Russian Math. Surveys 29: 5 (1974), 195–208
*From* Uspekhi Mat. Nauk 29: 5 (1974), 197–210

# NIKOLAI NIKOLAEVICH LUZIN

(on the ninetieth anniversary of his birth)

Nikolai Nikolaevich Luzin was born on December 9 (November 27) 1883 in Irkutsk.*

He first went to a private school and then to the Tomsk provincial grammar school (Gymnasium). In the upper classes of the Gymnasium Luzin read a great deal. He was particularly fascinated by books on pure philosophy. In mathematics he was not particularly interested until the very last years of study. This was because the system of instruction in force at that time was based on mechanical memory: it was required to learn the theorems by heart and to reproduce the proofs exactly. For Luzin this was torture. His progress in mathematics at the Gymnasium became worse and worse, so that his father was obliged to engage a tutor for him. Fortunately, the tutor turned out to be a very talented student of the recently opened Tomsk Polytechnic Institute. This student made a powerful impression on Luzin by showing him mathematics not as a system of rote learning, but as a system of reasoning guided by lively imagination.

On leaving the Gymnasium in 1901, Luzin went to the mathematics branch of the Faculty of Physics and Mathematics at the University of Moscow. This choice was motivated by Luzin's wish, at the time, to become an engineer, for which a solid mathematical foundation was necessary. (This is stated in Luzin's autobiography.)

In 1906 Luzin left the University of Moscow. In 1910 he took the master's examination and, having given two trial lectures – one of his own choice and one set by the Faculty – he obtained the post of Privatdozent (assistant lecturer) of the University of Moscow in the Department of Pure Mathematics. In the same year the university sent him to Göttingen

1) In the final editing of this article I read the valuable advice and comments on the manuscript made by P. S. Aleksandrov, B. V. Aleshin, I. Ya. Verchenko, B. V. Gnedenko, V. K. Gol'tsman, F. A. Kabakov, L. V. Keldysh, A. F. Lapko, V. S. Lyukshin, L. A. Lyusternik, A. A. Lyapunov, D. E. Men'shov, N. M. Nagornyi, P. S. Novikov, S. I. Novoselov, E. D. Solomentsev, B. S. Sotskov, L. N. Sretenskii, P. L. Ul'yanov and V. S. Fedorov. To all these I wish to express my deep gratitude.

* The Russian text gives the birthplace erroneously as Tomsk; see p. 179. Ed.

and Paris to perfect his mathematical education. In Göttingen Luzin wrote his first paper "Über eine Potenzreihe" and, at the insistence of Professor Landau, published it in 1911, at the age of 28.

Having returned to Moscow in 1914, Luzin proceeded to give basic and optional courses of lectures at the University of Moscow. Among those attending his optional courses a group rapidly sprung up, united by their interest in problems in the theory of functions – the basis of the future Moscow mathematics school. As well as giving lectures, Luzin worked on his dissertation "The integral and trigonometric series". It was printed in Moscow in 1915 [1] and was presented for the degree of Master of Pure Mathematics. The oral examination took place in the spring of 1916 and Luzin was awarded the degree of Doctor of Pure Mathematics [2]. The immediate award of the academic degree of Doctor had not been made during the preceding 60 years.

In 1917, the Board of the Faculty and the Council of the University appointed Luzin Professor of the University of Moscow in the Department of Pure Mathematics, in 1927 he was elected Corresponding Member of the USSR Academy of Sciences, and in 1929 a Full Member, first in the Department of Philosophy [3] and then in the Department of Pure Mathematics. From his election as full member to his death (February 28, 1950) Luzin worked in the Institutes of the USSR Academy of Sciences [4] – [7].

Luzin's main scientific activity was in the theory of functions. Following the work of Cantor, the development of the theory of functions and sets split into two branches – metric and descriptive. The metric branch investigates those properties of sets and functions that are connected with the idea of the measure of a set and an integral, while in the descriptive part it is only a question of the structural properties of these objects, the idea of the measure of a set being excluded.

In the metric theory of functions a detailed study is made of the fundamental ideas of mathematical analysis, such as limit, function, derivative, integral, and series.

