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MATHEMATICAL PROBLEMS OF NONLINEARITY

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The Thermodynamic Formalism and the Central Limit Theorem for Stochastic Perturbations of Circle Maps with a Break

A. Dzhalilov, D. Mayer, A. Aliyev

Let $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, be an orientation preserving circle homeomorphism with rotation number $\rho_T = [k_1, k_2, \ldots, k_m, 1, 1, \ldots]$, $m \geqslant 1$, and a single break point x_b. Stochastic perturbations $\overline{z}_{n+1} = T(\overline{z}_n) + \sigma\xi_{n+1}$, $\overline{z}_0 := z \in S^1$ of critical circle maps have been studied some time ago by Diaz-Espinoza and de la Llave, who showed for the resulting sum of random variables a central limit theorem and its rate of convergence. Their approach used the renormalization group technique. We will use here Sinai's et al. thermodynamic formalism approach, generalised to circle maps with a break point by Dzhalilov et al., to extend the above results to circle homemorphisms with a break point. This and the sequence of dynamical partitions allows us, following earlier work of Vul at al., to establish a symbolic dynamics for any point $z \in S^1$ and to define a transfer operator whose leading eigenvalue can be used to bound the Lyapunov function. To prove the central limit theorem and its convergence rate we decompose the stochastic sequence via a Taylor expansion in the variables ξ_i into the linear term $L_n(z_0) = \xi_n + \sum\limits_{k=1}^{n-1} \xi_k \prod\limits_{j=k}^{n-1} T'(z_j)$, $z_0 \in S^1$ and a higher order term, which is possible in a neighbourhood A_k^n of the points z_k, $k \leqslant n-1$, not containing the break points of T^n. For this we construct for a certain sequence $\{n_m\}$ a series of neighbourhoods $A_k^{n_m}$ of the points z_k which do not contain any break point of the map $T^{q_{n_m}}$, q_{n_m} the first return times of T. The proof of our results follows from the proof of the central limit theorem for the linearized process.

Keywords: circle map, rotation number, break point, stochastic perturbation, central limit theorem, thermodynamic formalism

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Akhtam Dzhalilov
adzhalilov21@gmail.com
Natural-Mathematical Science Department, Turin Polytechnic University
Kichik Halqa Yoli 17, Tashkent 100095, Uzbekistan

Dieter Mayer
dieter.mayer@tu-clausthal.de
Institut für Theoretische Physik, TU Clausthal
Leibnizstrasse 10, D-38678 Clausthal-Zellerfeld, Germany

Abdurahmon Aliyev
aliyev95@mail.ru
V. I. Romanovsky Institute of Mathematics, Academy of Sciences
Beruniy street 369, Tashkent 100170, Uzbekistan

1. Introduction

Dynamical systems theory is mostly interested in describing the typical behavior of orbits as time goes to infinity, and to understand how this behavior is modified under small perturbations of the system. In the present work we study stochastic perturbations of circle maps with one break point, using as the main tool the thermodynamic formalism. Ya. G. Sinai constructed in [19] the first example of a thermodynamic formalism for Anosov's flows, which was generalized later in the works of D. Ruelle [18], R. Bowen [2] and others for Smale's Axiom A systems. E B. Vul, Ya. G. Sinai and K. M. Khanin finally succeeded in establishing in [20] a thermodynamic formalism approach to Feigenbaum universality in families of critical interval maps, the first example of such an approach to nonhyperbolic systems, whereas the standard approach by Feigenbaum and others has been the renormalization group well known from statistical mechanics.

A natural generalization of smooth interval maps and circle diffeomorphisms are piecewise smooth circle homeomorphisms with break points (see [14]). Contrary to diffeomorphisms, the invariant measure of these circle homeomorphisms $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, with a break point x_b and an irrational rotation number is singular w.r.t. Lebesque measure [6]. The renormalizations of such maps are exponentially approximated by fractional-linear maps [13]. Consider two homeomorphisms T_1 and T_2 with the same irrational rotation number $\rho = \rho(T_1) = \rho(T_2)$, and with identical breakpoint $x_b = x_1 = x_2$. The question of the regularity of the conjugation Φ between T_1 and T_2 is called the rigidity problem. It has been intensively studied in the works of [11, 12] and others.

Theorem 1 (see [11]). *Let $T_1, T_2 \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, be circle homeomorphisms with break point x_b. Suppose that*

1) *they have the same rotation number $\rho(T_1) = \rho(T_2) = \rho$;*
2) *ρ is irrational and has a periodic continued fraction expansion of the form*

$$\rho = [k_1, k_2, \ldots, k_s, k_1, k_2, \ldots, k_s, \ldots], \quad s \geqslant 1.$$

Then the conjugating homeomorphism Φ between T_1 and T_2 belongs to the class $C^{1+\theta}(S^1)$, where $\theta > 0$ depends only on the rotation number ρ.

J. Crutchfield et al. and B. Shraiman et al. considered in [1] respectively [15] heuristically a renormalization group respectively a field theoretic path-integral approach for weak Gaussian noise perturbing one-dimensional maps with period doubling at the onset of chaos. The main result in those papers was that, after appropriately rescaling space and time, the Lyapunov exponent at the transition satisfies some scaling relations. Vul et al. developed in [16] a rigorous thermodynamic formalism approach for critical maps with period doubling. Among many other results these authors studied the effect of noise on the ergodic properties of these maps and showed that for systems with weak noise at the accumulation of period doubling there is a stationary measure, depending on the magnitude of the noise, which converges for vanishing noise to the invariant measure of the attractor.

O. Diaz-Espinosa and R. de la Llave studied in [9] stochastic perturbations of several systems using the renormalization group technique. Among others they proved a central limit theorem for critical circle maps with a golden mean rotation number and some mild conditions on the stochastic noise.

Before turning to the formulation of the main results of our work we recall the general setup and more details of the two main results of O. Diaz-Espinosa and R. de la Llave in [9].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T\colon S^1 \to S^1$ a homeomorphism of the circle $S^1 \to S^1$. Let the stochastic sequence be defined as

$$\overline{x}_{n+1} = T(\overline{x}_n) + \sigma\xi_{n+1}, \quad \overline{x}_0 := x \in S^1, \tag{1.1}$$

where (ξ_n) is a sequence of independent random variables with $p > 2$ finite moments satisfying the following conditions:

$$E\xi_n = 0; \tag{1.2}$$

$$\text{const} \leqslant (E|\xi_n|^2)^{1/2} \leqslant (E|\xi_n|^p)^{1/p} \leqslant \text{Const}. \tag{1.3}$$

In the following we set

$$x_n = T^n(x_0), \quad n \geqslant 1,$$

where T^n denotes the nth iteration of T. The linearized effective noise is defined as

$$L_n(x) = \xi_n + \sum_{k=1}^{n-1} \xi_k \prod_{j=k}^{n-1} T'(x_j), \quad x \in S^1. \tag{1.4}$$

Let $\omega_n(x, \sigma)$ be the stochastic process defined by

$$\omega_n(x, \sigma) = \frac{\overline{x}_n - x_n}{\sigma\sqrt{\operatorname{var}(L_n(x))}}. \tag{1.5}$$

For an arbitrary $z_0 \in S^1 \setminus \{T^i(x_b),\ i = 0, -1, -2, \ldots\}$ and each $s \geqslant 0$, $n \geqslant 1$ the Lyapunov functions $\Lambda_s(z_0, n)$ and $\widehat{\Lambda}(z_0, n)$ are defined as follows:

$$\Lambda_s(z_0, n) = 1 + \sum_{k=1}^{n-1} \prod_{j=k}^{n-1} \left|T'(z_j)\right|^s, \tag{1.6}$$

$$\widehat{\Lambda}(z_0, n) = \max_{1 \leqslant i \leqslant n-1} \sum_{k=1}^{i} \prod_{j=k}^{i} \left|T'(z_j)\right|. \tag{1.7}$$

Using the renormalization group approach, O. Diaz-Espinosa and R. de la Llave established in [9] a sufficient condition for the following CLT to hold for the sequence of random variables $\omega_n(x, \sigma_n)$ defined by certain one-dimensional maps.

Theorem 2 (see [9]). *Let $T\colon M \to M$ be a C^2 map for $M = R^1$, $I = [-1, 1]$ or S^1 and let $\{\xi_n,\ n = 1, 2, \ldots\}$ be a sequence of independent random variables with $p > 2$ finite moments. Suppose that for some $x \in M$ there is an increasing sequence of positive integers n_k such that*

$$\lim_{k\to\infty} \frac{\Lambda_p(x, n_k)}{(\Lambda_2(x, n_k))^{p/2}} = 0. \tag{1.8}$$

Let σ_k be a sequence of positive numbers. Furthermore, assume either of the following two conditions:

H1) *the noise satisfies conditions* (1.2) *and* (1.3) *with $p > 2$ and the sequence σ_k satisfies*

$$\lim_{k\to\infty} \frac{\sup\limits_{x\in S^1} |T''(x)| \| \max\limits_{1\leqslant j\leqslant n_k} \xi_j\|_p^2 (\widehat{\Lambda}(x, n_k))^6 \sigma_k}{\sqrt{\Lambda_2(x, n_k)}} = 0; \tag{1.9}$$

H2) *the noise satisfies conditions* (1.2) *and* (1.3) *with* $p \geqslant 4$ *and the sequence* σ_k *satisfies*

$$\lim_{k\to\infty} \frac{\sup\limits_{x\in S^1} |T''(x)| \| \max\limits_{1\leqslant j\leqslant n_k} \xi_j\|_p^2 (\widehat{\Lambda}(x,\, n_k))^3 \sigma_k}{\sqrt{\Lambda_2(x,\, n_k)}} = 0. \tag{1.10}$$

Then there exists a sequence of events $B_k \in \mathcal{F}$ *such that*

M1) $\lim\limits_{k\to\infty} P(B_k) = 1$;

M2) *The two processes defined by*

$$\omega_{n_k}(x,\, \sigma_k) = \frac{\overline{x}_{n_k} - x_{n_k}}{\sigma_k \sqrt{\mathrm{var}(L_{n_k}(x))}}, \tag{1.11}$$

$$\widetilde{\omega}_{n_k}(x,\, \sigma_k) = \frac{(\overline{x}_{n_k} - x_{n_k})\mathbf{1}_{B_k}}{\sqrt{\mathrm{var}((\overline{x}_{n_k} - x_{n_k})\mathbf{1}_{B_k})}} \tag{1.12}$$

converge in distribution to a standard Gaussian as $k \to \infty$.

If, furthermore, the sequence ξ_n *is supported on a compact set, then we can choose* $B_k = \Omega$ *for all* k.

For the rate of convergence to the Gaussian in this theorem these authors got the following result, where $\Phi(z)$ denotes the distribution of the standard Gaussian on the real line.

Theorem 3 (see [9]). *Let* T, ξ_n *be as in Theorem* 2 *and let* $s = \min(p,\, 3)$. *Assume that condition* (1.8) *holds at some* $x \in M$. *If* σ_k *is a sequence of positive numbers such that*

$$\frac{(\widehat{\Lambda}(x,\, n_k))^3}{\sqrt{\mathrm{var}(L_{n_k}(x))}} \sup_{x\in S^1} |T''(x)| \| \max_{1\leqslant j\leqslant n_k} \xi_j\|_s^2 \sigma_k \leqslant \left(\frac{\Lambda_s(x,\, n_k)}{(\Lambda_s(x,\, n_k))^{s/2}}\right)^2, \tag{1.13}$$

then we have

$$\sup_{z\in R} |P(\omega_{n_k}(x,\, \sigma_k)\mathbf{1}_{B_k} \leqslant z) - \Phi(z)| \leqslant A \frac{\Lambda_s(x,\, n_k)}{(\Lambda_s(x,\, n_k))^{s/2}}, \tag{1.14}$$

where the constant $A > 0$ *depends only on* x.

Our main result of the present paper is

Theorem 4. *Let* $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, *be a circle homeomorphism with a break point* x_b, $T'(x) \geqslant \mathrm{const} > 0$, $x \in [x_b,\, x_b{+}1]$ *and a rotation number* $\rho_T = [k_1,\, k_2,\, \ldots,\, k_m,\, 1,\, 1,\, \ldots]$, $m \geqslant 1$. *Consider a sequence of independent random variables* (ξ_n) *with* $p > 2$ *finite moments satisfying conditions* (1.2) *and* (1.3) *for some* $x \in S^1 \setminus \{T^i(x_b),\, i = 0,\, -1,\, -2,\, \ldots\}$. *Then*

1) *there exists a constant* $\gamma > 0$ *such that, if*

$$\lim_{n\to\infty} \sigma_n n^\gamma = 0, \tag{1.15}$$

the process $\omega_{q_n}(x,\, \sigma_{q_n})$ *defined by* (1.11) *converges in distribution to the standard Gaussian*;

2) *furthermore, there are constants $\tau > 0$ and $\kappa > 0$ depending on p and a constant $C_1 > 0$ such that, if $\sigma_n \leqslant C_1 n^{-\tau}$, then*

$$\sup_{z\in\mathbb{R}} \Big| P(\omega_{q_n}(x, \sigma_{q_n}) \leqslant z) - \Phi(z) \Big| \leqslant C q_n^{-\kappa},$$

where q_n is the first return time of T, the constant $C > 0$ depends only on x and $\Phi(z)$ is the distribution function of the standard Gaussian on $\mathbb{R}$.

REMARK 1. Our paper is strongly influenced by the work of O. Diaz-Espinosa and R. de la Llave in [9]. Following their ideas, we prove the analog of their Theorem 2 for circle maps with a break point with small modifications. For estimating the Lyapunov function, however, we are using the thermodynamic formalism for such maps.

We restrict our discussion to the simplest case of circle maps with one break point and eventually golden mean rotation number. Since the thermodynamic formalism in our approach can be extended to rotation numbers with arbitrary eventually periodic continued fraction expansions, our constructions can be in principle generalized to such cases. However, technically these become more involved and would lead to a longer paper.

We emphasize that the limit theorems in this paper are considered in the setup of sums of random variables.

2. Preliminaries and notations

Let T be an orientation-preserving circle homeomorphism with an irrational rotation number ρ_T. Then ρ_T can be uniquely expanded as a continued fraction, i. e., $\rho_T = 1/(k_1 + 1/(k_2 + + \cdots)) := [k_1, k_2, \ldots, k_n, \ldots)$. Denote by $\frac{p_n}{q_n} = [k_1, k_2, \ldots, k_n]$, $n \geqslant 1$, its nth convergent. The numbers q_n, $n \geqslant 1$ are called the **first return times** of T and satisfy the recurrence relations $q_{n+1} = k_{n+1} q_n + q_{n-1}$, $n \geqslant 1$, where $q_0 = 1$ and $q_1 = k_1$. Fix an arbitrary point $z_0 \in S^1$. Its forward orbit $O_T^+(z_0) = \{z_i = T^i(z_0), i = 0, 1, 2, \ldots\}$ defines a sequence of natural partitions of the circle. Indeed, denote by $I_0^{(n)}(z_0)$ the closed interval in S^1 with endpoints z_0 and $z_{q_n} = T^{q_n}(z_0)$. In the clockwise orientation of the circle the point z_{q_n} is then for n odd to the left of z_0, and for n even to its right. If $I_i^{(n)}(z_0) = T^i(I_0^{(n)}(z_0))$, $i \geqslant 1$, denote the iterates of the interval $I_0^{(n)}(z_0)$ under T, it is well known that the set $P_n(z_0)$ of intervals with mutually disjoint interiors, defined as

$$P_n(z_0) = \{I_i^{(n)}(z_0),\ 0 \leqslant i < q_{n+1}\} \cup \{I_j^{(n+1)}(z_0),\ 0 \leqslant j < q_n\},$$

determines a partition of the circle for any n. The partition $P_n(z_0)$ is called the nth **dynamical partition** of S^1 determined by the point z_0 and the map T.

Proceeding from partition $P_n(z_0)$ to $P_{n+1}(z_0)$, all the intervals $I_j^{(n+1)}(z_0)$, $0 \leqslant j \leqslant q_n - 1$, are preserved, whereas each of the intervals $I_i^{(n)}(z_0)$, $0 \leqslant i \leqslant q_{n+1} - 1$, is partitioned into $k_{n+2} + 1$ subintervals belonging to $P_{n+1}(z_0)$, such that

$$I_i^{(n)}(z_0) = I_i^{(n+2)}(z_0) \cup \bigcup_{s=0}^{k_{n+2}-1} I_{i+q_n+sq_{n+1}}^{(n+1)}(z_0).$$

Obviously, one has $P_1(z_0) \prec P_2(z_0) \prec \ldots \prec P_n(z_0) \prec \ldots$

The intervals $I_0^{(n)}(z_0)$, $I_0^{(n+1)}(z_0)$ are called **generators** of the partition $P_n(z_0)$.

Later we will use also the so-called **renormalization intervals** $J_i^{(n)}(z_0) = T^i(J_0^{(n)}(z_0)) = J_0^{(n)}(z_i)$, $i = 0, 1, 2, \ldots$, where $J_0^{(n)}(z_0) = I_0^{(n)}(z_0) \cup I_0^{(n+1)}(z_0)$ and $z_i = T^i(z_0)$. Define the *Poincaré map* π_n: $J_0^{(n)}(z_0) \to J_0^{(n)}(z_0)$ by

$$\pi_n(x) = \begin{cases} T^{q_{n+1}}x, & \text{if } x \in I_0^{(n)}(z_0), \\ T^{q_n}x, & \text{if } x \in I_0^{(n+1)}(z_0). \end{cases}$$

The following lemma plays a key role for studying the metrical properties of the homeomorphism T.

Lemma 1 (see [6]). *Let T be a circle homeomorphism with one break point x_b with jump ratio $c_T(x_b) = \sqrt{\frac{T(x_b-)}{T(x_b+)}} \neq 1$ and an irrational rotation number. Suppose $T \in C^1([x_b, x_b+1])$ and $\operatorname*{var}_{z\in[x_b, x_b+1]} \ln T' = \overline{v} < \infty$. Put $v = \overline{v} + 2|\ln c_T(x_b)|$. If $y_0 \in S^1$ and $T^i(y_0) \neq x_b$, $0 \leqslant i < q_n$, then*

$$e^{-v} \leqslant \prod_{s=0}^{q_n-1} DT(y_s) \leqslant e^{v} \tag{2.1}$$

holds.

Inequality (2.1) is called **Denjoy's inequality**.

It follows from Lemma 1 that the intervals of the dynamical partition $P_n(z_0)$ have exponentially small lengths. Indeed, one finds

Corollary 1. *Suppose the circle map T satisfies the conditions of Lemma 1. Then for an arbitrary element $I^{(n)}$ of the dynamical partition $P_n(z_0)$ the following bounds hold:*

$$\ell(I^{(n)}) \leqslant \text{const} \cdot \theta^n, \tag{2.2}$$

where $\theta = (1 + e^{-v})^{-1/2} < 1$ and l denotes Lebesque measure.

