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$\mathcal{I}^{\mathcal{K}}$-SEQUENTIAL TOPOLOGY

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Abstract: In the literature, $\mathcal{I}$-convergence (or convergence in $\mathcal{I}$) was first introduced in [11]. Later related notions of $\mathcal{I}$-sequential topological space and $\mathcal{I}^*$-sequential topological space were introduced and studied. From the definitions it is clear that $\mathcal{I}^*$-sequential topological space is larger(finer) than $\mathcal{I}$-sequential topological space. This rises a question: is there any topology (different from discrete topology) on the topological space $\mathcal{X}$ which is finer than $\mathcal{I}^*$-topological space? In this paper, we tried to find the answer to the question. We define $\mathcal{I}^{\mathcal{K}}$-sequential topology for any ideals $\mathcal{I}$, $\mathcal{K}$ and study main properties of it. First of all, some fundamental results about $\mathcal{I}^{\mathcal{K}}$-convergence of a sequence in a topological space $(\mathcal{X},\mathcal{T})$ are derived. After that, $\mathcal{I}^{\mathcal{K}}$-continuity and the subspace of the $\mathcal{I}^{\mathcal{K}}$-sequential topological space are investigated.

Keywords: Ideal convergence, $\mathcal{I}^{\mathcal{K}}$-convergence, Sequential topology, $\mathcal{I}^{\mathcal{K}}$-sequential topology.

1. Introduction

The notion of convergence of real or complex valued sequences was generalized using asymptotic density and was called statistical convergence by Fast [7] and Steinhause [20] in the same year 1951, independently. After some years P. Kostyrko, T. Šalát, W. Wilczyńki [11] gave a generalization of statistical convergence and called it as ideal convergence (or converges in ideal). Various fundamental properties (convergence in $\mathcal{I}$ and $\mathcal{I}^*$) were investigated. Later B.K. Lahiri and P. Das in [12] discussed convergence in $\mathcal{I}$ and in $\mathcal{I}^*$ and investigate some additional results related to mentioned concepts [4, 8–10, 15–17].

The concept of $\mathcal{I}^*$-convergence of functions was extended to $\mathcal{I}^{\mathcal{K}}$-convergence by M. Mačaj and M. Sleziak in [13] in 2011. The authors of [2, 3, 5, 6, 14] gave further properties and results about $\mathcal{I}^{\mathcal{K}}$-convergence.

In first part of this paper we introduce $\mathcal{I}^{\mathcal{K}}$-sequential topological (seq.-top.) space, which is a natural generalization of $\mathcal{I}^*$-seq.-top. space. Later we discuss the $\mathcal{I}^{\mathcal{K}}$-continuity of the function and in last two section we write about $\mathcal{I}^{\mathcal{K}}$-subspace and $\mathcal{I}^{\mathcal{K}}$-connectedness. We will use further the abbreviation T.S. for a topological space.

2. Definition and preliminaries

In this part, we give some known definitions and necessary results.

Definition 1 [7, 20]. *Let $\mathcal{A}\subset\mathbb{N}$, and for $m\in\mathbb{N}$ let the set*

$$\mathcal{A}_m := \{x\in\mathcal{A} : x<m\}$$

and $|\mathcal{A}_m|$ stand for the cardinality of $\mathcal{A}_m$. Natural density of $\mathcal{A}$ is defined by

$$\beta(\mathcal{A}) := \lim_{m\to\infty}\frac{|\mathcal{A}_m|}{m}$$

whenever the limit exists. A real sequence $\tilde{x} = (x_i)$ *is said to statistically converges to* x_0 *if for any* $\varepsilon > 0$,

$$\beta\big(\{n : |x_i - x_0| > \varepsilon\}\big) = 0$$

holds.

Definition 2 [11]. *Let* $\mathcal{I}$ *be any subfamily of* $\mathcal{P}(\mathbb{N})$, *with* $\mathcal{P}(\mathbb{N})$ *being the family of all subsets of* $\mathbb{N}$. *Then,* $\mathcal{I}$ *is called an ideal on* $\mathbb{N}$ *if the following requirements hold*:

(i) *finite union of sets in* $\mathcal{I}$ *is again in* $\mathcal{I}$;
(ii) *any subset of a set in* $\mathcal{I}$ *is in* $\mathcal{I}$.