Luzin's first papers were on the metric theory of functions [8]. They were devoted to an investigation of the most general case of convergence of trigonometric and power series and the properties of functions representable by them. He constructed an example of a power series with coefficients tending to zero that diverges everywhere on the circle of convergence, and an example of a trigonometric series with coefficients tending to zero that diverges almost everywhere. Luzin obtained a very important result on the structure of measurable functions, establishing one of the foundations on which the whole metric theory of functions is based. He proved that every measurable function becomes continuous if it is changed suitably on a set of arbitrarily small measure. This property of functions became known as Luzin's $C$-property.

In addition, Luzin proved that for every finite measurable function $f(x)$ there is a continuous function $F(x)$ such that $F'(x) = f(x)$ almost everywhere, that is, with the exclusion of a set of measure zero.

Luzin showed that these results have an immediate application in one of the most important problems of mathematical analysis – the Dirichlet problem. In the generalized form due to Luzin, this problem can be stated as follows: for a measurable function $f(\theta)$ defined on the unit circle it is required to find a function $u(r, \theta)$, harmonic inside the unit disc, such that the values of $u(r, \theta)$ along all paths that approach the boundary without touching it tend to the values $f(\theta)$ almost everywhere on the unit circle ([1], 85). In his dissertation Luzin solved this problem for the case when the given measurable function $f(\theta)$ is finite almost everywhere. It is easy to see that the condition on $f(\theta)$ to be measurable is necessary in the statement above, but Luzin at once conjectured that the condition on $f(\theta)$ to be finite almost everywhere is also necessary. Somewhat later, in 1925, Luzin, together with Privalov, completely proved the validity of this conjecture. However, if the condition on the approach to the boundary is weakened, so that the approach is made only along radii, then, as was shown in the same paper of 1925, the condition on $f(\theta)$ to be finite almost everywhere is no longer necessary. Even now the problem is not completely solved.

The generalized Dirichlet problem, in Luzin's formulation, proved very fruitful. Numerous investigators have studied this problem for domains of general form, for domains in space, for various ways of approaching the boundary, questions of uniqueness in this problem, and other problems. This work is closely connected with the study of boundary properties of analytic functions.

Luzin also obtained important results in the study of absolute convergence of trigonometric series. He showed that: 1) if a trigonometric series converges absolutely on a set of positive measure or on a set of the second category, then it converges absolutely everywhere; 2) if a trigonometric series converges absolutely at two points whose distance apart is incommensurable with $\pi$, then it converges absolutely on an everywhere dense set and, moreover, it converges almost everywhere or diverges almost everywhere.

Luzin's main results in the metric theory of functions are contained in his dissertation "The integral and trigonometric series".

The influence of Luzin's dissertation on the future development of the theory of functions cannot be overestimated. Its fundamental results, deep methods of investigation and fundamental statements of problems put it into the ranks of works with which it is difficult to compare any dissertation or monograph of the time. In Luzin's dissertation problems were formulated whose solutions have been the subject of numerous investigations, both in the USSR and abroad. We note, for example, that the problem posed in

Luzin's dissertation on the convergence almost everywhere of a trigonometric Fourier series of a function with integrable square did not yield a solution for 50 years, in spite of the efforts of the greatest mathematicians. Only in 1965 did the Swedish mathematician Carleson give a positive answer to this problem of Luzin.

Recently Luzin's dissertation has been reprinted [1]. The comments on it contain an account of the later development of the questions presented in it.

The dissertation particularly emphasizes the historical connection between the theory of trigonometric series and the development of the ideas of function and integral. The fundamental meaning of this work is best characterized by the words of the author himself: "Have the results of the theory of functions an important significance for other disciplines, and above all for classical analysis? We must bear in mind that in the present state of knowledge the method of classical analysis, the method of using analytic expressions, forms the basis of almost every mathematical discipline, and so any theory that has no direct or indirect contact with analytic expressions inevitably occupies an isolated position among the other branches of mathematics. Therefore, if we do not want the theory of functions of a real variable to be a closed theory, having no influence on other mathematical theories, we must connect analytic expressions on the one hand with the definitions and ideas of the theory of functions on the other hand" ([1], 50).

The ideas developed by Luzin in his dissertation had an immense influence on the future development of the metric theory of functions. Some of these ideas were applied by pupils of Luzin in various related fields of mathematics, physics, and engineering. For example, Luzin proved a theorem that gives an effective method of finding, from the values $u(\theta)$ of a harmonic function $u(x, y)$ on a circle of unit radius, the values $v(\theta)$ of the conjugate harmonic function $v(x, y)$. Using this theorem, Nekrasov succeeded in proving the well-known theorem on the existence of gravitational waves on the surface of an incompressible fluid.

