline{F}^{\frac{\delta}{k}}(2^k)\right\}^k<\sum_{k=n-1}^{\infty}\rho^k=\frac{\rho^n}{\rho(1-\rho)}\ .\end{aligned}$$

Let $x\to\infty$(so $n\to\infty$), then (3.6) holds. #

Remark. Combining the discussion in [13], we obtain

$$\text{(3.6) holds for any } 0<p<1\Longleftrightarrow\gamma_F=\infty.$$

P r o o f of Theorem 1(the case $\gamma_F > 1$): Let $\gamma_F > 1$ and $0 < p < 1 - \frac{1}{\gamma_F}$, then (3.6) holds by Lemma 3.1. For any $t > 0$, write $u = \frac{p\,t}{1-p}$ and

$$\frac{e^{tx}\overline{F}(x)}{\overline{V}(x)} = \left\{\frac{\overline{F}^p(x)}{\overline{V}(x)}\right\}^{1/p} \cdot e^{tx}\overline{V}^{\frac{1-p}{p}}(x) = \left\{\frac{\overline{F}^p(x)}{\overline{V}(x)}\right\}^{1/p} \left\{e^{ux}\overline{V}(x)\right\}^{\frac{1-p}{p}}. \tag{3.7}$$

By the fact that $F \in \mathcal{M} \Leftrightarrow F_e \in \mathcal{L}$ (see [14] or [15]), and noticing the definition of $\overline{V}(x)$, it follows from the property of Class $\mathcal{L}$ that $\lim_{x\to\infty} e^{ux}\overline{V}(x) = \infty$, which together with (3.6) and (3.7) yields

$$\lim_{x\to\infty} \frac{e^{tx}\overline{F}(x)}{\overline{V}(x)} = \infty, \quad \forall\, t > 0. \tag{3.8}$$

Writing $0 < t = 2s$, by (3.8), we have $e^{sx}\overline{F}(x) \geq \overline{V}(x)$ for large x. By the property of Class $\mathcal{L}$ again,

$$\lim_{x\to\infty} e^{tx}\overline{F}(x) = \lim_{x\to\infty} e^{2sx}\overline{F}(x) \geq \lim_{x\to\infty} e^{sx}\overline{V}(x) = \infty, \quad \forall\, t > 0.$$

Thus (3.1) holds. #

4. The tail behaviors of Class $\mathcal{M}^*$

Except the misunderstanding on tail behaviors of heavy-tailed distributions, which is mentioned in the last section, another one is the problem that for $F \in \mathcal{D}$, whether there exists some $t > 0$ such that

$$\lim_{x\to\infty} x^t\overline{F}(x) = \infty? \tag{4.1}$$

For instance, In order to obtained results on rates of approximations to ruin probabilities, Mikosch and Nagaev [7] assume that

"$F \in \mathcal{D}$ has non-zero and finite expectation μ, and there exists some $\alpha > 0$ such that

$$\liminf_{x\to\infty} x^\alpha\overline{F}(x) > 0." \tag{4.2}$$

In fact, according to the following Proposition 4.1, their assumption is unnecessary since all distributions in Class $\mathcal{D}$ satisfy the condition showed in (4.1) for some $t > 0$.

Proposition 4.1. *For any $F \in \mathcal{D}$, there exists some $t > 0$ such that* (4.1) *holds.*

The detailed proof of Proposition 4.1 can be found in [13].

As for Class $\mathcal{M}^*$, we also have the following theorem.

Theorem 1. *For $F \in \mathcal{M}^*$ with $\gamma_F > 1$, we have*

$$\lim_{x\to\infty} x^\gamma\overline{F}(x) = \infty\ , \quad \forall\ \gamma > \frac{\gamma_F\beta_F}{\gamma_F - 1}, \tag{4.3}$$

where γ_F is defined in (3.5).

Remark. It is similar to Theorem 1 that the conclusion doesn't always hold for $F \in \mathcal{M}^*$ with $\gamma_F = 1$. We shall give an example later, which is another illustration of the abnormality of non-negative distribution with $\gamma_F = 1$.