$\mathcal{I}$ is admissible if all singleton subsets of $\mathbb{N}$ belong to $\mathcal{I}$. The ideal $\mathcal{I}$ is non-trivial if $\mathcal{I} \neq \varnothing$ and $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$. A non-trivial ideal $\mathcal{I}$ is called proper if $\mathbb{N}$ is not in $\mathcal{I}$.

The family of finite subsets of the $\mathbb{N}$ is an admissible non-trivial ideal denoted by $\mathcal{F}in$ and the family of the subsets of $\mathbb{N}$ with natural density zero is also an admissible non-trivial ideal denoted by $\mathcal{I}_\beta$. The set of all non-trivial admissible ideals will be denoted as NA throughout the study.

Example 1. [11] Consider the decomposition of $\mathbb{N}$ as $\mathbb{N} = \bigcup_{j=1}^{\infty} \beta_j$ where all β_j are infinite subsets of $\mathbb{N}$ and are mutually disjoint. Take the family

$$\mathcal{I} = \{N \subset \mathbb{N} : N \text{ intersect only finite number of } \beta_j' s\}.$$

Then, $\mathcal{I}$ belongs to NA.

Definition 3 [19]. *Assume* $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$. *The collection* $\mathcal{F}$ *is a filter on* $\mathbb{N}$ *if*

(i) *a finite intersection of elements of* $\mathcal{F}$ *is in* $\mathcal{F}$ *and*
(ii) *if* $\mathcal{C} \in \mathcal{F} \wedge \mathcal{C} \subseteq \mathcal{D}$, *then* $\mathcal{D} \in \mathcal{F}$.

If empty set is not in $\mathcal{F}$ then $\mathcal{F}$ is proper. If $\mathcal{I} \in NA$ then the collection

$$\mathcal{F} = \{N \subset \mathbb{N} : N^C \in \mathcal{I}\}$$

is a filter on $\mathbb{N}$. It is known as the $\mathcal{I}$-associated filter.

Definition 4 [21]. *In a T.S.* $(\mathcal{X}, \mathcal{T})$ *a sequence* $\tilde{x} = (x_i) \subset \mathcal{X}$ *is called to converging in* $\mathcal{I}$ *to a point* $x \in \mathcal{X}$ *if*

$$\{i \in \mathbb{N} : \ x_i \in \upsilon\} \in \mathcal{F}(\mathcal{I})$$

holds for each neighborhood υ *of* x. *The point* x *is referred to as the ideal limit of the sequence* $\tilde{x} = (x_i)$ *and it is represented by* $x_i \xrightarrow{\mathcal{I}} x$ *(or* $\mathcal{I} - \lim x_i = x$*).*

Remark 1.

(i) Statistical and $\mathcal{I}_\beta-$ convergence are coincide.
(ii) Classical convergence and $\mathcal{F}in-$convergence are coincide.

Lemma 1 [1]. *Assume that* $\mathcal{I}, \mathcal{I}_1$ *and* $\mathcal{I}_2$ *be ideals on the set* $\mathbb{N}$ *and consider a T.S.* $(\mathcal{X}, \mathcal{T})$, *then*

1. *If* $\mathcal{I} \in NA$, *then every convergent sequence is* $\mathcal{I}$-*convergent sequence which converges to same point.*
2. *If* $\mathcal{I}_1 \subseteq \mathcal{I}_2$ *and* $(x_i) \subseteq \mathcal{X}$ *is a sequence which* $x_i \xrightarrow{\mathcal{I}_1} x$, *then* $x_i \xrightarrow{\mathcal{I}_2} x$.
3. *If* $\mathcal{X}$ *the Hausdorff space, then the limit of every convergent sequence is unique.*

3. $\mathcal{I}^{\mathcal{K}}$-convergence of sequence

In this part we will investigate some results related to $\mathcal{I}^{\mathcal{K}}$-convergence of sequences which is a generalized form of $\mathcal{I}^*$-convergence of sequences. If we consider $\mathcal{F}in$ instead of $\mathcal{K}$, then we will have $\mathcal{I}^*$-convergence.