Luzin's papers on the theory of functions of a complex variable are almost inseparable from his papers on the metric theory of functions of a real variable. Luzin writes: "My problem, as a person who devotes much time to functions of a real variable, consists in formulating complex variable problems as real variable problems and solving them by methods forged in the arsenal of functions of a real variable" [9].

Luzin's attention was particularly drawn to boundary properties of analytic functions. Here, above all, we must mention the well-known theorem of Luzin and Privalov (1919) that a set of boundary points of linear measure zero is invariant under a conformal mapping of two domains with rectifiable boundaries. This conjecture had already been made in 1916 by Golubev in his master's dissertation. In a more general formulation, which evolved from the investigations of Golubev, Luzin and Privalov, the problem of a numerical estimate of the variation of the measure of sets of boundary points under a conformal mapping of domains with rectifiable boundaries has attracted the attention of many outstanding specialists in the theory of functions of a complex variable.

The investigations of Luzin and Privalov into boundary uniqueness properties of analytic functions are classical in the theory of analytic functions. First of all, it was shown that an analytic (and even meromorphic) function is completely determined by its boundary values along all non-tangent paths on a set of positive measure. In other words, if a function $f(z)$, meromorphic in the unit disc, has zero boundary values along all non-tangent paths on a set $E$ of positive measure, then $f(z)$ is identically zero. On the other hand, Luzin constructed an example of a function $f(z)$, analytic in the unit disc and not identically zero, which, on radial approach to all points of some set $E$ of full measure $2\pi$, takes the boundary values zero; this set $E$ in Luzin's example was of Baire's first category. It turned out that the idea of a category was essential in boundary uniqueness properties of analytic functions.

In a sense, the following important uniqueness theorem of 1925 for radial boundary values due to Luzin and Privalov can be regarded as definitive: Let $E$ be a set of the second category on some arc $\sigma$ of the unit circle, where $E$ is a reduced set (that is, such that each portion of it has positive measure on $\sigma$); if the function $f(z)$, meromorphic inside the unit disc (or at least in some domain that lies inside this disc and adjoins $\sigma$), has radial boundary values zero everywhere on $E$, then $f(z)$ is identically zero. In other words, a meromorphic function is completely determined by its radial boundary values on a reduced set, provided that this set is of the second category. These classical propositions are the basis of the modern theory of boundary properties of analytic functions.

Luzin discovered a connection between boundary properties of analytic functions in the unit disc and the metric of Riemann surfaces onto which they map the disc, having established the following proposition, known as the "local principle of finite area": Let $D$ be any domain inside the unit disc that adjoins some arc $\sigma$ of the unit circle, and $w = f(z) = \sum_{n=0}^{\infty} c_n z^n$ a regular function in the unit disc; when the area of the Riemann surface which is the image of $D$ under the mapping $w = f(z)$ is finite, the series $\sum_{n=0}^{\infty} c_n z^n$ converges almost everywhere on $\sigma$. In connection with this proposition, Luzin stated the famous "Luzin problem".

In modern terminology, a point $P$ of the unit circle is called a Luzin point of a function $f(z)$ if $f(z)$ maps each disc touching the unit circle internally at $P$ into a domain of finite area of the Riemann surface of

$w = f(z)$. Luzin's conjecture of 1947 is that there are bounded analytic functions in the unit disc such that each point $P$ of the unit circle is a Luzin point for them. This conjecture of Luzin was completely confirmed in 1955, but investigations connected with this characteristically fruitful formulation of the question have extended up to the present time.

The results we have given by no means exhaust Luzin's work on the metric theory of functions and the theory of analytic functions.

However, in 1917 Luzin's main interest switched to descriptive set theory.

In the first decade of the twentieth century Borel, Baire and Lebesgue established the great importance for mathematical analysis of a class of sets called Borel sets or $B$-sets. These sets are obtained by repeated application to segments of the operations of countable summation and countable intersection.

All the mathematical constructions of the previous epoch had been in the main restricted by the framework of $B$-sets. Lebesgue succeeded in constructing an artificial and very complicated example of a set that is not a $B$-set, but the value of such constructions for mathematics was not at all clear. At about that time there appeared in set theory a whole series of varied and extremely peculiar constructions, based on Zermelo's axiom of choice, which lead to new concepts of fundamental importance.