Corollary 1 implies that the trajectory of every point $x \in S^1$ is dense in S^1. This, together with monotonicity of T, implies that the homeomorphism T is topologically conjugate to the linear rotation $T_\rho(x) = x + \rho \bmod 1$.

Definition 1. *Let $K > 1$ be a constant. We call two intervals I_1 nd I_2 of the circle S^1 K-comparable if the inequality $K^{-1}|I_2| \leqslant |I_1| \leqslant K|I_2|$ holds.*

Lemma 2. *Suppose the circle homeomorphism T satisfies the conditions of Lemma 1 and $z_0 \in S^1$. Then for an arbitrary interval $I^{(n)}$ of the dynamical partition $P_n(z_0)$ at least $n-1$ elements of $P_n(T, z_0)$ are e^v-comparable with $I^{(n)}$.*

Proof. Let $I_{j_0}^{(n)} \in P_n(T, z_0)$, $0 \leqslant j_0 < q_{n+1}$. First we assume $j_0 = 0$, i.e., $I_{j_0}^{(n)} = I_0^{(n)}$. Applying Denjoy's inequality (2.1), we see that the intervals $T^{q_0}(I_0^{(n)})$, $T^{q_1}(I_0^{(n)})$, $\ldots$, $T^{q_n}(I_0^{(n)})$ of the partition P_n are e^v-comparable with $I_0^{(n)}$.

If $0 < j_0 < q_{n+1}$, then $q_{i_0} \leqslant j_0 \leqslant q_{i_0+1}$, for some $0 \leqslant i_0 \leqslant n$.

In this case, the intervals

$$T^{-q_{i_0}} I_{j_0}^{(n)},\ T^{-q_{i_0-1}} I_{j_0}^{(n)},\ \ldots,\ T^{-q_0} I_{j_0}^{(n)},\ T^{q_0} I_{j_0}^{(n)},\ \ldots,\ T^{q_{n-i_0-1}} I_{j_0}^{(n)} \tag{2.3}$$

are elements of the partition $P_n(T, z_0)$. Applying again Denjoy's inequality (2.1), we can see that each interval in (2.3) is e^v-comparable with $I_{j_0}^{(n)}$.

For intervals $I_{j_0}^{(n+1)}$, $0 \leqslant j_0 < q_n$, the statement can be proved analogously. □

We recall the following definition introduced in [10].

Definition 2. *An interval $I = [\tau, t] \subset S^1$ is said to be q_n-small, and its endpoints q_n-close, if the intervals $T^i(I)$, $0 \leqslant i \leqslant q_n - 1$, are, except for the endpoints, pairwise disjoint.*

It follows from the structure of the dynamical partition that an interval $I = [\tau, t]$ is q_n-small if and only if either $\tau \prec t \preceq T^{q_{n-1}}(\tau)$ or $T^{q_{n-1}}(t) \preceq \tau \prec t$.

Lemma 3 (see [8]). *Suppose the homeomorphism T with an irrational rotation number ρ_T satisfies the conditions of Lemma 1 and the interval $I = (x, y) \subset S^1$ is q_n-small. Then for any $0 \leqslant k < q_n$ Finzi's inequality holds*:

$$e^{-v} \leqslant \frac{DT^k(x)}{DT^k(y)} \leqslant e^v, \tag{2.4}$$

where v is the total variation of $\log DT$ on S^1.

Next, we formulate the thermodynamic formalism for circle maps with a break point.

Let X_{br} be the set of strictly increasing pairs of functions ($f(x)$, $x \in [-1, 0]$, $g(x)$, $x \in [0, \alpha]$ for some $\alpha > 0$) satisfying the following conditions:

- $f(0) = \alpha$, $g(0) = -1$;
- $f(-1) = g(\alpha)$;
- $f(g(0)) = f(-1) < 0$;
- $f^{(2)}(g(0)) \geqslant 0$;
- $f(x) \in C^{2+\varepsilon}([-1, 0])$, $g(x) \in C^{2+\varepsilon}([0, \alpha])$ for all $\varepsilon > 0$;
- $f'_+(0) \neq g'_-(0)$.

These conditions allow us to construct a circle homeomorphism $G_{f,g}$ on $[-1, \alpha)$ from a pair $(f, g) \in X_{br}$ with a break point $x_b = 0$ (and possibly with a second break point $x_b = -1$ if $f'(-1) \neq g'(\alpha)$) as follows:

$$G_{f,g}(x) = \begin{cases} f(x) & \text{if } x \in [-1, 0), \\ g(x) & \text{if } x \in [0, \alpha). \end{cases}$$

Using the map $l\colon [-1, \alpha] \to S^1$ with $l(x) = \frac{x+1}{\alpha+1}$, we get a circle homeomorphism $l \circ G_{f,g} \circ l^{-1}$ on $S^1 = \mathbb{R} \mod 1$, which we denote for simplicity also by $G_{f,g}$ whenever its domain of definition is clear. We define the rotation number $\rho(G_{f,g})$ of $G_{f,g}$ by the rotation number of this circle homeomorphism when acting on S^1. Denote by $X_{br}(\omega)$ the subset of $(f, g) \in X_{br}$ with $\rho(G_{f,g}) = \omega = \frac{\sqrt{5}-1}{2}$ the golden mean. Recall the jump ratio $c = \sqrt{\frac{DG_{f,g}(0\_)}{DG_{f,g}(0+)}} = \sqrt{\frac{Df(0\_)}{Dg(0+)}}$ of $G_{f,g}$ at its break point $x_b = 0$.

It is clear that in the case $c = 1$ the homeomorphism $G_{f,g}$ is a smooth map as long as 0 is its only break point. We assume that $c \neq 1$. Define a renormalization operator $R_{br}\colon X_{br}(\omega) \to X_{br}(\omega)$ as follows:

$$R_{br}(f(x),\, g(x)) = \Big(\widetilde{f}(x),\ x \in [-1,\, 0];\ \widetilde{g}(x),\ x \in [0,\, \alpha']\Big),$$

where

$$\widetilde{f}(x) = -\alpha^{-1} f(g(-\alpha x)), \quad \widetilde{g}(x) = -\alpha^{-1} f(-\alpha x), \quad \alpha' = -\alpha^{-1} f(-1).$$

Since $D\widetilde{f}(0_-) = Df(-1_+)Dg(0_+)$ and $D\widetilde{g}(0_+) = Df(0_-)$, we find $\widetilde{c}(0) = \sqrt{\frac{DG_{\widetilde{f},\widetilde{g}}(0_-)}{DG_{\widetilde{f},\widetilde{g}}(0_+)}} = c^{-1}\sqrt{Df(-1_+)}$. On the other hand, $D\widetilde{f}(-1_+) = Df(f(-1))Dg(\alpha_-)$ and $D\widetilde{g}(\alpha'_-) = Df(f(-1))$, which leads to $\widetilde{c}(-1) = \sqrt{\frac{DG_{\widetilde{f},\widetilde{g}}(\alpha'_-)}{DG_{\widetilde{f},\widetilde{g}}(-1_+)}} = \frac{1}{\sqrt{Dg(\alpha_-)}}$, which in general is also different from 1. This shows that $G_{\widetilde{f},\widetilde{g}}$ has in general two break points, one at $x = x_b = 0$ and one at $x = G_{\widetilde{f},\widetilde{g}} x_b$. For the product of these two jump ratios one finds $\widetilde{c}(0)\widetilde{c}(-1) = \sqrt{\frac{Df(-1_+)}{Dg(\alpha_-)}}c^{-1}$. From the work of Khanin and Vul in [13] it is known that $R_{br}\colon X_{br}(\omega) \to X_{br}(\omega)$ has a unique periodic orbit $\{f_i(x,\, c_i),\, g_i(x,\, c_i),\, i = 1,\, 2\}$ of period two, which means

$$R_{br}(f_1(x,\, c_1),\, g_1(x,\, c_1)) = (f_2(x,\, c_2),\, g_2(x,\, c_2)),$$
$$R_{br}(f_2(x,\, c_2),\, g_2(x,\, c_2)) = (f_1(x,\, c_1),\, g_1(x,\, c_1)),$$

where the functions $f_i(x,\, c_i)$ and $g_i(x,\, c_i)$, $i = 1,\, 2$ have the following explicit form:

$$f_i(x,\, c_i) = \frac{(\alpha_i + c_i x)\beta_i}{\beta_i + (\beta_i + \alpha_i - c_i)x}, \tag{2.5}$$

$$g_i(x,\, c_i) = \frac{\alpha_i\beta_i(x_i - c_i)}{\alpha_i\beta_i c_i + (c_i - \alpha_i - c_i\beta_i)x}, \tag{2.6}$$

with

$$\alpha_1 = \frac{c - \beta_0^2}{1 + \beta_0}, \quad \alpha_2 = \frac{c^{-1} - \beta_0^2}{1 + \beta_0}, \quad c_1 = c, \quad c_2 = c^{-1}, \quad \beta_1 = \beta_2 = \beta_0,$$

β_0 the unique root of the equation

$$\beta^4 - \beta^3 - \beta^2\frac{(c+1)^2}{c} - \beta + 1 = 0,$$

belonging to the interval $(0,\, 1)$.

By using the pairs of functions $(f_i,\, g_i)$, $i = 1,\, 2$, we define circle homeomorphisms G_i: $[-1,\, \alpha_i] \to [-1,\, \alpha_i]$, $i = 1,\, 2$, as

$$G_i(x) = \begin{cases} f_i(x,\, c_i) & \text{if} \quad x \in [-1,\, 0), \\ g_i(x,\, c_i) & \text{if} \quad x \in [0,\, \alpha_i), \end{cases}$$

where we have used the fact that $f_i(0) = \alpha_i$ and $g_i(0) = -1$ respectively $f_i(-1) = g_i(\alpha_i) = -\beta_i$. Since $Df_1(0_-) = \frac{c(\alpha_1+\beta_0) - \alpha_1(\alpha_1+\beta_0)}{\beta_0}$ and $Dg_1(0_+) = \frac{(1-\beta_0)(c-\alpha_1)}{\alpha_1\beta_0 c}$, one finds $c_{G_1}(0) = \sqrt{\frac{Df_1(0_-)}{Dg_1(0_+)}} = \sqrt{\frac{(c+\beta_0)(c-\beta_0^2)}{(1+\beta_0)(1-\beta_0^2)}}$. But this equals 1 only iff $c = 1$. Hence, G_1 has a break point at $x_b = 0$. On the other hand, one finds $Df_1(-1_+) = \frac{\beta_0(\alpha_1+\beta_0)}{c-\alpha_1}$ respectively $Dg_1(\alpha_1) = \frac{c\beta_0(1-\beta_0)}{\alpha_1(c-\alpha_1)}$, which leads to $c_{G_1}(-1) = \sqrt{\frac{Dg_1(\alpha_{1\_})}{Df_1(-1_+)}} = \sqrt{\frac{(1+\beta_0)(1-\beta_0^2)c}{(c+\beta_0)(c-\beta_0^2)}}$. This equals 1 iff $c = 1$. This shows that the map G_1 has indeed two break points, namely, at $x = x_b = 0$ and at $x = G_1(x_b) = -1$. For the product of the jump ratios one finds $c_{G_1}(0)c_{G_1}(G_1(0)) = \sqrt{c}$.

Using the function $l_i(x) = \frac{1+x}{1+\alpha_i}$, the map $l \circ G_i \circ l^{-1}$ defines then a homeomorphism of the circle S^1 with breakpoints $x_b = \frac{1}{\alpha_i+1}$ and $x_b' = 0$.

We rename the homeomorphism of the circle S^1 corresponding to G_1 by G_{br} and denote by $B(G_{br})$ the set of all circle homeomorphisms which are $C^{1+\epsilon}$ conjugate to G_{br}.

The sequence of dynamical partitions $P_n(x_b)$, $n \geqslant 1$, allows us to introduce symbolic dynamics for the map G_{br}. For this we take an arbitrary point $x \in S^1 \setminus O^+_{G_{br}}(x_b)$ where $O^+_{G_{br}}(x_b)$ denotes the forward orbit of the break point x_b of G_{br}. For $n \geqslant 0$ put $a_{n+1} =: a_{n+1}(x) = a$ if $x \in I_j^{(n+1)}(x_b)$, $0 \leqslant j < q_n$. If, however, $x \in I_i^{(n)}(x_b)$, $0 \leqslant i < q_{n+1}$, we know from the construction of the partition P_{n+1} from P_n in case $\rho(G_{br}) = \omega = \frac{\sqrt{5}-1}{2}$ that either $x \in I_i^{(n+2)}(x_b)$, $0 \leqslant i < q_{n+1}$ or $x \in I_{i+q_n}^{(n+1)}(x_b)$, $0 \leqslant i < q_{n+1}$. In the first case we put $a_{n+1} = 0$ and in the second $a_{n+1} = 1$. In this way we get a one-to-one correspondence

$$\varphi\colon S^1 \setminus O^+_{G_{br}}(x_b) \leftrightarrow \{(a_1, \ldots, a_n, \ldots),\ a_n \in \{a, 0, 1\};\ a_{n+1} = a \iff a_n = 0,\ n \geqslant 1\} := \Theta_{\mathbb{A}},$$

where $\Theta_{\mathbb{A}}$ denotes the space of allowed infinite one-sided sequences of symbols from the alphabet $A = \{1, 0, a\}$ with the transition matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

that is,

$$\Theta_{\mathbb{A}} := \{\underline{a} = (a_1, \ldots, a_n, \ldots),\ a_i \in A,\ \mathbb{A}_{a_i, a_{i+1}} = 1 \text{ for } i \in \mathbb{Z}_+\}.$$

The triple $(\theta_{\mathbb{A}}, A, \sigma)$ with $\sigma\colon \theta_{\mathbb{A}} \to \theta_{\mathbb{A}}$ the shift map $(\sigma(\underline{a}))_i = a_{i+1}$ is called a subshift of finite type over the alphabet A with the transition matrix $\mathbb{A}$. Notice that every interval $I^{(n)}$ of the dynamical partition P_n corresponds to the unique finite word $(a_1, a_2, \ldots, a_n)$ of length n. In particular, for n even the words $(a, 0, a, 0, \ldots, a, 0)$ and $(0, a, 0, a, \ldots, 0, a)$ correspond to the atoms $I_0^{(n+1)}(x_b)$ respectively $I_0^{(n)}(x_b)$ of P_n.

In [5] A. Dzhalilov and J. Karimov constructed the thermodynamic formalism for maps in $B(G_{br})$ by using a closely related subshift of finite type as follows:

Denote by $\mathbb{A}^t = (\mathbb{A}^t_{i,j}), i, j \in A$ the transposed matrix of $\mathbb{A}$ and by $(\Theta_{\mathbb{A}^t}, A, \sigma)$ the subshift of finite type with alphabet A and transition matrix $\mathbb{A}^t$. Obviously, $\underline{b} = (b_1, \cdots, b_n, b_{n+1,\ldots}) \in \Theta_{\mathbb{A}^t}$ if and only if $b_{n+1} = 0$ iff $b_n = a$. Therefore, a finite sequence $(a_1, \cdots, a_n)$ is a subsequence of some $\underline{a} \in \Theta_{\mathbb{A}}$ iff $(a_n, a_{n-1}, \cdots, a_1)$ is a subsequnce of some $\underline{b} \in \Theta_{\mathbb{A}^t}$. Since $\mathbb{A}^3_{i,j} = (\mathbb{A}^t)^3_{j,i} > 0$ for all $i, j \in A$, the map σ is topological mixing in both $\Theta_{\mathbb{A}}$ and $\Theta_{\mathbb{A}^t}$.

In the following we denote by $\vec{a}$ the vector $(a_1, \ldots, a_n) \in A^n$: $\mathbb{A}_{a_i, a_{i+1}} = 1$, $1 \leqslant i \leqslant n-1$, and by $\overleftarrow{b} = (b_1, \ldots, b_n)$: $\mathbb{A}^t_{b_i, b_{i+1}} = 1$, $1 \leqslant i \leqslant n-1$.

Define for $x \in A$

$$\underline{\gamma}(x) := \begin{cases} (a,\, 0,\, a,\, 0,\, \ldots,\, a,\, 0,\, \ldots), & x = 0,\, 1, \\ (0,\, a,\, 0,\, a,\, \ldots,\, 0,\, a,\, \ldots), & x = a. \end{cases}$$

Theorem 5 (see [5]). *For any $G \in B(G_{br})$, there exists a function $U_{br} \colon \Theta_{\mathbb{A}^t} \to (-\infty, 0)$ continuous in the product topology, such that the following properties hold*:

1) *for any $\underline{b} = (b_1, \ldots, b_k, b_{k+1}, \ldots, b_n, \ldots)$, $\underline{c} = (b_1, \ldots, b_k, c_{k+1}, \ldots, c_n, \ldots) \in \Theta_{\mathbb{A}^t}$ there exist constants $C_1 > 0$ and $q \in (0, 1)$, not depending on $\underline{b}$, $\underline{c}$ and k, such that*

$$|U_{br}(\underline{b}) - U_{br}(\underline{c})| \leqslant C_1 \cdot q^k;$$

2) *let $\Delta_{s_n}^{(n)} \subset \Delta_{s_r}^{(r)}$, $1 \leqslant r < n$, $0 \leqslant s_r \leqslant q_{r+1} - 1$, $0 \leqslant s_n \leqslant q_{n+1} - 1$ and $\varphi(\Delta_{s_n}^{(n)}) = (a_1, \ldots, a_r, \ldots, a_n)$, $\varphi(\Delta_{s_r}^{(r)}) = (a_1, \ldots, a_r)$, then*

$$|\Delta_{s_n}^{(n)}| = (1 + \psi_r(a_1, \ldots, a_n))|\Delta_{s_r}^{(r)}| \exp\left\{\sum_{s=r}^{n} U_{br}(a_s, a_{s-1}, \ldots, a_r, \ldots, a_1, \underline{\gamma}(a_1))\right\},$$

where $|\psi_r(a_1, \ldots, a_n)| \leqslant C_2 \cdot q^r$, with the constant $C_2 > 0$ not depending on r, n and $(a_1, \ldots, a_n)$.