P r o o f of Theorem 1: For $0 < p < 1 - \frac{1}{\gamma_F}$, (3.6) holds by Lemma 3.1. Choose $u > \beta_F$ and write $t = \frac{(1-p)u}{p}$, then

$$\frac{x^t\overline{F}(x)}{\overline{V}(x)} = \left\{\frac{\overline{F}^p(x)}{\overline{V}(x)}\right\}^{1/p} \cdot x^t\overline{V}^{\frac{1-p}{p}}(x) = \left\{\frac{\overline{F}^p(x)}{\overline{V}(x)}\right\}^{1/p} \left\{x^u\overline{V}(x)\right\}^{\frac{1-p}{p}}.$$

Noticing $F_e = \frac{1}{\overline{V}(0)}V \in \mathcal{D}$ and using (3.6) and Proposition 4.1, we get

$$\lim_{x\to\infty} \frac{x^t\overline{F}(x)}{\overline{V}(x)} = \infty, \tag{4.4}$$

which shows that $x^t\overline{F}(x) > \overline{V}(x)$ holds for large x. Thus if $\gamma > \frac{\beta_F}{p}$, denoting $\gamma = \frac{(1-p)u}{p} + u = t + u$, then we have $u > \beta_F$ and

$$\lim_{x\to\infty} x^\gamma\overline{F}(x) = \lim_{x\to\infty} x^{t+u}\overline{F}(x) \geq \lim_{x\to\infty} x^u\overline{V}(x) = \infty .$$

To summarize, (4.3) holds for $\gamma = t + u > \frac{\gamma_F\beta_F}{\gamma_F - 1}$. #

Remark. By [13], we have $\gamma_F \leq \beta_F + 1$. When $\gamma_F = \beta_F + 1$, Theorem 1 holds for all $\gamma > \gamma_F$, which shows that the bound of γ is precise. And it is easy to get $\frac{\gamma_F\beta_F}{\gamma_F-1} \geq \beta_F + 1$.

As follows, we give a counterexample to explain that the conclusion of Theorem 1 doesn't always hold for "$F \in \mathcal{M}^*$ with $\gamma_F = 1$."

Example 1. *Let us use the method in Proposition 2.1 to construct counterexample. Let* $a_1 = 2$, $a_{n+1} = a_n + \frac{1}{n}a_n^n$, $n \geq 1$ *and define*

$$h(x) = 1, \quad x \in [0, a_1); \quad h(x) = a_n^{-n}, \quad x \in [a_n, a_{n+1}), \quad n \geq 1.$$

Clearly $0 < h(x) \downarrow 0$, $\int_{a_n}^{a_{n+1}} h(x)dx = \frac{1}{n}$, which yields $\int_0^\infty h(x)dx = \infty$. Thus $h(x)$ satisfies the conditions in Proposition 2.1. If we define $F \in \mathcal{M}$ by the method in Proposition 2.1, then F has expectation and $\frac{\overline{F}(x)}{\overline{V}(x)} = r_e(x) = h(x)$. Since $\limsup_{x\to\infty} xh(x) \leq \lim_{n\to\infty} a_{n+1}h(a_n) = 0$, i.e.

$$\beta_F = \limsup_{x\to\infty} \frac{x\overline{F}(x)}{\overline{V}(x)} = \limsup_{x\to\infty} xr_e(x) = 0,$$

we obtain $F \in \mathcal{M}^*$. And from [13], we get $\gamma_F = 1$. But for any $\gamma > 0$,

$$\liminf_{x\to\infty} x^\gamma\overline{F}(x) \leq \liminf_{x\to\infty} x^\gamma h(x) \leq \lim_{n\to\infty} a_n^r h(a_n) = \lim_{n\to\infty} a_n^{r-n} = 0 . \quad \#$$

From this example, we have

Proposition 4.2. *There exists* $F \in \mathcal{M}^*$ *with* $\gamma_F = 1$ *such that* $\lim_{x\to\infty} x^\gamma\overline{F}(x) = \infty$ *doesn't hold for any* $\gamma > 0$.

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ABSTRACT

Since the introduction of Class $\mathcal{M}$ and Class $\mathcal{M}^*$, they have played important roles in insurance to describe tail equivalence of ruin probability and tail behavior of the deficit at ruin. And in insurance and finance most of heavy-tailed distributions with finite and positive expectations belong to Class $\mathcal{M}$. So it is important to study tail behaviors of Class $\mathcal{M}$ and Class $\mathcal{M}^*$. In this paper, we obtain some results on essential tail behaviors of these two classes.

Key words: *class $\mathcal{M}$, class $\mathcal{M}^*$, class $\mathcal{D}$, tail behaviors*