The principle of choice (Zermelo's axiom) is as follows: given some infinite set consisting of sets $(A, B, C, \ldots)$, non-empty and pairwise disjoint, then there is a set that has exactly one element in common with each of the given sets $(A, B, C, \ldots)$. Putting it descriptively, from each of the given sets $(A, B, C, \ldots)$ we can choose one "representative".

This principle caused great argument among mathematicians: some regarded it as obvious, while others objected to it and did not regard theorems whose proofs depended on it as established. Hence arose the question of the possibility of proving theorems without using the axiom of choice. In 1939 Gödel proved that the axiom of choice does not contradict the remaining axioms of set theory. In 1963 Cohen established the independence of the axiom of choice from the other axioms.

Luzin formulated the problem of descriptive set theory thus: "The aim of set theory is a question of great importance: can we regard a line atomistically as a set of points: incidentally, this question is not new, but goes back to the Greeks" ([11], 22).

The problem of studying effective sets, that is, sets that can be constructed without using Zermelo's axiom, is fundamental in the investigations of Luzin and his school into the subject of descriptive set theory.

Luzin discovered, and he and his pupils investigated in detail, the so-called projective sets, obtained from $B$-sets lying in spaces of several dimensions, by successive application of the operation of projection and taking the complement.

The class of projective sets is significantly wider than that of $B$-sets.

However, it was discovered that many fundamental questions relating to properties of projective sets did not admit solutions. Among such problems were, for example, those on the cardinality and measurability of projective sets.

Luzin came to the conclusion that the classical tools of set theory were in principle insufficient to overcome the difficulties that arise in the study of projective sets, and in this connection he put forward the idea of the necessity of a radical revision of the foundations of set theory. In order to appreciate the importance of this idea, he had to take into consideration the fact that set theory is the logical basis of the whole of modern mathematics. At that time there was no way of confirming Luzin's concepts. However, his views turned out to be prophetic – they have now been completely confirmed. In his monograph [11], he collected together the theory of *B*-sets, *A*-sets, and projective sets. This book is extraordinarily rich in facts and new ideas. It was also a programme for later work not only in descriptive set theory, but also one of the branches of mathematical logic [12].

The result of the work in descriptive set theory is, on the one hand, the development of the subject itself and, on the other hand, the extension of its general concepts to various branches of mathematics. The results of the theory of *B*-sets and *A*-sets are constantly being used in the axiomatic description of various systems of mathematical ideas.

"In very recent times the author (Luzin), having returned to projective sets, has begun to look for similar difficulties and indeterminacies in problems of number theory.

Using the method of separation that he worked out in the theory of point sets, he has posed problems of similar difficulty for the sequence of natural numbers" ([18], vol.II, 709–722). (The quotation is from an unpublished commentary on Luzin's scientific work.)

The relationship between the ideas of *B*-sets and *A*-sets and the idea of recursive and recursively enumerable sets is well known. In very recent times set-theoretical concepts have begun to penetrate a number of branches of theoretical science, in particular cybernetics.

Working in the theory of functions of a real variable, Luzin took an interest in many young mathematicians. He inculcated in his pupils a taste for the deepest and most fundamental mathematical questions and guided them in difficult, but particularly perspicuous, directions. Under Luzin's influence, many of his pupils achieved success in the theory of functions itself or in related fields.

"In the early 1920's the majority of mathematics students felt that they were pupils of Luzin and thought highly of the scientific authority of D. F. Egorov, backed up by his deep and wide understanding of various branches of mathematics" ([20], 16). In this way, Luzin and his teacher Egorov were the founders of the Moscow mathematical school of the theory of functions (see [19]). Here is what P. S. Aleksandrov, Gnedenko and Stepanov wrote about this: "In 1911 Egorov proved his well-known theorem on sequences of measurable functions. Immediately afterwards Luzin published his theorem on the *C*-property of measurable functions. These two results fix the beginning of the Moscow school in the metric theory of functions of a real variable ..." ([20], 13).

Representatives of Luzin's school had a great influence on the development of branches of mathematics such as the theory of functions of a real or complex variable, mathematical logic, the theory of probability, topology, functional analysis, differential equations, and various branches of applied mathematics.

In 1931 Luzin began to work on a number of theoretical and practical problems that were new to him. We mention the Riquier – Janet theory in differential equations and its application to questions in geometry and automatic control theory.