Definition 5 [6]. *In a T.S. $(\mathcal{X},\mathcal{T})$ a sequence $\tilde{x}=(x_i)\subset\mathcal{X}$ is called to be $\mathcal{I}^*$-converging to $x_0\in\mathcal{X}$ if $\exists M\in\mathcal{F}(\mathcal{I})$ s.t. the sequence*

$$y_i := \begin{cases} x_i, & i\in M,\\ x, & i\notin M \end{cases}$$

is $\mathcal{F}in$ convergent to x.

That is, for each neighborhood υ of x,

$$\{i\in\mathbb{N}:\ y_i\in\upsilon\}\in\mathcal{F}(\mathcal{F}in),$$

or

$$\{i\in M:\ y_i\notin\upsilon\}\cup\{i\in M^C:\ y_i\notin\upsilon\}\in\mathcal{F}in.$$

So,

$$\{i\in M:\ x_i\notin\upsilon\}\cup\{i\in M^C:\ x\notin\upsilon\}\in\mathcal{F}in.$$

This implies that

$$\{i\in M:\ y_i\notin\upsilon\}\in\mathcal{F}in.$$

Therefore,

$$\{i\in M:\ y_i\in\upsilon\}\in\mathcal{F}(\mathcal{F}in).$$

It is clear that this definition is the same as the definition given in [6]. In the definition of $\mathcal{I}^*$-convergence of sequence if we consider an arbitrary ideal $\mathcal{K}$ instead of the ideal $\mathcal{F}in$ then it yields the definition of $\mathcal{I}^{\mathcal{K}}$-convergence of a sequence. That is, $\mathcal{I}^{\mathcal{K}}$-convergence is the generalized form of $\mathcal{I}^*$-convergence.

Definition 6 [13]. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider a T.S. $(\mathcal{X},\mathcal{T})$. The sequence $\tilde{x}=(x_i)\subset\mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to a point $x\in\mathcal{X}$ if $\exists M\in\mathcal{F}(\mathcal{I})$ s.t. the sequence*

$$y_i = \begin{cases} x_i, & i\in M,\\ x, & i\notin M, \end{cases}$$

$\mathcal{K}$-converges to x. We represent it as $\mathcal{I}^{\mathcal{K}}-\lim(x_i)=x$ or $x_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$.

Definition 7. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X},\mathcal{T})$ represent a T.S. Consider the sequences $\tilde{x}=(x_i)\subset\mathcal{X}$ and $\tilde{y}=(y_i)\subset\mathcal{X}$. Define a relation $\sim_{\mathcal{I}}$ as*

$$\tilde{x}\sim_{\mathcal{I}}\tilde{y}\Leftrightarrow\{i: x_i\neq y_i\}\in\mathcal{I}.$$

The relation $\sim_{\mathcal{I}}$ is an equivalence relation. That is,

1. $\forall\ \tilde{x}=(x_i)\subset\mathcal{X}$, $\{i: x_i\neq x_i\}=\varnothing\in\mathcal{I}\Rightarrow\tilde{x}\sim_{\mathcal{I}}\tilde{x}$.
2. Let $\tilde{x}\sim_{\mathcal{I}}\tilde{y}$. Since $\{i: y_i\neq x_i\}=\{i: x_i\neq y_i\}\in\mathcal{I}$, then $\tilde{y}\sim_{\mathcal{I}}\tilde{x}$.
3. Let $\tilde{x}\sim_{\mathcal{I}}\tilde{y}$ and $\tilde{y}\sim_{\mathcal{I}}\tilde{z}$. Then, $A:=\{i: x_i=y_i\}\in\mathcal{F}(\mathcal{I})$ and $B:=\{i: y_i=z_i\}\in\mathcal{F}(\mathcal{I})$. So, $\{i: x_i=z_i\}=A\cap B\in\mathcal{F}(\mathcal{I})$. Hence, $\tilde{x}\sim_{\mathcal{I}}\tilde{z}$ holds.