Here $\Delta^{(n)}$ are elements of the dynamical partition $P_n(G, x_b) := \{\Delta_i^{(n)} = G^i \Delta_0^{(n)}, 0 \leqslant i \leqslant q_{n+1} - 1\} \cup \{\Delta_j^{(n+1)} = G^j \Delta_0^{(n+1)}, 0 \leqslant j \leqslant q_n - 1\}$ with $\Delta_0^{(n)} = [x_{q_n}, x_b)$ respectively $\Delta_0^{(n+1)} = [x_b, x_{q_{n-1}})$ and $x_i = G^i x_b$, where $x_b = 0$ denotes the break point of G. The function U_{br} is defined as follows: for $\underline{b} = (b_1, \ldots, b_k, b_{k+1}, \ldots) \in \Theta_{\mathbb{A}^t}$ and any $k \geqslant 1$ denote for $i = 1, 2$ by $\vec{a}_k^i$ the finite sequence $\vec{a}_k^i = (b_k, \ldots, b_i)$. Choose $I(\vec{a}_k^i) \in P_{k-i+1}$ such that $\varphi(I(\vec{a}_k^i)) = \vec{a}_k^i$. Obviously, one has $I(\vec{a}_k^1) \subset I(\vec{a}_k^2)$. Define for $\overleftarrow{b}_k = (b_1, \ldots, b_k)$ the function $U_k = U_k(\overleftarrow{b}_k)$ as

$$U_k(\overleftarrow{b}_k) := \log \frac{|I(\vec{a}_k^1)|}{|I(\vec{a}_k^2)|}.$$

One can show that the thermodynamic limit $k \to \infty$ of the function U_k exists, namely, for $\underline{b} \in \Theta_{\mathbb{A}^t}$

$$U_{br}(\underline{b}) = \lim_{k \to \infty} U_k(\vec{b}_k)$$

is a well defined function on $\Theta_{\mathbb{A}^t}$ with values in $(-\infty, 0)$.

According to a well-known theorem in [14, Proposition 5.13], the following limit exists:

$$\ln \lambda_\beta = \lim_{n \to \infty} \frac{1}{n} \ln \left[\sum_{\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)} \exp\left\{\beta \sum_{s=1}^{n} U_{br}(\varepsilon_s, \varepsilon_{s-1}, \ldots, \varepsilon_1, \underline{\gamma}(\varepsilon_1))\right\}\right], \tag{2.7}$$

where λ_β is the leading eigenvalue of the transfer operators D_β: $C(\Theta_{\mathbb{A}^t}) \to C(\Theta_{\mathbb{A}^t})$ respectively its dual D_β^*: $M(\Theta_{\mathbb{A}^t}) \to M(\Theta_{\mathbb{A}^t})$

$$(D_\beta f)(\underline{b}) = \sum_{\underline{x} \in \sigma^{-1}(\underline{b}) \cap \Theta_{\mathbb{A}^t}} e^{\beta U_{br}(\underline{x})} f(\underline{x}), \tag{2.8}$$

$$(D_\beta^* \mu)(c, \underline{b}) = \mathbb{A}_{c, b_1}^t e^{\beta U_{br}(c, \underline{b})} \mu(\underline{b}). \tag{2.9}$$

$\sigma \colon \Theta_{\mathcal{A}^t} \to \Theta_{\mathcal{A}^t}$ is the shift map $\sigma(\underline{b})_i = b_{i+1}$, $i \geqslant 1$, $C(\Theta_{\mathcal{A}^t})$ denotes the space of continuous functions and $M(\Theta_{\mathcal{A}^t})$ the space of Borel measures on $\Theta_{\mathbb{A}^t}$. We used also the formal expression $d\mu(x) = \mu(x)\, dx$ for the measure $d\mu$.

3. Estimates for the Lyapunov function

In this section we estimate the Lyapunov functions. Let $T \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$, $\varepsilon > 0$, be a circle homeomorphism with a break point x_b, $T'(x) \geqslant \text{const} > 0$, $x \in S_1 \equiv [x_b, x_b + 1)$ and a rotation number $\rho_T = [k_1, k_2, \ldots, k_m, 1, 1, \ldots]$ for some $m \geqslant 1$. In this case, obviously, $q_{n+1} = q_n + q_{n-1}$, $n \geqslant m$. Consider the nth dynamical partition $P_n(x_b) = \{I_j^{(n)}(x_b),$ $0 \leqslant j < q_{n+1};\ I_i^{(n+1)}(x_b),\ 0 \leqslant i < q_n\}$ determined by the break point x_b under the map T. We denote by $I^{(n)}(x_b; z_0)$ the interval of the partition $P_n(x_b)$ containing the point z_0.

Theorem 6. *For any $\beta > 0$ and $z_0 \in S^1 \setminus \{T^i(x_b),\ i = 0, -1, -2, \ldots, -q_n + 1\}$, there exists $n_0 := n_0(\beta) > m$ such that for every $n > n_0$ the following inequalities hold*:

$$c_1|I^{(n)}(x_b;\, z_0)|^{\beta}\lambda_{-\beta}^{n} \leqslant \Lambda_\beta(z_0,\, q_n) \leqslant C_1|I^{(n)}(x_b;\, z_0)|^{\beta}\lambda_{-\beta}^{n}, \tag{3.1}$$

where $\lambda_{-\beta}$ is the leading eigenvalue of the transfer operator $D_{-\beta}$ in (2.8) *and the positive constants c_1, C_1 depend on the map T.*

Proof. Using the elements of the partition $P_n(x_b)$, we introduce the following sum:

$$S_{n,\beta}(z_0) = |I_0^{(n)}(x_b;\, z_0)|^{\beta}\left(\sum_{k=0}^{q_{n+1}-1}|I_k^{(n)}(x_b)|^{-\beta} + \sum_{s=0}^{q_n-1}|I_s^{(n+1)}(x_b)|^{-\beta}\right).$$

Then one has the following

Lemma 4. *For any $z_0 \in S^1 \setminus \{T^i(x_b),\ i = 0, -1, -2, \ldots, -q_n - q_{n+1} + 1\}$, $\beta > 0$ and $n \geqslant 1$*

$$c_2 \cdot S_{n,\beta}(z_0) \leqslant \Lambda_\beta(z_0,\, q_n + q_{n+1}) \leqslant C_2 \cdot S_{n,\beta}(z_0), \tag{3.2}$$

where the positive constants c_2 and C_2 depend only on the total variation of $\log T'$ and β.

Proof. For any $z_0 \in S^1 \setminus \{T^{-i}x_b,\ 0 \leqslant i < q_n + q_{n+1}\}$ and $\beta > 0$, we consider

$$\Lambda_\beta(z_0,\, q_n + q_{n+1}) = 1 + \sum_{k=1}^{q_n+q_{n+1}-1}\prod_{j=k}^{q_n+q_{n+1}-1}\left|DT(z_j)\right|^{\beta}. \tag{3.3}$$

We rewrite $\Lambda_\beta(z_0,\, q_n + q_{n+1})$ in the following form:

$$\Lambda_\beta(z_0,\, q_n + q_{n+1}) = \left|DT^{q_n+q_{n+1}-1}(z_1)\right|^{\beta}\left(1 + \sum_{k=1}^{q_n+q_{n+1}-1}\left|DT^k(z_1)\right|^{-\beta}\right). \tag{3.4}$$

Since

$$DT^{q_n+q_{n+1}-1}(z_1) = \frac{DT^{q_n+q_{n+1}}(z_0)}{DT(z_0)},$$

and using the bounds

$$K_1^{-1} \leqslant DT(x) \leqslant K_1, \quad x \in [x_b,\, x_b + 1], \quad K_1 = K_1(T) > 0,$$

respectively Denjoy's inequality (2.4), we obtain

$$K_1^{-1}e^{-2v} \leqslant DT^{q_n+q_{n+1}-1}(z_1) \leqslant K_1e^{2v}, \tag{3.5}$$

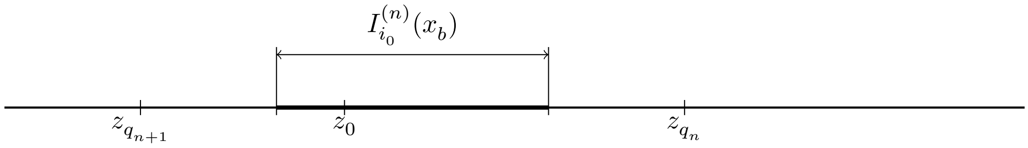
where $v = \operatorname*{var}_{x \in S^1} DT(x) + 2|\ln c_T(x_b)|$. Next, we estimate the sum in (3.4). Let $P_n(T, x_b)$ and $P_n(T, z_0)$ be the nth dynamical partitions determined by x_b respectively z_0 under the map T.

Here two cases are possible

Case I. $z_0 \in I_{i_0}^{(n)}(x_b)$, $0 \leqslant i_0 < q_{n+1}$,

Case II. $z_0 \in I_{j_0}^{(n+1)}(x_b)$, $0 \leqslant j_0 < q_n$.

We consider only case I, case II can be treated similarly.



Obviously,

$$1 + \sum_{k=1}^{q_{n+1}+q_n-1} \left|DT^k(z_1)\right|^{-\beta} = |DT(z_0)|^{\beta} \left(\sum_{k=1}^{q_{n+1}} |DT^k(z_0)|^{-\beta} + \sum_{k=q_{n+1}+1}^{q_{n+1}+q_n} |DT^k(z_0)|^{-\beta} \right) =$$
$$= |DT(z_0)|^{\beta} (S_n^{(1)}(\beta) + S_n^{(2)}(\beta)). \quad (3.6)$$

First we estimate the sum $S_n^{(1)}(\beta)$.

By assumption $z_0 \in I_{i_0}^{(n)}(x_b)$, $0 \leqslant i_0 < q_{n+1}$. In the case $0 \leqslant i_0 \leqslant q_{n+1} - 2$ we split the sum $S_n^{(1)}(\beta)$ into two parts

$$S_n^{(1)}(\beta) = \sum_{k=1}^{q_{n+1}-i_0-1} |DT^k(z_0)|^{-\beta} + \sum_{k=q_{n+1}-i_0}^{q_{n+1}} |DT^k(z_0)|^{-\beta}. \quad (3.7)$$

In the case $i_0 = q_{n+1} - 1$ we put

$$S_n^{(1)}(\beta) = \sum_{k=1}^{q_{n+1}} |DT^k(z_0)|^{-\beta}. \quad (3.8)$$

If $0 \leqslant i_0 \leqslant q_{n+1} - 2$ we estimate $DT^k(z_0)$ for $1 \leqslant k \leqslant q_{n+1} - i_0 - 1$ and $q_{n+1} - i_0 \leqslant k \leqslant q_{n+1}$ separately.

If $1 \leqslant k \leqslant q_{n+1} - i_0 - 1$ we have

$$I_{k+i_0}^{(n)}(x_b) = \int\limits_{I_{i_0}^{(n)}(x_b)} DT^k(t)dt.$$

It is clear that

$$\frac{I_{k+i_0}^{(n)}(x_b)}{DT^k(z_0)} = \int\limits_{I_{i_0}^{(n)}(x_b)} \frac{DT^k(t)}{DT^k(z_0)} dt.$$

Applying Finzi's inequality (2.4), we obtain

$$e^{-v} \frac{|I_{i_0}^{(n)}(x_b)|}{|I_{k+i_0}^{(n)}(x_b)|} \leqslant (DT^k(z_0))^{-1} \leqslant e^{v} \frac{|I_{i_0}^{(n)}(x_b)|}{|I_{k+i_0}^{(n)}(x_b)|}. \quad (3.9)$$

These inequalities imply

$$e^{-\beta v}|I_{i_0}^{(n)}(x_b)|^{\beta}\sum_{k=1}^{q_{n+1}-i_0-1}|I_{k+i_0}^{(n)}(x_b)|^{-\beta}\leqslant\sum_{k=1}^{q_{n+1}-i_0-1}|DT^k(z_0)|^{-\beta}\leqslant$$

$$\leqslant e^{\beta v}|I_{i_0}^{(n)}(x_b)|^{\beta}\sum_{k=1}^{q_{n+1}-i_0-1}|I_{k+i_0}^{(n)}(x_b)|^{-\beta}. \quad (3.10)$$

Next, consider the case $q_{n+1}-i_0<k\leqslant q_{n+1}$. Then, obviously,

$$DT^k(z_0)=DT^{q_{n+1}-i_0}(z_0)DT^{k-(q_{n+1}-i_0)}(z_{q_{n+1}-i_0}). \quad (3.11)$$

As for the estimate (3.9), we can show that

$$e^{-v}\frac{|I_{q_{n+1}}^{(n)}(x_b)|}{|I_{k+i_0}^{(n)}(x_b)|}\leqslant(DT^{k-(q_{n+1}-i_0)}(z_{q_{n+1}-i_0}))^{-1}\leqslant e^{v}\frac{|I_{q_{n+1}}^{(n)}(x_b)|}{|I_{k+i_0}^{(n)}(x_b)|}.$$

Since

$$e^{-v}|I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)|\leqslant|I_{k+i_0}^{(n)}(x_b)|=\int\limits_{I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)}DT^{q_{n+1}}(t)dt\leqslant e^{v}|I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)|$$

one finds

$$e^{-2v}\frac{|I_{q_{n+1}}^{(n)}(x_b)|}{|I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)|}\leqslant(DT^{k-(q_{n+1}-i_0)}(z_{q_{n+1}-i_0}))^{-1}\leqslant e^{2v}\frac{|I_{q_{n+1}}^{(n)}(x_b)|}{|I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)|}.$$

Inserting $k=q_{n+1}-i_0$ into (3.9), we get

$$e^{-v}\frac{|I_{i_0}^{(n)}(x_b)|}{|I_{q_{n+1}}^{(n)}(x_b)|}\leqslant(DT^{q_{n+1}-i_0}(z_0))^{-1}\leqslant e^{v}\frac{|I_{i_0}^{(n)}(x_b)|}{|I_{q_{n+1}}^{(n)}(x_b)|}.$$

The last inequalities together with (3.11) imply for $q_{n+1}-i_0\leqslant k\leqslant q_{n+1}$

$$e^{-3v}\frac{|I_{i_0}^{(n)}(x_b)|}{|I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)|}\leqslant(DT^k(z_0))^{-1}\leqslant e^{3v}\frac{|I_{i_0}^{(n)}(x_b)|}{|I_{k-(q_{n+1}-i_0)}^{(n)}(x_b)|}. \quad (3.12)$$

In the case $i_0=q_{n+1}-1$ one shows for $1\leqslant k\leqslant q_{n+1}$ as above the bounds

$$e^{-2v}\frac{|I_{q_{n+1}-1}^{(n)}(x_b)|}{|I_{k-1}^{(n)}(x_b)|}\leqslant(DT^k(z_0))^{-1}\leqslant e^{2v}\frac{|I_{q_{n+1}-1}^{(n)}(x_b)|}{|I_{k-1}^{(n)}(x_b)|}.$$

From (3.10), (3.12) and the last inequality we get the bounds for $S_n^{(1)}(\beta)$. Similarly, we can derive the following estimate:

$$e^{-3v\beta}|I_{i_0}^{(n)(x_b)}|^{\beta}\sum_{s=0}^{q_n-1}|I_s^{(n+1)}(x_b)|^{-\beta}\leqslant S_n^{(2)}(\beta)\leqslant e^{3v\beta}|I_{i_0}^{(n)(x_b)}|^{\beta}\sum_{s=0}^{q_n-1}|I_s^{(n+1)}(x_b)|^{-\beta}.$$

Using (3.5) and the estimates for $S_n^{(1)}(\beta)$ respectively the ones for $S_n^{(2)}(\beta)$, we get the assertion of Lemma 4. □

Next, we estimate the sum $S_{n,\beta}(z_0)$.

Let $n > m > 0$. Consider the mth renormalization interval $J_m(x_b) = [x_{q_{m+1}}, x_{q_m}) = I_0^{(m+1)}(x_b) \cup I_0^{(m)}(x_b)$.

Define

$$\widetilde{\tau}_{n-m}(x_b) := P_n(T, x_b) \cap J_m(x_b),$$
$$\widetilde{\tau}_{n-m}^{(m-1)}(x_b) := P_n(T, x_b) \cap [x_{q_{m+1}}, x_b),$$
$$\widetilde{\tau}_{n-m}^{(m)}(x_b) := P_n(T, x_b) \cap [x_b, x_{q_m}).$$

Obviously, $\widetilde{\tau}_{n-m}(x_b)$ is a partition of $J_m(x_b)$. We will compare $S_{n,\beta}(z_0)$ with the sum of the lengths of the intervals of this partition $\widetilde{\tau}_{n-m}(x_b)$.

$$\begin{aligned} S_{n,\beta}(z_0) = {} & |I_0^{(n)}(x_b;\, z_0)|^{\beta} \left(\sum_{I_{i_k}^{(n)}(x_b)\in\widetilde{\tau}_{n-m}^{(m+1)}} |I_{i_k}^{(n)}(x_b)|^{-\beta} + \sum_{I_{j_k}^{(n+1)}(x_b)\in\widetilde{\tau}_{n-m}^{(m+1)}} |I_{j_k}^{(n+1)}(x_b)|^{-\beta} \right) + \\ & + |I_0^{(n)}(x_b;\, z_0)|^{\beta} \left(\sum_{I_{i_k}^{(n)}(x_b)\in\widetilde{\tau}_{n-m}^{(m)}} |I_{i_k}^{(n)}(x_b)|^{-\beta} + \sum_{I_{j_k}^{(n+1)}(x_b)\in\widetilde{\tau}_{n-m}^{(m)}} |I_{j_k}^{(n+1)}(x_b)|^{-\beta} \right) + \\ & + |I_0^{(n)}(x_b;\, z_0)|^{\beta} \sum_{s=1}^{q_m-1} \left(\sum_{I_{i_k}^{(n)}(x_b)\in\widetilde{\tau}_{n-m}^{(m+1)}} |T^s(I_{i_k}^{(n)}(x_b))|^{-\beta} + \sum_{I_{j_k}^{(n+1)}(x_b)\in\widetilde{\tau}_{n-m}^{(m+1)}} |T^s(I_{j_k}^{(n+1)}(x_b))|^{-\beta} \right) + \\ & + |I_0^{(n)}(x_b;\, z_0)|^{\beta} \sum_{s=1}^{q_{m+1}-1} \left(\sum_{I_{i_k}^{(n)}(x_b)\in\widetilde{\tau}_{n-m}^{(m)}} |T^s(I_{i_k}^{(n)}(x_b))|^{-\beta} + \sum_{I_{j_k}^{(n+1)}(x_b)\in\widetilde{\tau}_{n-m}^{(m)}} |T^s(I_{j_k}^{(n+1)}(x_b))|^{-\beta} \right). \end{aligned} \tag{3.13}$$

We denote the first two sums in (3.13) by $|I_0^{(n)}(x_b;\, z_0)|^{\beta}\widetilde{D}_{n-m}(x_b,\, \beta)$ and the remaining part by $|I_0^{(n)}(x_b;\, z_0)|^{\beta}\widetilde{D}_{n-m}^{(1)}(x_b,\, \beta)$. Since $0 < \text{const} \leqslant T'(x) \leqslant \text{Const}$ for $x \in [x_b,\, x_b + 1]$, we have

$$0 < \text{const} \leqslant \min_{1\leqslant s\leqslant q_m} \inf_{S^1} DT^s(x) \leqslant \max_{1\leqslant s\leqslant q_m} \sup_{S^1} DT^s(x) \leqslant \text{Const}.$$

Hence, one finds for some constants c and C depending on n, m and z_0

$$c \cdot |I_0^{(n)}(x_b;\, z_0)|^{\beta}\widetilde{D}_{n-m}(x_b,\, \beta) \leqslant S_{n,\beta}(z_0) \leqslant C \cdot |I_0^{(n)}(x_b;\, z_0)|^{\beta}\widetilde{D}_{n-m}(x_b,\, \beta). \tag{3.14}$$

To estimate the sum $\widetilde{D}_{n-m}(x_b,\, \beta)$, we make use of the thermodynamic formalism.