At the start of his course in 1945 on partial differential equations, in the department of engineering of the USSR Academy of Sciences, Luzin said: "Even comparatively recently, a mathematician speaking about systems of partial differential equations in any number of unknown functions with any number of independent variables felt rather uncertain, since the laws governing such systems seemed so complicated and involved that the right conclusion appeared to be: in this situation anything is possible and there can be no laws. This situation arose as a result of the work of certain knowledgeable and skilful mathematicians who approached the subject from a direction different from that in which the real point of the question lay". (This is quoted from an unpublished manuscript.) In Luzin's opinion, the solution of this problem is possible only with the help of local analytic theory, the foundations of which were laid by Cauchy and Sonya Kovalevski, and the later development is contained in papers of Riquier and Janet.

A profound study of this theory led Luzin to a number of investigations: geometrical – the bending of surfaces on a principal base, and analytic – the theory of systems of differential equations, where he applied it to systems with one independent variable.

The bending of a surface on a principal base is a continuous bending of a surface under which the conjugacy of the net of certain curves on the surface is preserved. In 1866 the Moscow geometer Peterson first investigated such bendings of surfaces. Later these questions were studied by many authors at home and abroad. It is sufficient to mention that there are about 150 papers on this subject by as many as 40 authors. Already in 1911, Finikov had derived differential equations that determine all principal bases on any given surface, and Byushgens had obtained differential equations that determine surfaces which have a given linear element and admit a bending on a principal base. However, the question of solubility of these equations, in general, remained unclear. Several special cases were investigated, and no example was found in which the equations of Finikov and Byushgens were insoluble. This was the case up to 1938, when Luzin, by means of a subtle analysis of these equations, established that the existence of a principal base is rather rare. Namely, Luzin proved the following results ([18], vol.III, 7–97):

1) There are linear elements that have no principal base, that is, no surface with such a linear element admits a bending on a principal base. Also, there is an "overwhelming majority" of such linear elements, namely, two coefficients $E$ and $F$ (or $G$ and $F$) of $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ can be chosen quite arbitrarily, and the third coefficient can then be chosen with a high degree of arbitrariness.

2) Surfaces isometric to a given one (having a given linear element) that admit a bending on a principal base form a family with not more than 13 parameters, whereas it is known that the choice of any surface isometric to a given one is determined by the choice of two arbitrary functions of one argument.

"All this", said Luzin, "compels us to look at the property of a surface of having a principal base as a very special property that occurs only in exceptional cases ... On the other hand, we must not underrate the significance of this phenomenon. If a principal base were to constitute the singularity of some previously known classical family of surfaces, the interest in Peterson's phenomenon would be reduced to nothing, since the principal base would be simply a property of this family of surfaces. And in such a case this property would doubtless have been perceived long ago. But in fact this is not the case, and Peterson's phenomenon gives us a 'skew' classification of surfaces, still insufficiently understood, in which the presence of a principal base turns out to affect many classical families of surfaces without containing them completely. We must take into account the fact that the pursuit of generality is satisfied at the cost of a loss of both depth and the discovery of unexpected, admittedly special, but deep relations ...".

It is obvious that Luzin's result is fundamental for an extensive theory of bendings on a principal base, since it finally, after 70 years, determined its boundaries.

Having solved the fundamental questions of the theory of bending on a principal base, Luzin turned to a consideration of two problems.

1. The general problem. To find necessary and sufficient conditions for a given linear element to have surfaces with a principal base.

2. The restricted problem. To find necessary and sufficient conditions for a given surface to have a principal base.

In his published papers, Luzin did not obtain the compatibility equations in explicit form, and the aim of his last work [13], published posthumously, was to set up these equations in complete form for the restricted problem [14].

Although more than 20 years have passed since the publication of [13], the compatibility equations have not been found, and we are still waiting for a solution of the problem.

The work of Luzin in differential equations is connected with certain questions of automatic control theory. In studying these questions he first of all presented, by means of matrix theory, properties of systems of linear differential equations with constant coefficients

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \qquad (i = 1, 2, \ldots, n). \tag{1}$$

Here the $x_j$ are unknown functions, the $b_i$ are given functions of $t$ and the $a_{ij}$ are quadratic operators with constant coefficients of the form

$$a_{ij} = L_{ij}D^2 + M_{ij}D + N_{ij} \qquad \left(D = \frac{d}{dt}\right).$$

Luzin posed the following two fundamental problems.