Lemma 2. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X},\mathcal{T})$ and the sequences $\tilde{x}=(x_i)\subseteq\mathcal{X}$. Assume $x_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$ for any $x\in\mathcal{X}$ and $\tilde{t}=(t_i)\subseteq\mathcal{X}$ is a sequence s.t. $\tilde{x}\sim_{\mathcal{I}}\tilde{t}$. Then, the sequence $t_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$.*

P r o o f. Let $x_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$, then $\exists M\in\mathcal{F}(\mathcal{I})$ s.t. the following sequence

$$y_i=\begin{cases}x_i, & i\in M,\\ x, & i\notin M\end{cases}$$

is $\mathcal{K}$-convergent to x. Since $(x_i)\sim_{\mathcal{I}}(t_i)$. So $\forall i\in M$, $x_i=t_i$. Therefore, the following sequence

$$y_i=\begin{cases}t_i, & i\in M,\\ x, & i\notin M\end{cases}$$

is $\mathcal{K}$-convergent to x which shows that $t_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$ holds.

□

The Definition 7 gives the possibility that the definition of $\mathcal{I}^{\mathcal{K}}$-convergence of a sequence can be rewritten as follows:

Definition 8. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X},\mathcal{T})$. A sequence $\tilde{x}=(x_i)\subset X$ is $\mathcal{I}^{\mathcal{K}}$-convergent to the point $x\in\mathcal{X}$ if there exist a sequence $\tilde{t}=(t_i)\subset X$ s.t. $\tilde{x}\sim_{\mathcal{I}}\tilde{t}$ and $t_i\xrightarrow{\mathcal{K}}x$ holds.*

In the following lemma we demonstrate that Definition 6 and Definition 8 are equivalent for any ideals $\mathcal{I}$ and $\mathcal{K}$ and for any T.S. $(\mathcal{X},\mathcal{T})$.

Lemma 3. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X},\mathcal{T})$ and $\tilde{x}=(x_i)\subset\mathcal{X}$ be a sequence. Then, $x_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$ iff $\exists\,\tilde{t}=(t_i)\subset\mathcal{X}$ s.t. $\tilde{x}\sim_{\mathcal{I}}\tilde{t}$ and $t_i\xrightarrow{\mathcal{K}}x$ hold.*

P r o o f. Let $x_i\xrightarrow{\mathcal{I}^{\mathcal{K}}}x$ holds. Then, $\exists M\in\mathcal{F}(\mathcal{I})$ s.t. the following sequence

$$y_i=\begin{cases}x_i, & i\in M,\\ x, & i\notin M\end{cases}$$

is $\mathcal{K}$-convergent to x. Let us chose $(t_i)=(y_i)$ $\forall i\in\mathbb{N}$. Then, the proof will complete if we show that $\tilde{x}\sim_{\mathcal{I}}\tilde{y}$.

Consider the fact $\{i\in\mathbb{N}:x_i=y_i\}=\{i\in M:x_i=y_i\}\in\mathcal{F}(\mathcal{I})$. Hence, $\tilde{x}\sim_{\mathcal{I}}\tilde{t}$.

Conversely, let $\tilde{x}=(x_i)$ and $\tilde{t}=(t_i)$ be sequences s.t. $\tilde{x}\sim_{\mathcal{I}}\tilde{t}$ and $t_i\xrightarrow{\mathcal{K}}x$ hold. Since $\tilde{x}\sim_{\mathcal{I}}\tilde{t}$, then

$$M=\{i\in\mathbb{N}:x_i=t_i\}\in\mathcal{F}(\mathcal{I})$$

holds. Define a sequence

$$y_i=\begin{cases}x_i, & i\in M,\\ x, & i\notin M.\end{cases}$$

Since $x_i=t_i$ hold $\forall i\in M$, then we can write

$$t_i=\begin{cases}x_i, & i\in M,\\ x, & i\notin M.\end{cases}$$

Because $\tilde{t}=(t_i)$ is $\mathcal{K}$-convergent to x, the sequence $\tilde{y}=(y_i)$ is also $\mathcal{K}$-convergent to x. Hence, the sequence $\tilde{x}=(x_i)$ is $\mathcal{I}^{\mathcal{K}}$-convergent to the point x and this completes the proof. □

4. $\mathcal{I}^{\mathcal{K}}$-seq.-top. space

In this section, we are going to define a new topology on the $\mathcal{X}$ using the ideal $\mathcal{I}$ and $\mathcal{K}$ and investigate some properties of the new T.S. This topology will be an extended version of the $\mathcal{I}^*$-seq.-top. space which was discussed in [18]. If we take $\mathcal{I} = \mathcal{F}in$, then $\mathcal{I}^{\mathcal{K}}$-seq.-top. space is coincide with $\mathcal{I}^*$-T.S.