For this consider the first return map $\pi_m\colon J_0^{(m)}(x_b) \to J_0^{(m)}(x_b)$ given by

$$\pi_m(x) = \begin{cases} T^{q_m}(x), & \text{if } x \in I_0^{(m+1)}(x_b), \\ T^{q_{m+1}}(x), & \text{if } x \in I_0^{(m)}(x_b). \end{cases}$$

We can pass from $[x_{q_{m+1}}, x_{q_m})$ to the unit circle $S^1 \equiv [0, 1)$ by the affine map $z\colon [x_{q_{m+1}}, x_{q_m}) \to \to [0, 1)$ given by

$$z = \frac{x - x_{q_{m+1}}}{x_{q_m} - x_{q_{m+1}}}, \quad x_{q_{m+1}} \leqslant x < x_{q_m}.$$

We rewrite the maps T^{q_m} and $T^{q_{m+1}}$ in normalized coordinates

$$f_m(z) := \frac{T^{q_m}(x(z)) - x_{q_{m+1}}}{x_{q_m} - x_{q_{m+1}}} = \frac{T^{q_m}(x_{q_{m+1}} + z(x_{q_m} - x_{q_{m+1}})) - x_{q_{m+1}}}{x_{q_m} - x_{q_{m+1}}},$$
$$g_m(z) := \frac{T^{q_{m+1}}(x(z)) - x_{q_{m+1}}}{x_{q_m} - x_{q_{m+1}}} = \frac{T^{q_{m+1}}(x_{q_{m+1}} + z(x_{q_m} - x_{q_{m+1}})) - x_{q_{m+1}}}{x_{q_m} - x_{q_{m+1}}},$$

where $0 \leqslant z < 1$.

With $a_m := \frac{x_b - x_{q_{m+1}}}{x_{q_m} - x_{q_{m+1}}}$ we define the map

$$T_m(z) := \begin{cases} f_m(z), & \text{if } 0 \leqslant z < a_m, \\ g_m(z), & \text{if } a_m \leqslant z < 1. \end{cases}$$

Since $T_m(0) = f_m(0) = g_m(1) = T_m(1)$ and $T_m(a_m-) = f_m(a_m) = 1$ respectively $T_m(a_m+) = = g_m(a_m) = 0$, the map T_m defines a homeomorphism of the circle $S^1 = [0, 1)$. Notice that the rotation number ρ_{T_m} of T_m equals the rotation number ρ_T which is the "golden mean", i. e.,

$$\rho_{T_m} = \frac{\sqrt{5} - 1}{2} = [1, 1, 1, \ldots].$$

Since $DT_m(a_m-) = Df_m(a_m-) = DT^{q_m}(x_b-)$ and $DT_m(a_m+) = DT^{q_{m+1}}(x_b+)$, one finds for the jump ratio $c^2_{T_m}(a_m) = \frac{DT_m(a_m-)}{DT_m(a_m+)}$ of the map T_m at the point $a_m \in S^1$:

$$c^2_{T_m}(a_m) = c^2_T(x_b) \prod_{s=0}^{q_{m-1}-1} DT(x_{q_m+s})^{-1}.$$

On the other hand, since $Df_m(0+) = DT^{q_m}(x_{q_{m+1}}+)$ respectively $Dg_m(1-) = DT^{q_{m+1}}(x_{q_m}-) = = DT^{q_m}(x_{q_{m+1}}-)DT^{q_{m-1}}(x_{q_m}-)$, one finds for the jump ratio $c^2_{T_m}(0) = \frac{DT_m(1-)}{DT_m(0+)}$ of T_m at the point $z = 0 \equiv 1$:

$$c^2_{T_m}(0) = \prod_{s=0}^{q_{m-1}-1} DT(x_{q_m+s}).$$

Hence, we get $c^2_{T_m}(a_m)c^2_{T_m}(0) = c^2_T(x_b)$, which means that the map $T_m\colon S^1 \to S^1$ in general has two break points, namely, at $z = a_m$ and $z = 0 \equiv 1$.

Hence, for the homeomorphism $T_m \in B(G_{br})$ the statements of Theorem 5 hold true and therefore to T_m there corresponds the same potential U_{br}.

Denote by $\tau_{n-m}(a_m)$ the $(n-m)$th dynamical partition of the circle S^1 determined by the point a_m and the map T_m^{n-m}, $n > m$. We denote the elements of the partition $\tau_{n-m}(a_m)$ by $\Delta_i^{(n-m+1)}$ and $\Delta_j^{(n-m)}$, i. e.,

$$\tau_{n-m}(a_m) = \{\Delta_i^{(n-m+1)},\ 0 \leqslant i < q_{n-m};\ \Delta_j^{(n-m)},\ 0 \leqslant j < q_{n-m+1}\}.$$

The affine map $z\colon [x_{q_{m+1}}, x_{q_m}] \to S^1$ induces a $1-1$ correspondence $\widetilde{z}\colon \widetilde{\tau}_{n-m}(x_b) \to \tau_{n-m}(a_m)$ of the intervals $\Delta^{(n-m)}$ of the partition $\widetilde{\tau}_{n-m}(x_b)$ and the intervals $I^{(n-m)})$ of the dynamical partition $\tau_{n-m}(a_m)$ of S^1 such that

$$|\Delta^{(n-m)}| = |\widetilde{z}(I^{(n-m)})| = \frac{|I^{(n-m)}|}{|[x_{q_{m-1}}, x_{q_m})|}.$$

Define $D_{n-m}(\beta)$ as the following sum:

$$D_{n-m}(\beta) := \sum_{i=0}^{q_{n-m}-1} |\Delta_i^{(n-m+1)}|^{-\beta} + \sum_{j=0}^{q_{n-m+1}-1} |\Delta_j^{(n-m)}|^{-\beta}.$$

Then, obviously,

$$D_{n-m}(\beta) = \frac{1}{|[x_{q_{m-1}}, x_{q_m})|}\widetilde{D}_{n-m}(x_b, \beta). \tag{3.15}$$

Using the same arguments as for an analogous result in [3, Theorem 2.4], it can be shown that

$$\lim_{n\to\infty} \frac{D_{n-m}(\beta)}{\lambda_{-\beta}^{n-m}} = r(\beta) > 0. \tag{3.16}$$

Summarizing (3.14), (3.15) and (3.16), we obtain

$$c\cdot r(\beta)|I_0^{(n)}(x_b;\, z_0)|^{\beta}\lambda_{-\beta}^{n-m} \leqslant S_{n,\beta}(z_0) \leqslant C\cdot r(\beta)|I_0^{(n)}(x_b;\, z_0)|^{\beta}\lambda_{-\beta}^{n-m}. \tag{3.17}$$

The last inequalities and Lemma 4 imply the statement of Theorem 6. □

Lemma 5. *For arbitrary real numbers β and δ with $1 \leqslant \delta < \beta$ the following inequality holds*:

$$\lambda_{-\beta}^{\delta} < \lambda_{-\delta}^{\beta},$$

where λ_t is the leading eigenvalue of the transfer operator D_t defined in (2.8).

Proof. Using the bounds (3.1), we obtain

$$\frac{c_1}{C_1}\left(\frac{\lambda_{-\beta}^{\delta}}{\lambda_{-\delta}^{\beta}}\right)^{n-m} \leqslant \frac{(\Lambda_\beta(x,\, q_n))^{\delta}}{(\Lambda_\delta(x,\, q_n))^{\beta}} \leqslant \frac{C_1}{c_1}\left(\frac{\lambda_{-\beta}^{\delta}}{\lambda_{-\delta}^{\beta}}\right)^{n-m}. \tag{3.18}$$

We will show

$$\lim_{n\to\infty} \frac{(\Lambda_\beta(x,\, q_n))^{\delta}}{(\Lambda_\delta(x,\, q_n))^{\beta}} = 0.$$

This together with (3.18) proves $\lambda_{-\beta}^{\delta} < \lambda_{-\delta}^{\beta}$ and hence Lemma 5.

Using Lemma 4, we have

$$0 \leqslant \lim_{n\to\infty} \frac{(\Lambda_\beta(x,\, q_n))^{\delta}}{(\Lambda_\delta(x,\, q_n))^{\beta}} \leqslant \mathrm{Const} \lim_{n\to\infty} \frac{(S_{n,\beta}(x))^{\delta}}{(S_{n,\delta}(x))^{\beta}} =$$

$$= \text{Const} \lim_{n\to\infty} \frac{|I_0^{(n)}(x_b;\, z_0)|^{\beta\delta} \left(\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\beta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\beta} \right)^{\delta}}{|I_0^{(n)}(x_b;\, z_0)|^{\beta\delta} \left(\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\delta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\delta} \right)^{\beta}} =$$

$$= \text{Const} \lim_{n\to\infty} \left(\frac{\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\beta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\beta}}{\left(\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\delta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\delta} \right)^{\frac{\beta}{\delta}}} \right)^{\delta}. \tag{3.19}$$

For any $\beta \in R^1$ we denote

$$\widetilde{S}_{n,\beta} = \sum_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\beta} + \sum_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\beta}.$$

We have

$$\lim_{n\to\infty} \left(\frac{\widetilde{S}_{n,\beta}}{\widetilde{S}_{n,\delta} \cdot \widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}} \right)^{\delta} = \lim_{n\to\infty} \left(\frac{\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\beta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\beta}}{\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\delta} \widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\delta} \widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}} \right)^{\delta}. \tag{3.20}$$

Next, for every $0 \leqslant s < q_{n+1}$

$$\frac{|I_s^{(n)}(x_b)|^{-\beta}}{|I_s^{(n)}(x_b)|^{-\delta} \widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}} =$$

$$= \left(\frac{|I_s^{(n)}(x_b)|^{-\delta}}{\widetilde{S}_{n,\delta}} \right)^{(\beta-\delta)/\delta} = \left(\frac{|I_s^{(n)}(x_b)|^{-\delta}}{\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\delta} + \sum\limits_{k=0}^{q_n-1} |I_k^{(n+1)}(x_b)|^{-\delta}} \right)^{(\beta-\delta)/\delta} <$$

$$< \left(\frac{|I_s^{(n)}(x_b)|^{-\delta}}{\sum\limits_{s=0}^{q_{n+1}-1} |I_s^{(n)}(x_b)|^{-\delta}} \right)^{(\beta-\delta)/\delta} < \left(\frac{1}{ne^{-v\delta}} \right)^{(\beta-\delta)/\delta} = \frac{e^{v(\beta-\delta)}}{n^{(\beta-\delta)/\delta}}.$$

Here we have used the assertion of Lemma 2.

Thus, we have

$$|I_s^{(n)}(x_b)|^{-\beta} < \frac{e^{v(\beta-\delta)}}{n^{(\beta-\delta)/\delta}} |I_s^{(n)}(x_b)|^{-\delta} \widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}. \tag{3.21}$$

Analogously, we can show, for $0 \leqslant k < q_n$,

$$|I_k^{(n+1)}(x_b)|^{-\beta} < \frac{e^{v(\beta-\delta)}}{n^{(\beta-\delta)/\delta}} |I_k^{(n+1)}(x_b)|^{-\delta} \widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}. \tag{3.22}$$

Using the bounds (3.21) and (3.22), we have

$$\left(\frac{\sum\limits_{s=0}^{q_{n+1}-1}|I_s^{(n)}(x_b)|^{-\beta}+\sum\limits_{k=0}^{q_n-1}|I_k^{(n+1)}(x_b)|^{-\beta}}{\sum\limits_{s=0}^{q_{n+1}-1}|I_s^{(n)}(x_b)|^{-\delta}\widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}+\sum\limits_{k=0}^{q_n-1}|I_k^{(n+1)}(x_b)|^{-\delta}\widetilde{S}_{n,\delta}^{(\beta-\delta)/\delta}}\right)^{\delta}<\frac{e^{v\delta(\beta-\delta)}}{n^{\beta-\delta}}.$$

This bound and relations (3.19) and (3.20) prove

$$\lim_{n\to\infty}\frac{(\Lambda_\beta(x,\,q_n))^\delta}{(\Lambda_\delta(x,\,q_n))^\beta}=0$$

and hence also Lemma 5. □

Lemma 6. *For any* $z_0\in S^1\setminus\{T^i(x_b),\ i=0,\,-1,\,-2,\,\ldots,\,-q_n-q_{n+1}+1\}$, *and* $n\geqslant 1$

$$\widehat{\Lambda}(z_0,\,q_{n+1})\leqslant \mathrm{Const}\cdot n\cdot\lambda_{-1}^n, \tag{3.23}$$

where the positive constant Const *depends only on the total variation of* $\log T'$.

Proof. One can easily see (see [9, Remark 2.1]) that

$$\Lambda_1(z_0,\,n+t)=|DT^t(z_n)|\Lambda_1(z_0,\,n)+\Lambda_1(z_n,\,t). \tag{3.24}$$

Let $1\leqslant i_n\leqslant q_{n+1}$ realize the maximum of

$$\widehat{\Lambda}(z_0,\,q_{n+1})=\max_{1\leqslant i\leqslant q_{n+1}-1}\sum_{k=1}^{i}\prod_{j=k}^{i}|T'(z_j)|=\sum_{k=1}^{i_n-1}\prod_{j=k}^{i_n-1}\left|T'(z_j)\right|=\Lambda_1(z_0,\,i_n)-1,$$

and decompose it as

$$i_n=b_{l_m}q_{l_m}+b_{l_{m-1}}q_{l_{m-1}}+\cdots+b_{l_1}q_{l_1},$$

where $1\leqslant b_j\leqslant k_j$, $0\leqslant l_1<l_2<\cdots<l_m\leqslant n$, since $\rho=[k_0,\,k_1,\,\ldots,\,k_n,\,\ldots]$ is of bounded type.

$$\begin{aligned}\Lambda_1(z_0,\,i_n)&=\Lambda_1(z_0,\,b_{l_m}q_{l_m}+b_{l_{m-1}}q_{l_{m-1}}+\cdots+b_{l_1}q_{l_1})=\\&=|DT^{i_n-q_{l_m}}(z_{q_{l_m}})|\Lambda_1(z_0,\,q_{l_m})+\Lambda_1(z_{q_{l_m}},\,i_n-q_{l_m})=\\&=\sum_{j=1}^{b_{l_m}}|DT^{i_n-jq_{l_m}}(z_{j\cdot q_{l_m}})|\Lambda_1(z_{(j-1)\cdot q_{l_m}},\,q_{l_m})+\Lambda_1(z_{b_{l_m}\cdot q_{l_m}},\,i_n-b_{l_m}\cdot q_{l_m})=\\&=\sum_{j=1}^{b_{l_m}}|DT^{i_n-jq_{l_m}}(z_{j\cdot q_{l_m}})|\Lambda_1(z_{(j-1)\cdot q_{l_m}},\,q_{l_m})+\\&+\sum_{j=1}^{b_{l_{m-1}}}|DT^{i_n-b_{l_m}q_{l_m}-jq_{l_{m-1}}}(z_{b_{l_m}\cdot q_{l_m}+j\cdot q_{l_{m-1}}})|\Lambda_1(z_{b_{l_m}\cdot q_{l_m}+(j-1)\cdot q_{l_{m-1}}},\,q_{l_{m-1}})+\cdots+\\&+\sum_{j=1}^{b_{l_1}}|DT^{(b_{l_1}-j)q_{l_1}}(z_{i_n-(b_{l_1}-j)}\cdot q_{l_1})|\Lambda_1(z_{i_n-(b_{l_1}-j+1)\cdot q_{l_1}},\,q_{l_1}).\end{aligned} \tag{3.25}$$

Applying Theorem 6, we obtain

$$\Lambda_1(z_0, i_n) \leqslant C_1\lambda_{-1}^{l_m}\sum_{j=1}^{b_{l_m}}|DT^{i_n-jq_{l_m}}(z_{j\cdot q_{l_m}})||I^{(l_m)}(x_b;\, z_{(j-1)\cdot q_{l_m}})|+$$
$$+\,C_1\lambda_{-1}^{l_{m-1}}\sum_{j=1}^{b_{l_{m-1}}}|DT^{i_n-b_{l_m}q_{l_m}-jq_{l_{m-1}}}(z_{b_{l_m}\cdot q_{l_m}+j\cdot q_{l_{m-1}}})|I^{(l_{m-1})}(x_b;\, z_{b_{l_m}\cdot q_{l_m}+(j-1)\cdot q_{l_{m-1}}})|+\dots+$$
$$+\,C_1\lambda_{-1}^{l_1}\sum_{j=1}^{b_{l_1}}|DT^{(b_{l_1}-j)q_{l_1}}(z_{i_n-(b_{l_1}-j)q_{l_1}})|I^{(l_1)}(x_b;\, z_{i_n-(b_{l_1}-j+1)q_{l_1}})|. \qquad (3.26)$$

Next, we estimate

$$DT^{i_n-N}(z_N)|I^{(l_j)}(x_b;\, z_{N-q_{l_j}})| =$$
$$= \frac{DT^{i_n-N}(z_N)|I^{(l_j)}(x_b;\, z_N)|}{\int\limits_{I^{(l_j)}(x_b;\, z_N)} DT^{i_n-N}(t)dt}\cdot\frac{|I^{(l_j)}(x_b;\, z_{N-q_{l_j}})|}{|I^{(l_j)}(x_b;\, z_N)|}\int\limits_{I^{(l_j)}(x_b;\, z_N)} DT^{i_n-N}(t)dt$$

for $i_n < q_{n+1}$, $N < q_{n+1}$ and $1 \leqslant j \leqslant m$. Using Finzi's inequality, we get

$$\frac{DT^{i_n-N}(z_N)|I^{(l_j)}(x_b;\, z_N)|}{\int\limits_{I^{(l_j)}(x_b;\, z_N)} DT^{i_n-N}(t)dt} = \frac{|I^{(l_j)}(x_b;\, z_N)|}{\int\limits_{I^{(l_j)}(x_b;\, z_N)}\frac{DT^{i_n-N}(t)}{DT^{i_n-N}(z_N)}dt} \leqslant \mathrm{Const}_1.$$

Using Denjoy's inequality, we obtain

$$\frac{|I^{(l_j)}(x_b;\, z_{N-q_{l_j}})|}{|I^{(l_j)}(x_b;\, z_N)|} \leqslant \mathrm{Const}_2$$

and

$$\int\limits_{I^{(l_j)}(x_b;\, z_N)} DT^{i_n-N}(t)dt < 1,$$

where Const_1 and Const_2 only depend on the total variation of $\ln DT$. Since ρ is of bounded type, $b_{l_j} \leqslant k_{l_j} \leqslant \mathrm{Const}_3$.