1) To find whether the system (1) is compatible or incompatible.

2) Knowing that the system (1) is compatible, to find all its solutions and to "estimate the arbitrariness".

First of all, he showed that any system (1) can be written in the Riquier – Janet normal form. This form of normal system leads to the discovery of necessary and sufficient conditions for the existence of a solution of (1) with zero initial conditions.

Later, Luzin made a detailed study of (1) with the conditions $b_1 = f(t)$, $b_i = 0$ $(i = 2, 3, \ldots, n)$, that is, the system

$$\sum_{j=1}^{n} a_{ij}x_j = \begin{cases} f(t) & (i = 1), \\ 0 & (i > 1) \end{cases} \qquad (i = 1, 2, \ldots, n). \tag{2}$$

He showed that a necessary and sufficient condition for the unknown function $x_1(t)$ to be independent of the arbitrary analytic function $f(t)$ on the right-hand side of the first equation of the system is that the adjoint $\Delta_{11}$ of the matrix $\| a_{ij} \|$ should be identically zero.

Henceforth, the identity $\Delta_{11} \equiv 0$ will be called the "criterion of absolute invariance for the solution $x_1(t)$". A development of the property of absolute invariance of a solution of (2) is the theorem on "invariance up to $\varepsilon$" [15]. These criteria have been widely applied in the construction of automatic control systems both at home and abroad. It says much for the practical importance of this problem that in 1958, 1962, 1966, and 1971 special conferences were held in Kiev on the theory of invariance and its application in automatic systems.

Thirty years have passed since the publication of Luzin's articles on differential equations connected with control theory, yet there is still no detailed account of them.

Luzin also obtained important results connected with Krylov's method for the secular equation, Chaplygin's method for the approximate integration of differential equations, and so on [15].

The name "secular equation" is derived from its origin in celestial mechanics: investigations of secular inequalities in the motion of planets lead to the secular equation. The secular equation is also called the characteristic equation or the frequency equation, since in problems in the theory of oscillations the roots of this equation are the squares of the angular frequencies of the system.

Working out the secular determinant, that is, calculating the coefficients of the secular equation, is a very laborious operation. Langrange, Laplace, Jacobi and Leverrier have worked on this problem. However, Krylov and Luzin made notable progress with it.

The main point of Luzin's investigations into the secular equation is that he gave an exhaustive algebraic and geometric analysis of the transformation of the classical secular equation. This transformation had been realized earlier by Krylov by means of differential equations. Luzin's investigations, devoted to cases of degeneracy of the secular equation, are of particular interest.

In connection with the problem on the motion of a train, Chaplygin proposed an approximate method of integrating differential equations, based on the use of differential inequalities. Chaplygin gave an algorithm for constructing approximating pairs. Luzin found a rule for the best choice of an original approximating pair and determined the speed of convergence of Chaplygin's process. He proved, first geometrically and then analytically, that the speed of convergence of Chaplygin's process significantly exceeds the speed of convergence of a geometric progression and even of the sequence of factorials $1/n!$; this process converges like the series with the general term $2^{-2^n}$.

Luzin was an ardent propagandist for Chaplygin's method. It is sufficient to point to the fact that Luzin gave a survey of articles devoted to Chaplygin's method and mentioned the present state of the problem, and in his textbook Integral calculus (1949) he devoted a special chapter (Chapter 12) to Chaplygin's method.

Luzin continued to take an active interest in Chaplygin's method for the rest of his life. On the basis of all the investigations that had been carried out, he proposed to write a monograph on Chaplygin's method; it is to be hoped that his pupils will carry out his wishes at last.

Zhukovskii and Chaplygin gave an approximate numerical solution of the differential equation of the motion of a train for various special cases. Luzin, who was interested in this problem, approached it from the qualitative point of view. He reduced the differential equation of the motion of a train to the form

$$\frac{du}{ds} = \Psi(u) + \Phi(s).$$

Here, as he showed, $\Psi(u)$ is a monotonically decreasing function of $u$ in a certain interval that vanishes for a certain value $u_0$; furthermore

$\Psi''(u) > G > 0$; the function $\Phi(s)$ is bounded and uniformly continuous in the interval $(-\infty, \infty)$.