Definition 9. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. Then*

1. *A set $F \subseteq \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-closed, if for each $(x_i) \subseteq F$ with $x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$, then $x \in F$.*
2. *A set $V \subset \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-open, if its complement V^C is $\mathcal{I}^{\mathcal{K}}$-closed.*

Remark 2. Consider the T.S. $(\mathcal{X}, \mathcal{T})$. An $O \subset \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-open iff each sequence in $\mathcal{X} - O$ has $\mathcal{I}^{\mathcal{K}}$-limit in $\mathcal{X} - O$.

P r o o f. The proof is evident from Definition 9. Therefore, it is omitted here. □

Definition 10. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. For any subset $A \subseteq \mathcal{X}$ define a set $\overline{A}^{\mathcal{I}^{\mathcal{K}}}$ (it is called $\mathcal{I}^{\mathcal{K}}$-closure of A) by*

$$\overline{A}^{\mathcal{I}^{\mathcal{K}}} := \{x \in \mathcal{X} : \exists (x_i) \subseteq A,\ x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x\}.$$

It is clear that $\overline{\varnothing}^{\mathcal{I}^{\mathcal{K}}} = \varnothing$, $\overline{\mathcal{X}}^{\mathcal{I}^{\mathcal{K}}} = \mathcal{X}$, and $A \subseteq \overline{A}^{\mathcal{I}^{\mathcal{K}}}$ holds $\forall A \subseteq \mathcal{X}$.

Remark 3. A subset C of the T.S. $\mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$ closed set iff $\overline{C}^{\mathcal{I}^{\mathcal{K}}} = C$.

P r o o f. Proof is obvious from the Definition 10. So, it is omitted here. □

Lemma 4. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and let $(\mathcal{X}, \mathcal{T})$ represent a T.S. For any subset $A \subseteq \mathcal{X}$, $\mathcal{I}^{\mathcal{K}}$-closure of A is $\mathcal{I}^{\mathcal{K}}$-closed.*

P r o o f. We must show that

$$\overline{(\overline{A}^{\mathcal{I}^{\mathcal{K}}})}^{\mathcal{I}^{\mathcal{K}}} = \overline{A}^{\mathcal{I}^{\mathcal{K}}}.$$

It is clear that

$$\overline{A}^{\mathcal{I}^{\mathcal{K}}} \subset \overline{(\overline{A}^{\mathcal{I}^{\mathcal{K}}})}^{\mathcal{I}^{\mathcal{K}}}.$$

Let $x \in \overline{(\overline{A}^{\mathcal{I}^{\mathcal{K}}})}^{\mathcal{I}^{\mathcal{K}}}$. Then, there exist a sequence $(x_i) \subset \overline{A}^{\mathcal{I}^{\mathcal{K}}}$ s.t. $x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$ holds. Since $(x_i) \subset \overline{A}^{\mathcal{I}^{\mathcal{K}}}$, then there exist sequences $(x_i^n) \subset A$ s.t. $x_i^n \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x_i$. Therefore there exist the sets $M_n \in \mathcal{F}(\mathcal{I})$ s.t.

$$\{i \in M_n : x_i^n \notin \upsilon^n\} \in \mathcal{K}$$

for each neighborhood υ^n of x_i. Choose m_1 the i where x_i^1 is belonging to neighborhood υ^1 of x_1, similarly m_2 the i where x_i^2 is belonging to neighborhood υ^2 of x_2. If we continue this process and take m_p the i where x_i^p is belonging to neighborhood υ^n of x_p. The obtained sequence (x_{m_p}) belongs to A. The theorem will be proved if we show that $x_{m_p} \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$. Since $x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$, so $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the sequence

$$y_i = \begin{cases} x_i, & i \in M, \\ x, & i \notin M, \end{cases} \quad y_i \stackrel{\mathcal{K}}{\to} x.$$