From these inequalities and (3.26) one obtains

$$\Lambda_1(z_0, i_n) < \lambda_{-1}^{l_m}C_1\cdot\mathrm{Const}_1\cdot\mathrm{Const}_2\sum_{j=1}^{m}b_{l_j} \leqslant \lambda_{-1}^{l_m}C_1\cdot\mathrm{Const}_1\cdot\mathrm{Const}_2\cdot\mathrm{Const}_3\cdot m \leqslant \mathrm{Const}\cdot n\cdot\lambda_{-1}^{n}.$$

□

4. Barycentric coefficients

A universal bound for $T^n\colon S^1 \to S^1$ is a constant that does not depend on n and the point $y \in S^1$.

Let $[a, b]$ be an interval in S^1 and $c \in [a, b]$. The barycentric coefficient of c in $[a, b]$ is the ratio $\frac{|[a, c]|}{|[a, b]|}$, where $|I|$ denotes the length of the interval I.

The intervals $[a, b]$, $[b, c]$ and $[a, c]$ are comparable with each other if and only if the barycentric coefficient of c in $[a, b]$ is universally bounded in $(0, 1)$.

We denote by $I^{(n)}(z)$ the interval of the nth dynamical partition $P_n(T, x_0)$, $x_0 \in S^1$, containing z.

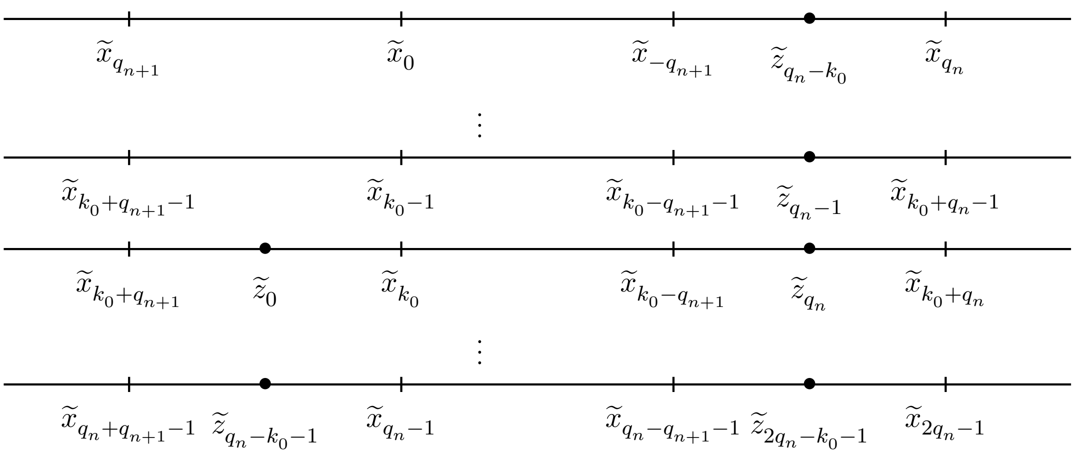
Proposition 1. *Let T be a circle homeomorphism with an irrational rotation number ρ of bounded type satisfying the conditions of Lemma 1. Suppose $x_0 := x_b \in S^1$ is the unique break point of T. Choose a point $z_0 \in S^1$ whose orbit $\mathbb{O}_T(z_0)$ is disjoint from the one of x_0. There exists a subsequence of integers $\{n_m,\ m = 1, 2, \ldots\}$ such that the barycentric coefficient of the points $z_k := T^k z_0$ in $I^{(n_m)}(z_k)$, $0 \leqslant k < q_{n_m+1}$ is universally bounded in $(0, 1)$, in other words, there exists a universal constant $0 < K_1 < 1$ such that the length of each of the two connected components of $I^{(n_m)}(z_k) \setminus z_k$ is at least $K_1|I^{(n_m)}(z_k)|$.*

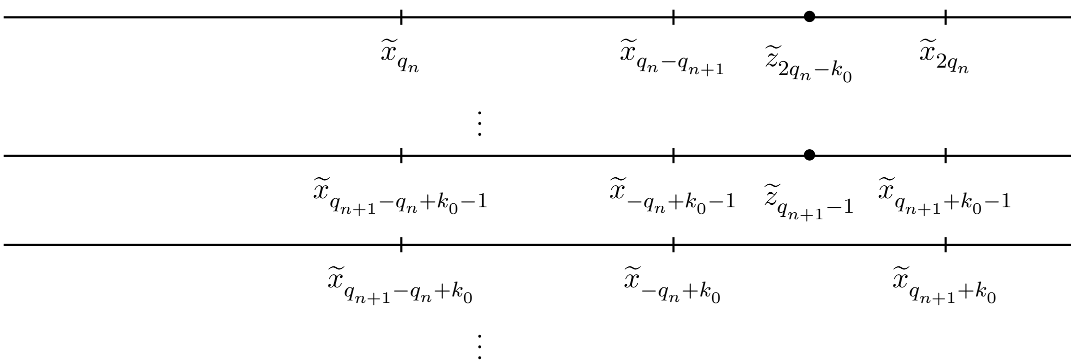
Let $T_\rho x = x+\rho \mod 1$ be the rotation of S^1 by the irrational angle $\rho \in (0, 1)$. Suppose that the orbits $\mathbb{O}_{T_\rho}(\widetilde{x}_0)$, $\mathbb{O}_{T_\rho}(\widetilde{z}_0)$ of $\widetilde{x}_0 := x_0 = x_b$ and $\widetilde{z}_0 = z_0$ are disjoint. Consider the dynamical partition $P_n(T_\rho, \widetilde{x}_0)$ of S^1 determined by $\widetilde{x}_0$ and T_ρ and let $\widetilde{z}_k = T_\rho^k \widetilde{z}_0$, $\widetilde{x}_k = T_\rho^k \widetilde{x}_0$, $0 \leqslant k < q_{n+1}$. Then $\widetilde{z}_0 \in \widetilde{I}_{k_0}^{(n+1)}(\widetilde{x}_0) = T_\rho^{k_0}[\widetilde{x}_{q_{n+1}}, \widetilde{x}_0)$ for some $0 \leqslant k_0 < q_n$ or $\widetilde{z}_0 \in \widetilde{I}_{i_0}^{(n)}(\widetilde{x}_0) = T_\rho^{i_0}([\widetilde{x}_0, \widetilde{x}_{-q_{n+1}})\cup$ $\cup[\widetilde{x}_{-q_{n+1}}, \widetilde{x}_{q_n}))$ for some $0 \leqslant i_0 < q_{n+1}$, i.e., $\widetilde{z}_0 \in T_\rho^{i_0}([\widetilde{x}_0, \widetilde{x}_{-q_{n+1}}))$ or $\widetilde{z}_0 \in T_\rho^{i_0}([\widetilde{x}_{-q_{n+1}}, \widetilde{x}_{q_n}))$.

In the following we need some lemmas describing the location of the points $\widetilde{z}_k$, $0 \leqslant k < q_{n+1}$, in intervals of $P_n(T_\rho, \widetilde{x}_0)$ which we formulate next.

Lemma 7 (Case 1). *If $\widetilde{z}_0 \in \widetilde{I}_{k_0}^{(n+1)}(\widetilde{x}_0)$ for some $0 \leqslant k_0 < q_n$, then the points $\widetilde{z}_k$, $0 \leqslant k <$ $< q_{n+1}$, belong to the following intervals of the dynamical partition $P_n(T_\rho, \widetilde{x}_0)$:*

- $\widetilde{z}_k \in \widetilde{I}_{k_0+k}^{(n+1)}(\widetilde{x}_0)$, $0 \leqslant k < q_n - k_0$,
- $\widetilde{z}_k \in \widetilde{I}_{k-q_n+k_0}^{(n)}(\widetilde{x}_0)$, $q_n - k_0 \leqslant k < q_{n+1}$.





Lemma 8 (Case 2). *Let $\widetilde{z}_0 \in T_\rho^{i_0}([\widetilde{x}_0, \widetilde{x}_{-q_{n+1}}))$ for some $0 \leqslant i_0 < q_{n+1} - q_n$. Then the points $\widetilde{z}_k$, $0 \leqslant k < q_{n+1}$, belong to the following intervals of the dynamical partition $P_n(T_\rho, \widetilde{x}_0)$:*

- $\widetilde{z}_k \in T_\rho^{k+i_0}([\widetilde{x}_0, \widetilde{x}_{-q_{n+1}})) \subset \widetilde{I}^{(n)}_{k+i_0}(\widetilde{x}_0)$, $0 \leqslant k < q_{n+1} - i_0$;
- $\widetilde{z}_k \in \widetilde{I}^{(n+1)}_{k-q_{n+1}+i_0}(\widetilde{x}_0)$, $q_{n+1} - i_0 \leqslant k < q_{n+1}$.

Lemma 9 (Case 2′). *If $\widetilde{z}_0 \in T_\rho^{i_0}([\widetilde{x}_0, \widetilde{x}_{-q_{n+1}}))$ for some $q_{n+1} - q_n \leqslant i_0 < q_{n+1}$, then the points $\widetilde{z}_k$, $0 \leqslant k < q_{n+1}$, belong to the following intervals of the dynamical partition $P_n(T_\rho, \widetilde{x}_0)$:*

- $\widetilde{z}_k \in T_\rho^{k+i_0}([x_0, \widetilde{x}_{-q_{n+1}})) \subset \widetilde{I}^{(n)}_{k+i_0}(\widetilde{x}_0)$, $0 \leqslant k < q_{n+1} - i_0$,
- $\widetilde{z}_k \in \widetilde{I}^{(n+1)}_{k-q_{n+1}+i_0}(\widetilde{x}_0)$, $q_{n+1} - i_0 \leqslant k < q_{n+1} + q_n - i_0$,
- $\widetilde{z}_k \in \widetilde{I}^{(n)}_{k-q_{n+1}-q_n+i_0}(\widetilde{x}_0)$, $q_{n+1} + q_n - i_0 \leqslant k < q_{n+1}$.

Lemma 10 (Case 3). *If $\widetilde{z}_0 \in T_\rho^{i_0}([\widetilde{x}_{-q_{n+1}}, \widetilde{x}_{q_n}))$ for some $0 \leqslant i_0 < q_{n+1}$, then the points of $\widetilde{z}_k$, $0 \leqslant k < q_{n+1}$, belong to the following intervals of the dynamical partition $P_n(T_\rho, \widetilde{x}_0)$:*

- $\widetilde{z}_k \in T_\rho^{k+i_0}([\widetilde{x}_{-q_{n+1}}, \widetilde{x}_{q_n})) \subset \widetilde{I}^{(n)}_{k+i_0}(\widetilde{x}_0)$, $0 \leqslant k < q_{n+1} - i_0$,
- $\widetilde{z}_k \in T_\rho^{k+i_0}([\widetilde{x}_{-q_{n+1}}, \widetilde{x}_{q_n})) \subset \widetilde{I}^{(n)}_{k-q_{n+1}+i_0}(\widetilde{x}_0)$, $q_{n+1} - i_0 \leqslant k < q_{n+1}$.

Denote by $\widetilde{I}^{(n)}(\widetilde{z}_k)$ the interval of $P_n(T_\rho, \widetilde{x}_0)$ which contains $\widetilde{z}_k$.

Lemma 11. *Let $T_\rho x = x + \rho \mod 1$ be the rotation of the circle with ρ irrational of bounded type. Choose a point $\widetilde{z}_0 \in S^1$ such that the orbits $\mathbb{O}_{T_\rho}(\widetilde{x}_0)$ and $\mathbb{O}_{T_\rho}(\widetilde{z}_0)$ are disjoint. Then there exists a subsequence $\{n_m\}$ of integers such that the barycentric coefficient of the points $\widetilde{z}_k$ in $\widetilde{I}^{(n_m)}(\widetilde{z}_k) \in P_{n_m}(T_\rho, \widetilde{x}_0)$, $0 \leqslant k < q_{n_m+1}$ is universally bounded in* $(0, 1)$.

Proof. Denote by $\widetilde{P}_n(T_\rho, \widetilde{x}_0)$ the partition generated by the points $\{\widetilde{x}_{-q_{n+1}}, \widetilde{x}_{-q_{n+1}+1}, \ldots, \widetilde{x}_0, \ldots, \widetilde{x}_{q_n+q_{n+1}-1}\}$. First, we prove that there exists a subsequence of positive integers $\{n_m\}$, such that the barycentric coefficient of the point $\widetilde{z}_0$ in the interval $\widetilde{\Delta}^{(n_m)}(\widetilde{z}_0) \in \widetilde{P}_{n_m}(T_\rho, \widetilde{x}_0)$ is universally bounded in $(0, 1)$.

Suppose on the contrary that there is no such subsequence.

Fact 1. *For sufficiently large n the point $\widetilde{z}_0$ is always at least in one of the two intervals $\widetilde{\Delta}_-^{(n+4)}$ and $\widetilde{\Delta}_+^{(n+4)}$ of $\widetilde{P}_{(n+4)}(T_\rho, \widetilde{x}_0)$ contained and intersecting the boundary of $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$.* But if $\widetilde{z}_0$ does not belong to both these intervals, the point $\widetilde{z}_0$ splits the interval $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$ into two intervals $\widetilde{\Delta}_+^{(n)}(\widetilde{z}_0)$ respectively $\widetilde{\Delta}_-^{(n)}(\widetilde{z}_0)$, the length of both these intervals being larger than $|\widetilde{\Delta}_-^{(n+4)}|$ respectively $|\widetilde{\Delta}_+^{(n+4)}|$. Hence, $\frac{|\widetilde{\Delta}_\pm^{(n+4)}|}{|\widetilde{\Delta}^{(n)}(\widetilde{z}_0)|} \leqslant \frac{|\widetilde{\Delta}_\pm^{(n)}(\widetilde{z}_0)|}{|\widetilde{\Delta}^{(n)}(\widetilde{z}_0)|} \leqslant 1 - \frac{|\widetilde{\Delta}_\pm^{(n+4)}|}{|\widetilde{\Delta}^{(n)}(\widetilde{z}_0)|}$ and therefore $0 < c < \frac{|\widetilde{\Delta}_\pm^{(n+4)}|}{|\widetilde{\Delta}^{(n)}(\widetilde{z}_0)|} < C < 1$ where these constants, c and C, do not depend on n. This shows that the barycentric coefficient of any $\widetilde{y}_0 \in \widetilde{\Delta}^{(n)}(\widetilde{z}_0)$ is universally bounded in $(0, 1)$.

Fact 2. *The point $\widetilde{z}_0$ cannot be contained for all n in the same of the two intervals $\widetilde{\Delta}_\pm^{(n+4)}$ of the partition $\widetilde{P}_{n+4}(T_\rho, \widetilde{x}_0)$ contained and intersecting the boundary of $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$.* Otherwise the point $\widetilde{z}_0$ must converge as $n \to \infty$ to the limit boundary point of the intervals $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$, which, however, belongs to the orbit $\mathbb{O}_{T_\rho}(\widetilde{x}_0)$ in contradiction to our assumption that the orbits of $\widetilde{z}_0$ and $\widetilde{x}_0$ are disjoint. Hence, there exists a subsequence n_m such that $\widetilde{z}_0$ belongs alternatively to an interval in $\widetilde{P}^{(n+4)}(\rho, \widetilde{x}_0)$ contained in and intersecting the left respectively the right boundary of $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$. Therefore, the point $\widetilde{z}_0$ must be separated from both the boundary points of $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$ by at least one interval in $\widetilde{P}^{(n+6)}(\rho, \widetilde{x}_0)$ contained in and intersecting the boundary of $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$. Otherwise there would be another subsequence n_l with $\widetilde{z}_{n_l}$ approaching the limit boundary point of the intervals $\widetilde{\Delta}^{(n_l)}(\widetilde{z}_0)$ which, however, is on $\mathbb{O}_{T_\rho}(\widetilde{x_0})$, contrary to our assumption. That shows that the barycentric coefficient of the point $\widetilde{z}_0$ in $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$ is universally bounded in $(0, 1)$. This holds true also for the interval $\widetilde{\Delta}^{(n)}(\widetilde{z}_0)$ replaced by the interval $\widetilde{I}^{(n)} \in P_n(T_\rho, \widetilde{x}_0)$ with $\widetilde{x}_0 = x_0$.

Next, we prove the barycentric coefficient of $\widetilde{z}_k$ in $\widetilde{I}^{(n)}(\widetilde{z}_k) \in P_n(T_\rho, \widetilde{x}_0)$ for all $0 \leqslant k \leqslant q_n - 1$ to be universally bounded in $(0, 1)$.

For the point $\widetilde{z}_0$ the following three cases are possible:

1) $\widetilde{z}_0 \in \widetilde{I}_{k_0}^{(n+1)}(\widetilde{x}_0)$, $0 \leqslant k_0 \leqslant q_n - 1$;

2) $\widetilde{z}_0 \in [T_\rho^{k_0}(\widetilde{x}_0), T_\rho^{k_0 - q_{n+1}}(\widetilde{x}_0)) \subset \widetilde{I}_{k_0}^{(n)}(\widetilde{x}_0)$, $0 \leqslant k_0 \leqslant q_{n+1} - 1$;

3) $\widetilde{z}_0 \in [T_\rho^{k_0 - q_{n+1}}(\widetilde{x}_0), T_\rho^{k_0 + q_n}(\widetilde{x_0})) \subset \widetilde{I}_{k_0}^{(n)}(\widetilde{x_0})$, $0 \leqslant k_0 \leqslant q_{n+1} - 1$.

Case 1. $\widetilde{z}_0 \in \widetilde{I}_{k_0}^{(n+1)}(\widetilde{x_0})$, $0 \leqslant k_0 \leqslant q_n - 1$ and $\widetilde{z}_0$ is separated from both boundary points of $\widetilde{I}_{k_0}^{(n+1)}(\widetilde{x_0})$. Then, for $0 \leqslant k \leqslant q_n - k_0 - 1$, $\widetilde{z}_k$ is also separated from the boundary points of $\widetilde{I}_{k+k_0}^{(n+1)}(\widetilde{x_0})$. For $q_n - k_0 \leqslant k \leqslant q_{n+1} - 1$, $\widetilde{z}_k$ is clearly separated from the boundary points of $T_\rho^k \widetilde{I}_{q_n - 1}^{(n+1)}(\widetilde{x_0})$. But since $T_\rho^k \widetilde{I}_{q_n-1}^{(n+1)}(\widetilde{x_0}) \subset \widetilde{I}^{(n)_{k-q_n-k_0}}(\widetilde{x_0})$, the barycentric coefficient of $\widetilde{z}_k$ in $\widetilde{I}^{(n)_{k-q_n-k_0}}(\widetilde{x_0})$ is universally bounded in $(0, 1)$.