Luzin proved that under these conditions the equation has one and only one limiting solution $U(s)$ in the interval $(-\infty, \infty)$ between definite limits. Any other solution $u(s)$ tends asymptotically to the limiting solution $U(s)$, as $s \to \infty$. If $\Phi(s)$ is periodic, then $U(s)$ is also periodic (with the same period). He also obtained this theorem for the system (2).

At the suggestion of the Geophysics Institute, Luzin carried out a critical analysis of methods of weather forecasting based on meteorological observations over a large interval of time, using his investigations into trigonometric series [17]. He wrote: "The aim of the present article is a review of methods of analysis that have gained wide recognition for the determination of the periodicity of empirically given curves. The technique of these methods was worked out a long time ago, and in practical applications it frequently gives excellent results. Nevertheless, it is still not clear what are the limits within which the applicability of these methods is completely reliable and beyond which they may give fantastic results".

Luzin showed a lively interest in the history of mathematics. In his articles "Differential calculus" ([18], vol.III, 292–318), "Functions" ([18], vol.III, 319–341), and "Newton's theory of limits" ([18], vol.III, 373–400) he reveals the fundamental ideas of mathematical analysis in their historical aspect, from the earliest times to the present day. In his articles "Isaac Newton as mathematician and scientist (1642–1727)" (on the 300th anniversary of his birth) ([18], vol.III, 401–414), and "Euler (1707–1783)" (on the 150th anniversary of his death) ([18], vol.III, 351–372) he gives a detailed testimonial to the greatest geniuses of mankind.

The originality of Luzin's work consists not only in the new statement of questions but also in the method: they are distinguished by the very intense nature of the geometric presentation. Luzin was able to find in very complicated and abstract questions a simple geometric core, which in many cases also predicted a solution of the problem. It is sufficient to recall the sieve method and the method of projection which he introduced in the descriptive theory of functions, or the framework for using the method of Riquier and Janet, which he introduced in the theory of differential equations.

Luzin was not only an outstanding scientist, but also a remarkable teacher. His presentation was always very elegant and at first sight apparently unnecessarily simple – the result of his great pedagogic talent. The solution of the large problems that he undertook is distinguished by their subtlety, elegance, and simplicity of presentation. We merely have to take his textbook "Theory of functions of a real variable", where, in a fascinating and brilliant way, he tells a wide circle of readers – students, middle school pupils and lovers of mathematics – about a large number of abstract and complicated concepts of modern function theory.

Finally, his elementary textbooks "Differential Calculus" and "Integral

Calculus", written for colleges of higher education, have run to more than 20 editions. They can be seen even now in students' hands.

In connection with the publication of Luzin's articles [1], [9], [11], which were not printed earlier, and also the surveys of his research [8], [10], [12], [14], [15], we can testify that questions posed there were further developed in the works of other scholars. We can confidently say that they are studied and will be studied by generations of mathematicians and engineers and will be for a long time the source of scientific work.

In completing this short survey of Luzin's papers, we must mention with satisfaction the three-volume collection of his works [18]. However, it is unfortunate that not all his articles on differential equations (for example, on the theory of invariance) and numerical methods appear in this collection. Many of these articles are of great practical importance.

In conclusion, it is appropriate to recall the words of Keldysh and Novikov: "Thanks to his exceptional intuition and his ability to see deeply into the heart of a question, Luzin frequently predicted mathematical facts whose proof turned out to be possible only after many years and required the creation of completely new mathematical methods. He was one of the outstanding mathematicians and thinkers of our time, who made a great contribution to knowledge and founded a large and very powerful school of Soviet mathematicians" ([12], 102).

The works of Luzin, with their wealth of content and deep analysis of the fundamental ideas of mathematics, the generality of their results, their multiplicity of new methods, and the elegance and clarity of their presentation, have put him into the ranks of scholars who have obtained worldwide fame.

*P. I. Kuznetsov*

## References

[1] N. N. Luzin, *Integral i trigonometricheskii ryad* (The integral and trigonometric series), Moscow 1915. Second edition, edited with comments by N. K. Bari and D. E. Men'shov, with additional articles by N. K. Bari, V. V. Golubev and L. A. Lyusternik, Gostekhizdat, Moscow–Leningrad 1951. Also in Collected works, vol.I, Izdat. Akad. Nauk SSSR, Moscow 1953, 48–212. MR **15**–591.