Case 2. $\widetilde{z}_0 \in [T_\rho^{k_0}(\widetilde{x_0}), T_\rho^{k_0 - q_{n+1}}(\widetilde{x_0})) \subset \widetilde{I}_{k_0}^{(n)}(\widetilde{x_0})$, $0 \leqslant k_0 \leqslant q_{n+1} - 1$ and $\widetilde{z}_0$ is separated from the points $T_\rho^{k_0}(\widetilde{x_0})$ and $T_\rho^{k_0 - q_{n+1}}(\widetilde{x_0})$. Then, for $0 \leqslant k \leqslant q_{n+1} - k_0 - 1$, z_k is separated from the points $T_\rho^{k+k_0}(\widetilde{x_0})$ and $T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})$. Since $[T_\rho^{k+k_0}(\widetilde{x_0}), T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})) \subset \widetilde{I}_{k+k_0}^{(n)}(\widetilde{x_0})$, the barycentric coefficient of $\widetilde{z}_k$ in $\widetilde{I}_{k+k_0}^{(n)}(\widetilde{x_0})$ is universally bounded. For $q_{n+1} - k_0 \leqslant k \leqslant q_{n+1} - 1$, $\widetilde{z}_k$

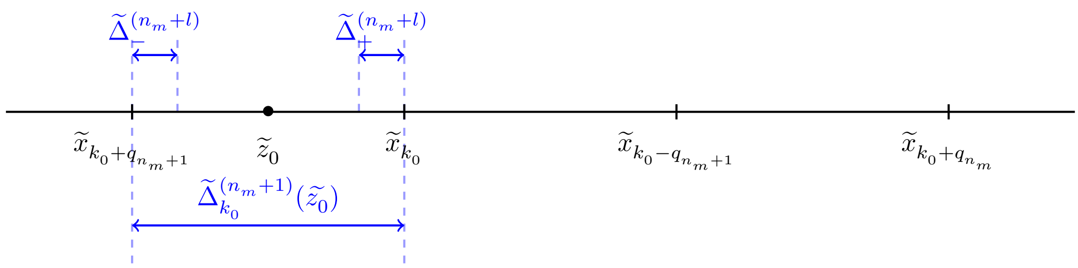
is separated from the points $T_\rho^{k+k_0}(\widetilde{x_0})$ and $T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})$, but $[T_\rho^{k+k_0}(\widetilde{x_0}), T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})) = \widetilde{I}^{(n+1)}_{k+k_0-q_{n+1}}(\widetilde{x_0})$. Then the barycentric coefficient of $\widetilde{z}_k$ in $\widetilde{I}^{(n+1)}_{k+k_0-q_{n+1}}(\widetilde{x_0})$ is universally bounded in $(0, 1)$.

Case 3. $\widetilde{z}_0 \in [T_\rho^{k_0-q_{n+1}}(\widetilde{x_0}), T_\rho^{k_0+q_n}(\widetilde{x_0})) \subset \widetilde{I}^{(n)}_{k_0}(\widetilde{x_0})$, $0 \leqslant k_0 \leqslant q_{n+1}-1$ and $\widetilde{z}_0$ is separated from the points $T_\rho^{k_0-q_{n+1}}(\widetilde{x_0})$ and $T_\rho^{k_0+q_n}(\widetilde{x_0})$. For $0 \leqslant k \leqslant q_{n+1}-k_0-1$, $\widetilde{z}_k$ is separated from the points $T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})$ and $T_\rho^{k+q_n}(\widetilde{x_0})$. Since $[T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0}), T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})) \subset \widetilde{I}^{(n)}_{k+k_0}(\widetilde{x_0})$, the barycentric coefficient of $\widetilde{z}_k$ in $\widetilde{I}^{(n)}_{k+k_0}(\widetilde{x_0})$ is universally bounded. For $q_{n+1}-k_0 \leqslant k \leqslant q_{n+1}-1$, $\widetilde{z}_k$ is separated from the points $T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0})$ and $T_\rho^{k+q_n}(\widetilde{x_0})$, but $[T_\rho^{k+k_0-q_{n+1}}(\widetilde{x_0}), T_\rho^{k+q_n}(\widetilde{x_0})) = \widetilde{I}^{(n)}_{k+k_0-q_{n+1}}(\widetilde{x_0})$. Then the barycentric coefficient of $\widetilde{z}_k$ in $\widetilde{I}^{(n+1)}_{k+k_0-q_{n+1}}(\widetilde{x_0})$ is universally bounded in $(0, 1)$. □

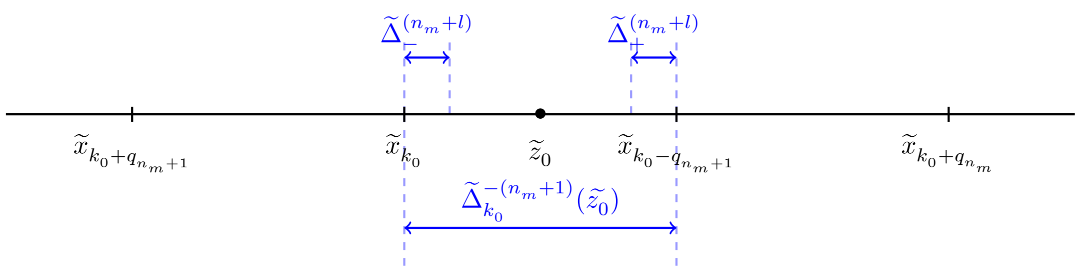
We proved in Lemma 11 that there exists a subsequence of positive integers $\{n_m\}$, such that the barycentric coefficient of the point $\widetilde{z}_0$ in the interval $\widetilde{\Delta}^{(n_m)}(\widetilde{z}_0) \in \widetilde{P}_{n_m}(T_\rho, \widetilde{x}_0)$ is universally bounded in $(0, 1)$.

This means that there exists a positive integer $l := l(z_0)$, such that the point $\widetilde{z}_0$ cannot be in the two intervals $\widetilde{\Delta}_-^{(n_m+l)}$ and $\widetilde{\Delta}_+^{(n_m+l)}$ of $\widetilde{P}_{(n_m+l)}(T_\rho, \widetilde{x}_0)$ contained in and intersecting the boundary of $\widetilde{\Delta}^{(n_m)}(\widetilde{z}_0)$. There are then three possible cases:

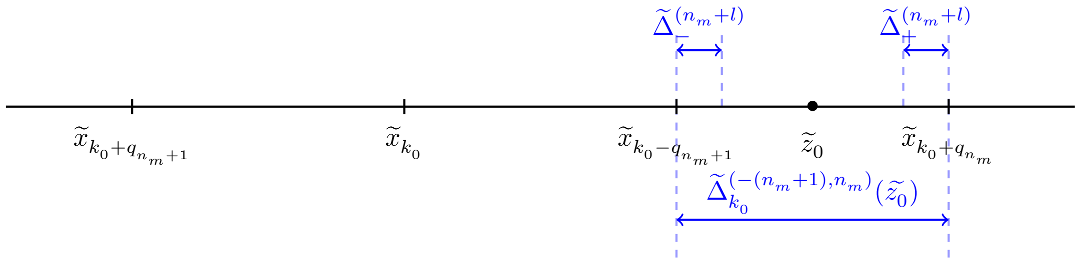
Case 1. $\widetilde{z}_0 \in \widetilde{\Delta}^{(n_m+1)}_{k_0}(\widetilde{z}_0) := T_\rho^{k_0}[\widetilde{x}_{q_{n_m+1}}, \widetilde{x}_0] \in \widetilde{P}_{n_m}(T_\rho, \widetilde{x}_0)$, for some k_0, $0 \leqslant k_0 \leqslant q_{n_m}-1$.



Case 2. $\widetilde{z}_0 \in \widetilde{\Delta}^{-(n_m+1)}_{k_0} := T_\rho^{k_0}[\widetilde{x}_0, \widetilde{x}_{-q_{n_m+1}}] \in \widetilde{P}_{n_m}(T_\rho, \widetilde{x}_0)$ for some k_0, $0 \leqslant k_0 \leqslant q_{n_m+1}-1$. Obviously, $\widetilde{\Delta}^{-(n_m+1)}_{k_0} \subset \widetilde{\Delta}^{(n_m)}_{k_0}$.



Case 3. $\widetilde{z}_0 \in \widetilde{\Delta}^{-(n_m+1),n_m}_{k_0} := T_\rho^{k_0}[\widetilde{x}_{-q_{n_m+1}}, \widetilde{x}_{q_{n_m}}] \in \widetilde{P}_{n_m}(T_\rho, \widetilde{x_0})$ for some k_0, $0 \leqslant k_0 \leqslant q_{n_m+1}-1$. Obviously, $\widetilde{\Delta}^{-(n_m+1),n_m}_{k_0} \subset \widetilde{\Delta}^{(n_m)}_{k_0}$.



One also can say that there exists at least one interval of the partition $\widetilde{P}_{(n_m+l)}(T_\rho, \widetilde{x}_0)$ between the points $\widetilde{z}_k$ and $\widetilde{x}_{-j}$, $0 \leqslant j \leqslant q_{n_m+1} - 1$.

Lemma 12 (see [7]). *Let T be a circle homeomorphism with an irrational rotation number of bounded type satisfying the conditions of Lemma 1. Put $\theta_\pm := (1+e^{\pm v})^{-1/2}$ and let n and $l \geqslant 2$ be positive integers. Then there exist universal positive constants C_1, C_2, $C_1 < C_2$ such that for arbitrary $I^{(n)} \in P_n(T, x_0)$ and $I^{(n+l)} \in P_{n+l}(T, x_0)$ with $I^{(n+l)} \subset I^{(n)}$ the following bounds hold:*

$$C_1\theta_+^l \leqslant \frac{I^{(n+l)}}{I^{(n)}} \leqslant C_2\theta_-^l.$$

Proof of Proposition 1. Consider the dynamical partition $P_n(T, x_0)$ and the finite part $\mathbb{O}_T^{n+1}(z_0) = \{z_k, 0 \leqslant k < q_{n+1}\}$ of the orbit of z_0 under the map T. $I^{(n)}(z_k)$ denotes again the interval in $P_n(T, x_0)$ which contains z_k. Since T and T_ρ are topologically conjugate, Lemma 11 shows that there exists for $l \geqslant 4$ a subsequence $\{n_m\}$ in $\mathbb{N}$ such that the point z_k cannot be in the two boundary intervals of the partition $P_{n_m+l}(T, x_0)$ contained in $I^{(n_m)}(z_k)$. Hence, according to Lemma 12, the barycentric coefficient of z_k in $I^{(n_m)}(z_k)$ is universally bounded in (0, 1). □

5. Proof of Theorem 4

Take any point z_0 not belonging to the orbit $\mathbb{O}_T(x_b)$ of the break point x_b. Before giving the main steps in the proof of our main theorem, we emphasize an important point: a key ingredient in the proof of this theorem is the Taylor expansion of the process $\overline{z}_{q_{n+1}}(z_0, \sigma)$

$$\overline{z}_{q_{n+1}}(z_0, \sigma) = T^{q_{n+1}}(z_0) + \sigma L_{q_{n+1}}(z_0) + \sigma^2 Q_{q_{n+1}}(z_0, \sigma), \tag{5.1}$$

where the linear term $L_{q_{n+1}}$ is the sum of independent random variables as defined in (1.4). However, we can use Taylor's formula (5.1) only in case the neighborhood $U^{(n)}$ of the point $z_{q_{n+1}} = T^{q_{n+1}}(z_0)$ does not contain any break point of $T^{q_{n+1}}$.

Let us briefly sketch the main steps of the proof of Theorem 4, which will overcome this difficulty.

I. By Lemma 4.1, there exists an increasing sequence of natural numbers $\{n_m, m = 1, 2, \ldots\}$ such that for $0 \leqslant k < q_{n_m+1}$ the interval $I^{(n_m)}(z_k)$ of the partition $P^{(n_m)}(T, x_b)$ contains only the point $z_k := T^k(z_0)$ from the finite orbit $\{z_i = T^i z_0, 0 \leqslant i < q_{n_m+1}\}$. Moreover, the barycentric coordinates of the points z_k are strictly separated from 0 and 1 by universal constants, i. e., the points z_k are uniformly separated from the boundaries $\partial I^{(n_m)}(z_k)$.

For every $m \geqslant 1$ we will construct a series of neighborhoods $A_k^{n_m} \subset I^{(n_m)}(z_k)$, $0 \leqslant k < q_{n_m+1}$, of the points z_k, such that every interval $A_k^{n_m}$ does not contain any break point of $T^{q_{n_m+1}}$.

II. For the stochastic sequence

$$\overline{z}_k = T(\overline{z}_{k-1}) + \sigma_{q_{n_m+1}}\xi_k, \quad \overline{z}_0 = z_0$$

we show that the probabilities of the events

$$B_{n_m} := \{\overline{z}_1 \in A_1^{n_m},\, \overline{z}_2 \in A_2^{n_m},\, \ldots,\, \overline{z}_{q_{n_m+1}-1} \in A^{n_m}_{q_{n_m+1}-1}\}$$

tend to 1 as $m \to \infty$.

III. For fixed $z_0 \in S^1 \setminus \{x_b\}$ the Taylor expansion of the process $\overline{z}_{q_{n_m+1}}(z_0, \sigma_{q_{n_m+1}-1})$ in the variables $\xi_1, \xi_2, \ldots, \xi_{q_{n_m+1}-1}$ allows us to decompose it under the condition $\overline{z}_j \in A_j^{n_m}$, $1 \leqslant j \leqslant q_{n_m+1} - 1$, as

$$\overline{z}_{q_{n_m+1}-1}(z_0, \sigma_{q_{n_m+1}-1}) = \\ = T^{q_{n_m+1}-1}(z_0) + \sigma_{q_{n_m+1}-1}L_{q_{n_m+1}-1}(z_0) + \sigma^2_{q_{n_m+1}-1}Q_{q_{n_m+1}-1}(z_0, \sigma_{q_{n_m+1}-1}), \quad (5.2)$$

where the linear term $L_{q_{n_m+1}}$ is the sum of independent random variables defined by (1.4).

VI. We prove the CLT for this linear part $L_{q_{n_m+1}-1}$ which finally leads to the proof of Theorem 4.

To achieve this program, we formulate and prove in a first step several lemmas.

For this take a point $z_0 \in S^1 \setminus \mathbb{O}_T(x_b)$ and the sequence of increasing natural numbers n_m, $m = 1, 2, \ldots$ determined by Proposition 1. Consider the two partitions $\widetilde{P}_{n_m}(T, x_b)$ (generated by the points $\{x_{-q_{n_m+1}}, \ldots, x_0, \ldots, x_{q_{n_m}+q_{n_m+1}-1}\}$) respectively $\widetilde{P}_{n_m+l}(T, x_b)$. Each interval $\Delta^{(n_m)}(z_k) \in \widetilde{P}_{n_m}(T, x_b)$, $0 \leqslant k < q_{n_m+1}$ contains at least $q_l \geqslant q_4$ and hence at least three intervals of the partition $\widetilde{P}_{n_m+l}(T, x_b)$. By Proposition 1, the point z_k cannot be located in the intervals $\Delta_{\pm}^{n_m+l}$ of the partition $\widetilde{P}_{n_m+l}(T, x_t)$ contained in and intersecting the interval $\Delta^{(n_m)}(z_k)$.

Next, we introduce certain connected intervals $A_k^{(n_m)}$ composed of intervals of the partition $\widetilde{P}_{2n_m+l+1}(x_b)$ and containing the point z_k. For this let $\Delta^{(2n_m+l+1)}(z_0)$ be the element of the partition $\widetilde{P}_{2n_m+l+1}(x_b)$ which contains z_0. As above, there are again three possible cases for this interval: $\Delta^{(2n_m+l+1)}(z_0) = \Delta_{t_0}^{2n_m+l+2} = [x_{t_0+q_{2n_m+l+2}}, x_{t_0}]$, $\Delta^{(2n_m+l+1)}(z_0) = \Delta_{t_0}^{-(2n_m+l+2)} = [x_{t_0}, x_{t_0-q_{2n_m+l+2}}]$ respectively $\Delta^{(2n_m+l+1)}(z_0) = \Delta_{t_0}^{-(2n_m+l+2),2n_m+l+1} = [x_{t_0-q_{2n_m+l+2}}, x_{t_0+q_{2n_m+l+1}}]$. Thereby, $t_0 = t_0(k_0, l)$ is chosen such that $\Delta^{(2n_m+l+1)}(z_0) \subset \Delta_{k_0}^{n_m+1}(z_0) \setminus (\Delta_-^{n_m+l} \cup \Delta_+^{n_m+l})$. Then we define

- $A_0^{(n_m)} := \Delta^{(2n_m+l+1)}(z_0) \in \widetilde{P}_{2n_m+l+1}(x_b)$,
- for every k, $1 \leqslant k < q_{n_m+1}$, we set

$$A_k^{(n_m)} := A_k^-(n_m) \cup T(A_{k-1}^{(n_m)}) \cup A_k^+(n_m),$$

where $A_k^-(n_m)$ and $A_k^+(n_m)$ are the left and right neighbors of $T(A_{k-1}^{(n_m)})$ in the partition $\widetilde{P}_{2n_m+l+1}(x_b)$, respectively.

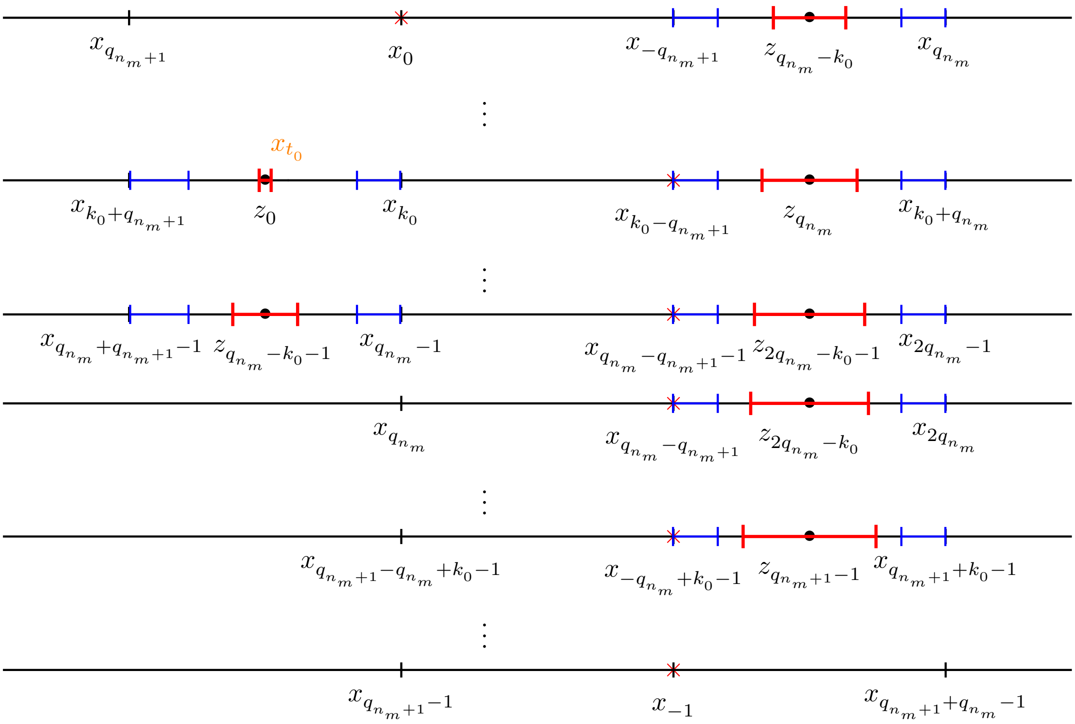
Lemma 13. $A_k^{(n_m)} \subset \Delta^{(n_m)}(z_k)$, *for all* $0 \leqslant k < q_{n_m+1}$.

Proof. Let the sequence $\{n_m, m = 1, 2, \ldots\}$ be determined by Lemma 4.1. Consider the partitions $\widetilde{P}_{n_m}(T, x_b)$, $\widetilde{P}_{n_m+l}(T, x_b)$ and $\widetilde{P}_{2n_m+l+1}(T, x_b)$. Recall that every interval of $\widetilde{P}_{n_m}(T, x_b)$ contains at least three intervals of $\widetilde{P}_{n_m+l}(T, x_b)$.

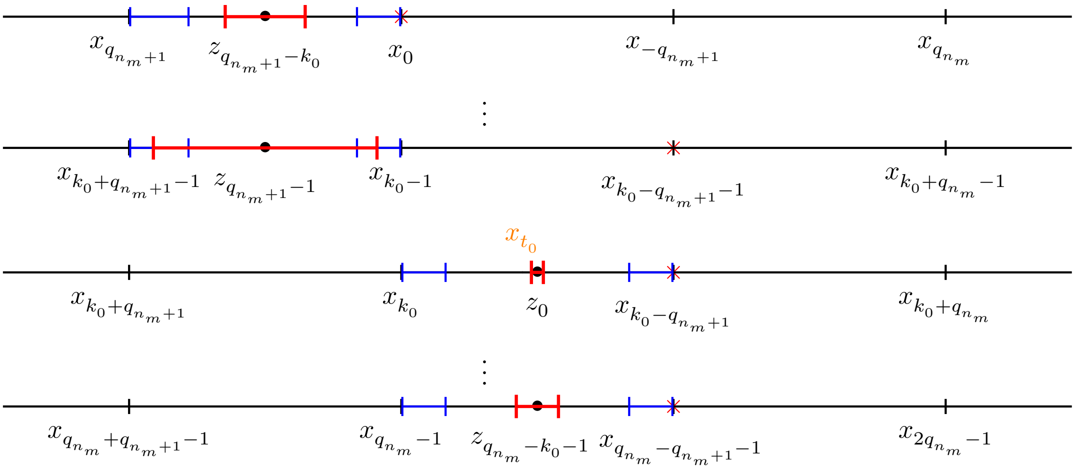
Furthermore, every interval of $\widetilde{P}_{n_m+l}(T, x_b)$ contains at least q_{n_m+1} intervals of $\widetilde{P}_{2n_m+l+1}(T, x_b)$. This implies that

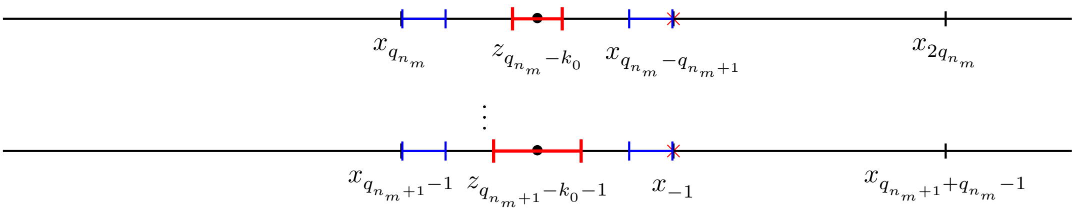
$$A_k^{(n_m)} \subset \Delta^{(n_m)}(z_k) \in P_{n_m}(T, x_b), \quad 0 \leqslant k < q_{n_m+1}. \qquad \square$$

The intervals $A_k^{(n_m)}$ in red in case I.

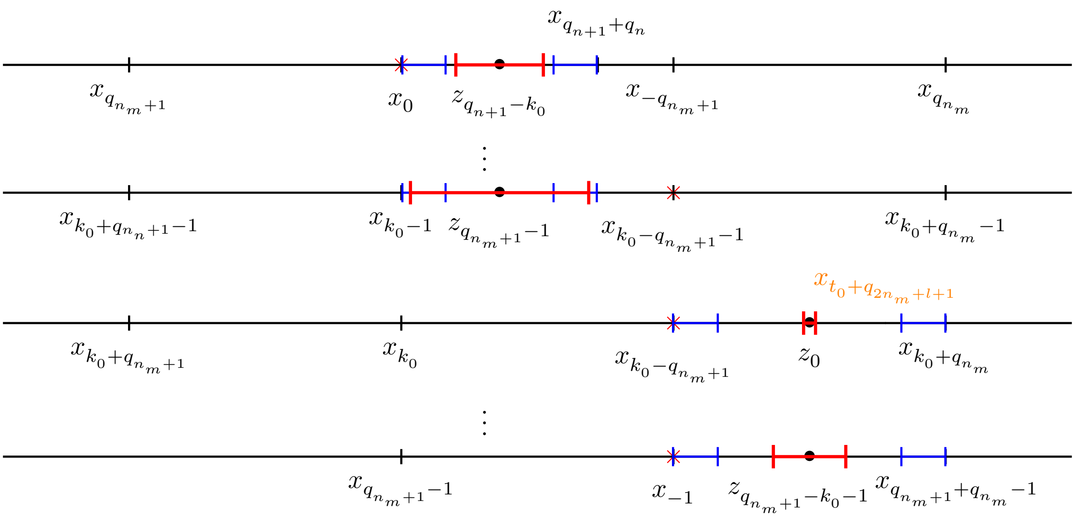


The intervals $A_k^{(n_m)}$ in red in case II.





The intervals $A_k^{n_m}$ in red in case III.



Lemma 14. *Let T be a circle homeomorphism, $(\xi_n)_{n=1}^{\infty}$ be a sequence of independent random variables satisfying the conditions of Theorem 4 and let $\theta_+ = (1+e^v)^{-1/2} < 1$.*

Suppose that the sequence $\left\{\sigma^2_{q_{n+1}},\ n \geqslant 1\right\}$ fulfills the condition

$$\lim_{n\to\infty} \frac{q_{n+1}\sigma^2_{q_{n+1}}}{\theta_+^{2n+4}} = 0.$$

Then the probability of the event

$$B_{n_m} := \{\overline{z}_1 \in A_1^{(n_m)},\ \overline{z}_2 \in A_2^{(n_m)},\ \ldots,\ \overline{z}_{q_{n_m+1}-1} \in A^{(n_m)}_{q_{n_m+1}-1}\} \tag{5.3}$$

tends to 1 as m tends to infinity, where

$$\overline{z}_k = T(\overline{z}_{k-1}) + \sigma_{q_{n_m+1}}\xi_k, \quad 1 \leqslant k \leqslant q_{n_m+1}-1, \quad \overline{z}_0 = z_0.$$

Proof. It is clear that

$$\mathbf{P}(B_{n_m}) = \mathbf{P}\left(\bigcap_{k=1}^{q_{n_m+1}-1}\{\overline{z}_k \in A_k^{n_m}\}\right) =$$

$$= \mathbf{P}\left(\overline{z}_1 \in A_1^{n_m}\right)\prod_{k=2}^{q_{n_m+1}-1}\mathbf{P}\left(\{\overline{z}_k \in A_k^{n_m}\}|\bigcap_{i=1}^{k-1}\{\overline{z}_i \in A_i^{n_m}\}\right). \tag{5.4}$$

Next, we estimate the factors of the last product.

Applying Chebyshev's inequality, we obtain

$$\mathbf{P}(\overline{z}_1 \in A_1^{n_m}) = \mathbf{P}(T(z_0) + \sigma_{q_{n_m+1}}\xi_1 \in A_1^- \cup TA_0^{n_m} \cup A_1^+) \geqslant$$
$$\geqslant \mathbf{P}(|\sigma_{q_{n_m+1}}\xi_1| \leqslant \min\{|A_1^-|, |A_1^+|\}) \geqslant 1 - \frac{\sigma^2_{q_{n_m+1}} \operatorname{Var}\xi_1}{\min\{|A_1^-|^2, |A_1^+|^2\}}.$$

Since A_1^- and A_1^+ are elements of P_{2n_m+7}, we have $\min\{|A_1^-|^2, |A_1^+|^2\} \geqslant C\theta_+^{2n_m+7}$. Using the last bound, we get

$$\mathbf{P}(\overline{z}_1 \in A_1^{n_m}) \geqslant 1 - \frac{C\sigma^2_{q_{n_m+1}}}{\theta_+^{2n_m+7}}. \tag{5.5}$$

For the other k, $2 \leqslant k < q_{n_m+1}$, analogyously we have

$$\mathbf{P}\left(\{\overline{z}_k \in A_k^{n_m}\} \Big/ \bigcap_{i=1}^{k-1}\{\overline{z}_i \in A_i^{n_m}\}\right) =$$
$$= \mathbf{P}\left(\{T(\overline{z}_k) + \sigma_{q_{n_m+1}}\xi_k \in A_k^- \cup TA_{k-1}^{n_m} \cup A_k^+\} \ / \ \{\overline{z}_{k-1} \in A_{k-1}^{n_m}\}\right) \geqslant$$
$$\geqslant \mathbf{P}(|\sigma_{q_{n_m+1}}\xi_k| \leqslant \min\{|A_k^-|, |A_k^+|\}) \geqslant 1 - \frac{\sigma^2_{q_{n_m+1}} \operatorname{Var}\xi_k}{\min\{|A_k^-|^2, |A_k^+|^2\}},$$

and therefore, as before,

$$\mathbf{P}\left(\{\overline{z}_k \in A_k^{n_m}\} \Big/ \bigcap_{i=1}^{k-1}\{\overline{z}_i \in A_i^{n_m}\}\right) \geqslant 1 - \frac{C\sigma^2_{q_{n_m+1}}}{\theta_+^{2n_m+7}}. \tag{5.6}$$

Summarizing (5.4), (5.5) and (5.6), we get

$$\mathbf{P}(B_{n_m}) = \mathbf{P}\left(\bigcap_{k=1}^{q_{n_m+1}-1}\{\overline{z}_k \in A_k^{(n_m)}(z_k)\}\right) \geqslant \left(1 - \frac{C\sigma^2_{q_{n_m+1}}}{\theta_+^{2n_m+7}}\right)^{q_{n_m+1}}. \tag{5.7}$$

By assumption,

$$\lim_{m\to\infty} \frac{q_{n_m+1}\sigma^2_{q_{n_m+1}}}{\theta_+^{2n_m+7}} = 0.$$

This, together with (5.6), implies that

$$\mathbf{P}(B_{n_m}) \to 1$$

as $m \to \infty$. □

We fix a point $z_0 \in S^1 \setminus \{x_b\}$. Assuming $\overline{z}_j \in A_j^{n_m}$, $1 \leqslant j \leqslant q_{n_m+1} - 1$, we can use the Taylor expansion for the random process $\overline{z}_{q_{n_m+1}}(z_0, \sigma_{q_{n_m+1}-1})$

$$\overline{z}_{q_{n_m+1}-1}(z_0, \sigma_{q_{n_m+1}-1}) = T^{q_{n_m+1}-1}(z_0) + \sigma L_{q_{n_m+1}-1}(z_0) + \sigma^2_{q_{n_m+1}-1} Q_{q_{n_m+1}-1}(z_0, \sigma_{q_{n_m+1}-1}), \tag{5.8}$$

where the linear term $L_{q_{n_m+1}}$ is the sum of independent random variables defined by (1.4).

To the linear process

$$y_n(z_0, \sigma) = T^n(z_0) + \sigma L_n(z_0) \tag{5.9}$$

we can apply the straightforward extension of the central limit theorem as proved in [9, Lemma 3.1].

Lemma 15 (see [9]). *Let $T \in C^2(S^1 \setminus \{x_b\})$ be a circle homeomorphism with a break point x_b and let $\{\xi_n\}$ be a sequence of independent random variables with $p > 2$ moments satisfying conditions* (1.2) *and* (1.3)*. Assume condition* (1.8) *holds for some point $z_0 \in S^1 \setminus \{\mathbb{O}_T(x_b)\}$ and some increasing sequence $\{n_k\}$ of positive integers, then*

$$l_{n_k}(z_0) = \frac{L_{n_k}(z_0)}{\sqrt{\operatorname{var} L_{n_k}(z_0)}}$$

converges in distribution to the standard Gaussian as $k \to \infty$. Moreover, there is a universal constant C such that

$$\sup_{t \in R} |P(l_{n_k}(z_0) < t) - \Phi(t)| \leqslant C \frac{\Lambda_{\min(p,\,3)}(x,\, n_k)}{(\Lambda_2(x,\, n_k))^{\min(p,\,3)/2}}. \tag{5.10}$$

Condition (1.8) holds indeed in our case, namely, using (3.1), Lemma 5 and $p > 2$, we get

$$\lim_{n\to\infty} \frac{\Lambda_p(z_0,\, q_n)}{(\Lambda_2(z_0,\, q_n))^{p/2}} = \lim_{n\to\infty} \frac{|I_0^{(n)}(x_b;\, z_0)|^p \lambda_{-p}^n}{(|I_0^{(n)}(x_b; z_0)|^2 \lambda_{-2}^n)^{p/2}} = 0. \tag{5.11}$$

Next, we treat the nonlinear part of the process (5.8). For this consider the process

$$\omega_{q_{n_m+1}-1} = \frac{\overline{z}_{q_{n_m+1}-1} - T^{q_{n_m+1}-1}(z_0)}{\sigma_{q_{n_m+1}-1}\sqrt{\operatorname{var}(L_{q_{n_m+1}-1}(z_0))}} = \\ = \frac{L_{q_{n_m+1}-1}(z_0)}{\sqrt{\operatorname{var}(L_{q_{n_m+1}-1}(z_0))}} + \sigma_{q_{n_m+1}-1}\frac{Q_{q_{n_m+1}-1}(z_0, \sigma_{q_{n_m+1}-1})}{\sqrt{\operatorname{var}(L_{q_{n_m+1}-1}(z_0))}}.$$

Lemma 15 implies that $\frac{L_{q_{n_m+1}-1}(z_0)}{\sqrt{\operatorname{var}(L_{q_{n_m+1}-1}(z_0))}}$ converges weekly (in distribution) to the standard Gaussian.

Next, we will show that the random process $\sigma_{q_{n_m+1}-1}\frac{Q_{q_{n_m+1}-1}(z_0, \sigma_{q_{n_m+1}-1})}{\sqrt{\operatorname{var}(L_{q_{n_m+1}-1}(z_0))}}$ converges to 0 in probability. For this we introduce the following constants:

$$K_1 = \sup_{x\in S^1} \frac{1}{|T'(x)|}, \quad K_2 = \exp\left(K_1 \sup_{x\in S^1} |T''_+(x)|\right), \quad K = K_1 \cdot K_2 \sup_{x\in S^1} |T''_+(x)|.$$

Lemma 16. *Suppose a circle map T satisfies the conditions of Theorem* 4 *and the sequence σ_n satisfies relation* (1.15)*. Let D_{n_m} be the event*

$$D_{n_m} = \left\{K \cdot \sigma_{q_{n_m+1}} \max_{1\leqslant i\leqslant q_{n_m+1}} |\xi_i| \left(\widehat{\Lambda}(z_0,\, q_{n_m+1})\right)^2 < \frac{1}{2}\right\}. \tag{5.12}$$

Under the condition of the event B_{n_m} in (5.3) *the following inequality holds*:

$$\mathbf{P}\left(\left|\frac{\sigma_{q_{n_m+1}} Q_{q_{n_m+1}}(z_0,\, \sigma_{q_{n_m+1}})}{\sqrt{\operatorname{var} L_{q_{n_m+1}}(z_0)}}\mathbf{1}_{D_{n_m}}\right| > \varepsilon\right) \leqslant$$

$$\leqslant \left(\frac{2K\sigma_{q_{n_m+1}}\left(\widehat{\Lambda}(x_0,\, q_{n+1})\right)^3\left(E(\max\limits_{1\leqslant i\leqslant q_{n_m+1}}|\xi_i|^p)\right)^{2/p}}{\varepsilon\sqrt{\Lambda_2(z_0,\, q_{n_m+1})}}\right)^{p/2}.$$

Proof. In a first step we estimate $|\sigma^2_{q_{n_m+1}} Q_{q_{n_m+1}}(z_0,\, \sigma_{q_{n_m+1}})|$.

Using (1.4) we obtain the recurrence relation

$$L_{k+1}(z_0) = L_k(z_0)T'(z_k) + \xi_{k+1}. \tag{5.13}$$

Let $\overline{z}_k \in A_k^{n_m}$, $0 \leqslant k < q_{n_m+1}$, then we have

$$\begin{aligned}\overline{z}_{k+1}(z_0,\, \sigma_{q_{n_m+1}}) &= T(\overline{z}_k) + \sigma_{q_{n_m+1}}\xi_{k+1} = \\ &= T(z_k + \sigma_{q_{n_m+1}} L_k(z_0) + \sigma^2_{q_{n_m+1}} Q_k(z_0)) + \sigma_{q_{n_m+1}}\xi_{k+1} = \\ &= z_{k+1} + T'(\widehat{z}_k)(\sigma_{q_{n_m+1}} L_k(z_0) + \sigma^2_{q_{n_m+1}} Q_k(z_0)) + \sigma_{q_{n_m+1}}\xi_{k+1},\end{aligned} \tag{5.14}$$

where $|\widehat{z}_k - z_k| \leqslant |\sigma_{q_{n_m+1}} L_k(z_0) + \sigma^2_{q_{n_m+1}} Q_k(z_0)| = |\overline{z}_k - z_k| \leqslant |A_k^{n_m}|$.

On the other hand,

$$\overline{z}_{k+1}(z_0,\, \sigma_{q_{n_m+1}}) = z_{k+1} + \sigma_{q_{n_m+1}} L_{k+1}(z_0) + \sigma^2_{q_{n_m+1}} Q_{k+1}(z_0).$$

The last relation, together with (5.14) and (5.13), implies that

$$\sigma^2_{q_{n_m+1}} Q_{k+1}(z_0) = \sigma_{q_{n_m+1}} L_k(z_0)(T'(\widehat{z}_k) - T'(z_k)) + \sigma^2_{q_{n_m+1}} Q_k(z_0)T'(\widehat{z}_k).$$

Iterating the last recurrence relation, we obtain

$$\left|\sigma^2_{q_{n_m+1}} Q_{k+1}(z_0)\right| = \sigma_{q_{n_m+1}} \sum_{i=1}^{k} \frac{|T'(\widehat{z}_i) - T'(z_i)|}{T'(\widehat{z}_i)} |L_i(z_0)| \prod_{s=i}^{k} T'(\widehat{z}_s). \tag{5.15}$$

Next, we estimate the right-hand side of (5.15).