[2] D. F. Egorov, Report on N. N. Luzin's dissertation "The integral and trigonometric series", presented for the degree of Master of Pure Mathematics, Uspekhi Mat. Nauk 8:2 (1953), 105–110. MR **14**–1051. Also in Collected works, vol.III, 1959, 433–439.

[3] A. N. Krylov, Note on the scientific work of N. N. Luzin, in the book *Zapiski ob uchenykh trudakh diestvitel'nykh chlenov AN SSSR po Otdeleniyu gumanitarnykh nauk, izbrannykh 12 yanvarya i 13 fevralya 1929 goda* (Notes on the scientific work of Members of the USSR Academy of Sciences in the Department of Humanities, selected on January 12 and February 13, 1929), USSR Academy of Sciences 1930, 48–64.

[4] Nikolai Nikolaevich Luzin, *Materialy k biobliografii uchenykh SSSR, seriya matematiki* (Material for the biography of scholars in the USSR, mathematical series), No.3, 1948.
[5] N. K. Bari and V. V. Golubev, Biography of N. N. Luzin, in the book *N. N. Luzin, Sobranie sochinenii* (Collected works), vol.III, 1959, 468–483.
[6] In memory of N. N. Luzin, Uspekhi Mat. Nauk **5**:4 (1950), 15–18.
[7] Academician Nikolai Nikolaevich Luzin, Mathematics in school, no.3 (1950), 1–2.
[8] N. K. Bari and L. A. Lyusternik, The work of N. N. Luzin on the metric theory of functions, Uspekhi Mat. Nauk **6**:6 (1951), 28–46. MR **14**–122. Also in Collected works, vol.III, 1959, 440–460.
[9] N. N. Luzin, Articles on the theory of functions of a complex variable, Uspekhi Mat. Nauk 7:2 (1952), 3–6. MR **13**–810. Also in Collected works, vol.III, 1959, 418–422.
[10] V. S. Fedorov, The work of N. N. Luzin on the theory of functions of a complex variable, Uspekhi Mat. Nauk 7:2 (1952), 7–16. MR **13**–810. Also in Collected works, vol.III, 1959, 423–432.
[11] N. N. Luzin, *Lektsii ob analiticheskikh mnozhestvakh i ikh prilozheniyakh* (Lectures on analytic sets and their applications), edited, with preface and notes, by L. V. Keldysh and P. S. Novikov, Izdat. Fizmatgiz, Moscow 1959. Also in Collected works, vol.II, 1958, 11–269.
[12] L. V. Keldysh and P. S. Novikov, The work of N. N. Luzin in descriptive set theory, Uspekhi Mat. Nauk **8**:2 (1953), 93–104. MR **14**–1051.
[13] N. N. Luzin, On a regular solution of the problem of bending a surface on a principal base, Uspekhi Mat. Nauk 8:2 (1953), 83–92. MR **15**–158. Also in Collected works, vol.III, 1959, 111–122.
[14] L. N. Sretenskii, Remarks on a posthumous work of N. N. Luzin on the integration of the equations of bending of surfaces on a principal base, Uspekhi Mat. Nauk **8**:2 (1953), 75–82. MR **15**–158. Also in Collected works, vol.III, 1959, 461–467.
[15] V. K. Gol'tsman and P. I. Kuznetsov, The work of N. N. Luzin on differential equations and numerical methods, Uspekhi Mat: Nauk 7:2 (1952), 17–30. MR **13**–810.
[16] N. N. Luzin, On the method of approximate integration of Academician S. A. Chaplygin, Uspekhi Mat. Nauk **6**:6 (1951), 3–27. MR **14**–323. Also in Collected works, vol.III, 1959, 181–208.
[17] N. N. Luzin, On the theoretical basis of periodograms, Collected works, vol.III, 1959, 218–291.
[18] N. N. Luzin, Collected works, Izdat. Akad. Nauk SSSR, Moscow, vol.I, 1953, vol.II, 1958, vol.III, 1959.
[19] P. I. Kuznetsov, Dmitrii Fedorovich Egorov (on the centenary of his birth), Uspekhi Mat. Nauk **26**:5 (1971), 169–206.
= Russian Math. Surveys **26**:5 (1971), 125–164.
[20] P. S. Aleksandrov, B. V. Gnedenko and V. V. Stepanov, Mathematics at the University of Moscow in the 20th century, Istor.-Mat. Issled., no.1 (1948).

Translated by E.J.F. Primrose