For $1 \leqslant i \leqslant k$ one finds

$$|T'(\widehat{z}_i) - T'(z_i)| \leqslant \sup_{x\in S^1} |T''_+(z)||\widehat{z}_i - z_i| \leqslant \sup_{z\in S^1} |T''_+(z)||\sigma_{q_{n_m+1}} L_i(z_0) + \sigma^2_{q_{n_m+1}} Q_i(z_0)|. \tag{5.16}$$

But

$$\frac{1}{|T'(\widehat{z}_i)|} \leqslant K_1. \tag{5.17}$$

Therefore,

$$\prod_{s=i}^{k} T'(\widehat{z}_s) = \prod_{s=i}^{k} T'(z_s) \prod_{s=i}^{k} \frac{T'(\widehat{z}_s)}{T'(z_s)} = \prod_{s=i}^{k} T'(z_s) \prod_{s=i}^{k} \left(1 + \frac{T'(\widehat{z}_s) - T'(z_s)}{T'(z_s)}\right) \leqslant$$
$$\leqslant \prod_{s=i}^{k} T'(z_s) \prod_{s=i}^{k} \left(1 + K_1 \sup_{z\in S^1} T''_+(z)|\widehat{z}_i - z_i|\right) \leqslant \exp\left(\sum_{s=i}^{k} K_1 \sup_{z\in S^1} T''_+(z)|\widehat{z}_i - z_i|\right) \prod_{s=i}^{k} T'(z_s) \leqslant$$
$$\leqslant \exp\left(K_1 \sup_{z\in S^1} |T''_+(z)|\right) \prod_{s=i}^{k} T'(z_s) = K_2 \prod_{s=i}^{k} T'(z_s). \quad (5.18)$$

Using (5.15) and (5.16)–(5.18), we have for $0 \leqslant k < q_{n_m+1}$

$$\left|\sigma^2_{q_{n_m+1}} Q_{k+1}(z_0)\right| \leqslant$$
$$\leqslant K_1 K_2 \sup_{z\in S^1} |T''_+(z)| \sigma_{q_{n_m+1}} \sum_{i=1}^{k} |L_i(z_0)| |\sigma_{q_{n_m+1}} L_i(z_0) + \sigma^2_{q_{n_m+1}} Q_i(z_0)| \prod_{s=i}^{k} T'(z_s) \leqslant$$
$$\leqslant K\sigma^2_{q_{n_m+1}} \sum_{i=1}^{k} |L_i(z_0)|^2 \prod_{s=i}^{k} T'(z_s) + K\sigma_{q_{n_m+1}} \sum_{i=1}^{k} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| |L_i(z_0)| \prod_{s=i}^{k} T'(z_s) \leqslant$$
$$\leqslant K\sigma^2_{q_{n_m+1}} \left(\max_{1\leqslant i\leqslant k} |\xi_i| \widehat{\Lambda}(z_0,\, k)\right)^2 \sum_{i=1}^{k} \prod_{s=i}^{k} T'(z_s) +$$
$$+ K\sigma_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| \max_{1\leqslant i\leqslant k} |\xi_i| \widehat{\Lambda}(z_0, k) \sum_{i=1}^{k} \prod_{s=i}^{k} T'(z_s). \quad (5.19)$$

It is then clear that

$$\max_{1\leqslant i\leqslant k+1} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| \leqslant K\sigma^2_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\xi_i|^2 \left(\widehat{\Lambda}(z_0, k)\right)^3 +$$
$$+ \max_{1\leqslant i\leqslant k} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| K\sigma_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\xi_i| \left(\widehat{\Lambda}(z_0,\, k)\right)^2 \leqslant K\sigma^2_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\xi_i|^2 \left(\widehat{\Lambda}(z_0, k)\right)^3 +$$
$$+ \max_{1\leqslant i\leqslant k+1} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| K\sigma_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\xi_i| \left(\widehat{\Lambda}(z_0,\, k)\right)^2. \quad (5.20)$$

Hence, we have the following bound:

$$\max_{1\leqslant i\leqslant k+1} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| \leqslant K\sigma^2_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\xi_i|^2 \left(\widehat{\Lambda}(z_0,\, k)\right)^3 +$$
$$+ \max_{1\leqslant i\leqslant k+1} |\sigma^2_{q_{n_m+1}} Q_i(z_0)| K\sigma_{q_{n_m+1}} \max_{1\leqslant i\leqslant k} |\xi_i| \left(\widehat{\Lambda}(z_0,\, k)\right)^2. \quad (5.21)$$

Now we can prove Lemma 16.

Using (5.12) and (5.21), we get

$$|\sigma^2_{q_{n_m+1}} Q_{q_{n_m+1}}(z_0) \mathbf{1}_{D_{n_m}}| \leqslant \max_{1\leqslant i\leqslant q_{n_m+1}} |\sigma^2_{q_{n_m+1}} Q_i(z_0) \mathbf{1}_{D_{n_m}}| \leqslant$$
$$\leqslant 2K\sigma^2_{q_{n_m+1}} \max_{1\leqslant i\leqslant q_{n_m+1}} |\xi_i|^2 \left(\widehat{\Lambda}(z_0,\, q_{n_m+1})\right)^3. \quad (5.22)$$

Consequently,

$$\left(E|\sigma^2_{q_{n_m+1}} Q_{q_{n_m+1}}(z_0)\mathbf{1}_{D_{n_m}}|^{p/2}\right)^{2/p} \leqslant 2K\sigma^2_{q_{n_m+1}}\left(\widehat{\Lambda}(z_0,\, q_{n_m+1})\right)^3\left(E(\max_{1\leqslant i\leqslant q_{n_m+1}}|\xi_i|^p)\right)^{2/p}.$$

Letting $\varepsilon > 0$ and using Chebeshev's inequality, we obtain

$$\mathbf{P}\left(\left|\frac{\sigma_{q_{n_m+1}} Q_{q_{n_m+1}}(z_0,\, \sigma_{q_{n_m+1}})}{\sqrt{\operatorname{var} L_{q_{n_m+1}}(z_0)}}\mathbf{1}_{D_{n_m}}\right| > \varepsilon\right) \leqslant$$

$$\leqslant \left(\frac{2K\sigma_{q_{n_m+1}}\left(\widehat{\Lambda}(z_0,\, q_{n_m+1})\right)^3\left(E(\max_{1\leqslant i\leqslant q_{n_m+1}}|\xi_i|^p)\right)^{2/p}}{\varepsilon\sqrt{\operatorname{var} L_{q_{n_m+1}}(z_0)}}\right)^{p/2}.$$

Conditions (1.2) and (1.3) imply

$$\operatorname{const}\Lambda_2(z_0,\, q_{n_m+1}) \leqslant \operatorname{var} L_{q_{n+1}}(z_0) \leqslant \operatorname{Const}\Lambda_2(z_0,\, q_{n_m+1}).$$

This and the previous bounds imply the assertion of Lemma 16. □

Lemma 17. *Suppose a circle map T satisfies the conditions of Theorem 4 and the sequence σ_n satisfies the relation*

$$\lim_{n\to\infty} \sigma^p_{q_{n+1}} q_{n+1}\lambda_{-1}^{2np} n^{2p} = 0. \tag{5.23}$$

Then the probabilities of the events D_{n_m} defined in (5.12) *tend to* 1 *as $m \to \infty$.*

Proof. Consider the probability

$$\mathbf{P}(\Omega \setminus D_{n_m}) = \mathbf{P}\left\{C\sigma_{q_{n_m+1}}\max_{1\leqslant i\leqslant q_{n_m+1}}|\xi_i|\left(\widehat{\Lambda}(z_0,\, q_{n_m+1})\right)^2 \geqslant \frac{1}{2}\right\} =$$

$$= \mathbf{P}\left\{\max_{1\leqslant i\leqslant q_{n_m+1}}|\xi_i| \geqslant \frac{1}{2C\sigma_{q_{n_m+1}}\left(\widehat{\Lambda}(z_0,\, q_{n_m+1})\right)^2}\right\} \leqslant$$

$$\leqslant E(\max_{1\leqslant i\leqslant q_{n_m+1}}|\xi_i|^p)\left(2C\sigma_{q_{n_m+1}}\left(\widehat{\Lambda}(z_0, q_{n_m+1})\right)^2\right)^p. \tag{5.24}$$

Condition (1.3) shows that

$$\operatorname{const} \leqslant E(\max_{1\leqslant i\leqslant q_{n+1}}|\xi_i|^p) \leqslant q_{n+1}\operatorname{Const},$$

and Lemma 6 implies $\widehat{\Lambda}(z_0,\, q_{n_m+1}) \leqslant \operatorname{Const}\cdot n_m \cdot \lambda_{-1}^{n_m}$.

Using these bounds, we get the following estimate for $\mathbf{P}(\Omega \setminus D_{n_m})$:

$$\mathbf{P}(\Omega \setminus D_n) \leqslant \operatorname{Const}_1 \cdot q_{n_m+1} \cdot n_m^{2p} \cdot \lambda_{-1}^{2pn_m} \cdot \sigma^p_{q_{n_m+1}},$$

which implies Lemma 17. □

Lemma 18. *Assume a circle map T satisfies the conditions of Theorem 4 and the sequence σ_n satisfies the relation*

$$\lim_{n\to\infty} \sigma_{q_{n+1}} \cdot n^3 \cdot q_{n+1}^{2/p} \cdot \lambda_{-1}^{5n/2} \cdot \theta_+^{-2n} = 0.$$

Then

$$\lim_{m\to\infty} \frac{\sigma_{q_{n_m+1}} \left(\widehat{\Lambda}(x_0,\, q_{n_m+1})\right)^3 \left(E(\max\limits_{1\leqslant i\leqslant q_{n_m+1}} |\xi_i|^p)\right)^{2/p}}{\sqrt{\Lambda_2(z_0,\, q_{n_m+1})}} = 0.$$

Proof. In the proof of Lemma 17 we showed

$$\text{const} \leqslant E(\max_{1\leqslant i\leqslant q_{n+1}} |\xi_i|^p) \leqslant q_{n+1}\text{Const}$$

respectively

$$\widehat{\Lambda}(z_0,\, q_{n_m+1}) \leqslant C \cdot n_m \cdot \lambda_{-1}^{n_m}.$$

Since $\lambda_{-1} \leqslant \lambda_{-2}$, Theorem 6 shows

$$c_1 |I_0^{(n_m+1)}(x_b;\, z_0)|^2 \lambda_{-2}^{n_m} \leqslant \Lambda_2(z_0,\, q_{n_m+1}) \leqslant C_1 |I_0^{(n_m+1)}(x_b;\, z_0)|^2 \lambda_{-2}^{n_m}.$$

On the other hand, Lemma 12 implies

$$|I_0^{(n_m+1)}(x_b;\, z_0)| \geqslant C_1 \theta_+^{n_m+1}.$$

Using these inequalities, we find

$$\lim_{m\to\infty} \frac{\sigma_{q_{n_m+1}} \left(\widehat{\Lambda}(x_0,\, q_{n_m+1})\right)^3 \left(E(\max\limits_{1\leqslant i\leqslant q_{n_m+1}} |\xi_i|^p)\right)^{2/p}}{\sqrt{\Lambda_2(z_0,\, q_{n_m+1})}} \leqslant \text{Const} \frac{\sigma_{q_{n_m+1}} \cdot n_m^3 \cdot \lambda_{-1}^{3n_m} q_{n+1}^{2/p}}{\sqrt{\theta_+^{2n_m+2} \lambda_{-2}^{n_m}}} \leqslant$$

$$\leqslant \text{Const} \frac{\sigma_{q_{n_m+1}} \cdot n_m^3 \cdot \lambda_{-1}^{3n_m} q_{n+1}^{2/p}}{\theta_+^{2n_m+2} \lambda_{-1}^{n_m/2}} = \text{Const}\, \sigma_{q_{n_m+1}} \cdot n_m^3 \cdot \lambda_{-1}^{5n_m/2} q_{n_m+1}^{2/p} \theta_+^{-2n_m-2}.$$

□

For the proof of Theorem 4 we make use of

$$\lim_{n\to\infty} \frac{\ln q_n}{n} = \ln \rho_T^{-1},$$

where $\rho_T = [k_1,\, k_2,\, \ldots,\, k_m,\, 1,\, 1,\, \ldots]$, $m \geqslant 1$. Thus, for arbitrary $0 < \varepsilon < \frac{1}{2}$ there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\rho_T^{-n/2} < \rho_T^{-n(1-\varepsilon)} < q_n < \rho_T^{-n(1+\varepsilon)} < \rho_T^{-3n/2}.$$

Define

$$\gamma := \max\left\{\frac{2}{p} - \frac{5\ln\lambda_{-1} + 4\ln\theta_+^{-1} + 6}{\ln\rho_T},\, \frac{1}{2} + \frac{2\ln\theta_+}{\ln\rho_T}\right\}.$$

In order to prove the first part of Theorem 4, it is enough to verify the conditions of Lemmas 14, 17 and 18.

1. Condition of Lemma 14. For sufficiently large n

$$\frac{q_{n+1}\sigma^2_{q_{n+1}}}{\theta_+^{2n+4}} = q_{n+1}\sigma^2_{q_{n+1}}\theta_+^{-2n-4} = q_{n+1}\sigma^2_{q_{n+1}}\rho_T^{\frac{\ln\theta_+^{-1}}{\ln\rho_T}(2n-4)} <$$

$$< \left(q_{n+1}^{\frac{1}{2}-\frac{\ln\theta_+^{-1}}{\ln\rho_T}\frac{n-2}{(n+1)(1-\varepsilon)}}\sigma_{q_{n+1}}\right)^2 < \left(q_{n+1}^{\gamma}\sigma_{q_{n+1}}\right)^2.$$

2. Condition of Lemma 17.

$$\sigma^p_{q_{n+1}}q_{n+1}\lambda_{-1}^{2np}n^{2p} = \left(\sigma_{q_{n+1}}\cdot q_{n+1}^{1/p}\cdot\lambda_{-1}^{2n}\cdot n^2\right)^p < \left(\sigma_{q_{n+1}}\cdot q_{n+1}^{2/p}\cdot\lambda_{-1}^{5n/2}\cdot n^3\cdot\theta_+^{-2n}\right)^p.$$

3. Condition of Lemma 18. For sufficiently large n

$$\sigma_{q_{n+1}}\cdot q_{n+1}^{2/p}\cdot\lambda_{-1}^{5n/2}\cdot n^3\cdot\theta_+^{-2n} = \sigma_{q_{n+1}}\cdot q_{n+1}^{2/p}\cdot\rho_T^{\frac{\ln\lambda_{-1}}{\ln\rho_T}\cdot\frac{5n}{2}}\cdot\rho_T^{\frac{3\ln n}{\ln\rho_T}}\cdot\rho_T^{2n\cdot\frac{\ln\theta_+^{-1}}{\ln\rho_T}} <$$

$$< \sigma_{q_{n+1}}\cdot q_{n+1}^{2/p}\cdot q_{n+1}^{-\frac{\ln\lambda_{-1}}{\ln\rho_T}\cdot\frac{5n}{2(n+1)(1-\varepsilon)}}\cdot q_{n+1}^{-\frac{3\ln n}{\ln\rho_T(n+1)(1-\varepsilon)}}\cdot q_{n+1}^{-\frac{2n}{(n+1)(1-\varepsilon)}\cdot\frac{\ln\theta_+^{-1}}{\ln\rho_T}} <$$

$$< \sigma_{q_{n+1}}\cdot q_{n+1}^{2/p-\frac{\ln\lambda_{-1}}{\ln\rho_T}\cdot\frac{5n}{(n+1)}-\frac{4n}{(n+1)}\cdot\frac{\ln\theta_+^{-1}}{\ln\rho_T}-\frac{6\ln n}{(n+1)\ln\rho_T}} < \sigma_{q_{n+1}}\cdot q_{n+1}^{\gamma}.$$

Thus,

$$\sigma_{q_{n+1}}\cdot n^3\cdot\lambda_{-1}^{5n/2}q_{n+1}^{2/p}\theta_+^{-2n-2} < \sigma_{q_{n+1}}\cdot q_{n+1}^{\gamma}. \tag{5.25}$$

For the second part of Theorem 4 it is enough to show that the rate of convergence to 0 of $\sigma_{q_{n_m+1}-1}\frac{Q_{q_{n_m+1}-1}(z_0,\sigma_{q_{n_m+1}-1})}{\sqrt{\operatorname{var}(L_{q_{n_m+1}-1}(z_0))}}$ is bounded by the right-hand side in the estimate in (5.10), which means

$$\sigma_{q_{n+1}}\cdot n^3\cdot\lambda_{-1}^{5n/2}q_{n+1}^{2/p}\theta_+^{-2n-2} < \frac{\Lambda_{\min(p,3)}(x,\,q_n)}{(\Lambda_2(x,\,q_n))^{\min(p,3)/2}}.$$

To prove this inequality, we use (5.25) and Theorem 6 to show

$$\mathrm{const}\cdot\left(\frac{\lambda^2_{-\min(p,3)}}{\lambda_{-2}^{\min(p,3)}}\right)^{n/2} \leqslant \frac{\Lambda_{\min(p,3)}(x,\,q_n)}{(\Lambda_2(x,\,q_n))^{\min(p,3)/2}}.$$

It is therefore enough to get

$$\sigma_{q_{n+1}}\cdot q_{n+1}^{\gamma} \leqslant \mathrm{const}\cdot\left(\frac{\lambda^2_{-\min(p,3)}}{\lambda_{-2}^{\min(p,3)}}\right)^{n/2}, \tag{5.26}$$

where $\sigma_{q_{n+1}} \leqslant C_1\cdot q_{n+1}^{-\tau}$. Let us choose $s=\min(p,\,3)$ and $\tau \geqslant \gamma + \frac{2\ln\lambda_{-s}-s\ln\lambda_{-2}}{3\ln\rho_T}$. Then

$$\sigma_{q_{n+1}} \leqslant C_1\cdot q_{n+1}^{-\tau} \leqslant C_1\cdot q_{n+1}^{-\gamma-\frac{2\ln\lambda_{-s}-s\ln\lambda_{-2}}{3\ln\rho_T}} \leqslant C_1\cdot q_{n+1}^{-\gamma}\cdot\rho_T^{\frac{2\ln\lambda_{-s}-s\ln\lambda_{-2}}{3\ln\rho_T}\cdot\frac{3(n+1)}{2}},$$

from which (5.26) follows. □

To finish the proof, we can apply Lemma 3.2 in [9] by noting that there the quantity $\|f''\|_{C_0}$ has to be replaced in our case by $\sup\limits_{z\in[x_b,\,x_b+1]} |T''(z)|$ which exists for $T \in C^{2+\epsilon}(S^1 \setminus \{x_b\})$. Then, following the arguments of the proof of the CLT in Section 3.3 of [9] leads finally to the convergence of the process $\omega_{n_k}(z_0,\, \sigma_{n_k})$ to the standard Gaussian distribution.

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