So,

$$\{i \in M : x_i \notin \upsilon\} \in \mathcal{K}$$

for each neighborhood υ of x. Now,

$$\{i \in M : \upsilon^n \not\subset \upsilon\} \subseteq \{i \in M : x_i \notin \upsilon\} \in \mathcal{K}.$$

Therefore,

$$\{i \in M : \upsilon^n \not\subset \upsilon\} \in \mathcal{K}$$

and

$$\{i \in M : x_{m_p} \notin \upsilon\} \subset \{i \in M : \upsilon^n \nsubseteq U\} \in \mathcal{K}$$

hold. So, $x_{m_p} \xrightarrow{\mathcal{I}^{\mathcal{K}}} x$ and $x \in \overline{A}^{\mathcal{I}^{\mathcal{K}}}$. □

Definition 11. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ represent a T.S. Then, for $A \subset \mathcal{X}$, $\mathcal{I}^{\mathcal{K}}$-interior of A is defined as*

$$A^{\circ\mathcal{I}^{\mathcal{K}}} := A - (\overline{\mathcal{X} - A}^{\mathcal{I}^{\mathcal{K}}}).$$

Proposition 1. *Let $\mathcal{V}$ be a subset of T.S. $\mathcal{X}$, then $\mathcal{V}$ is $\mathcal{I}^{\mathcal{K}}$-open iff $\mathcal{V}^{\circ\mathcal{I}^{\mathcal{K}}} = \mathcal{V}$.*

P r o o f. Let $\mathcal{V}$ be an $\mathcal{I}^{\mathcal{K}}$-open set. Then, $\mathcal{X} - \mathcal{V}$ is $\mathcal{I}^{\mathcal{K}}$-closed set and

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - \mathcal{V}) = \mathcal{X} - \mathcal{V}$$

holds. So, we have

$$\mathcal{V}^{\circ\mathcal{I}^{\mathcal{K}}} = \mathcal{V} - (\mathcal{X} - \mathcal{V}) = \mathcal{V}.$$

Conversely assume that

$$\mathcal{V}^{\circ\mathcal{I}^{\mathcal{K}}} = \mathcal{V}$$

holds. From the definition of $\mathcal{I}^{\mathcal{K}}$-interior of $\mathcal{V}$ we have

$$\mathcal{V} = \mathcal{V} - (\overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^{\mathcal{K}}}).$$

Hence,

$$\mathcal{V} \cap \overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^{\mathcal{K}}} = \varnothing.$$

Consequently

$$\overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^{\mathcal{K}}} \subset \mathcal{X} - \mathcal{V}.$$

Thus,

$$\overline{\mathcal{X} - \mathcal{V}}^{\mathcal{I}^{\mathcal{K}}} = \mathcal{X} - \mathcal{V}$$

is satisfied. Therefore, $\mathcal{X} - \mathcal{V}$ is $\mathcal{I}^{\mathcal{K}}$-closed and $\mathcal{V}$ is $\mathcal{I}^{\mathcal{K}}$-open. □

Definition 12 [21]. *A sequence (x_i) in a T.S. $\mathcal{X}$ is $\mathcal{I}$-eventually in a subset A of $\mathcal{X}$ if*

$$\{i \in \mathbb{N} : x_i \in A\} \in \mathcal{F}(\mathcal{I}).$$

Definition 13. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X},\mathcal{T})$. A sequence $\tilde{x}=(x_i)\subseteq\mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-eventually in a subset $\mathcal{V}$ of $\mathcal{X}$. If there exist a sequence $\tilde{y}=(y_i)\subseteq\mathcal{X}$ s.t. $\tilde{y}\sim_I\tilde{x}$ and $\tilde{y}$ is $\mathcal{K}$-eventually in $\mathcal{V}$.*

In the next theorem, we will provide a sequence characterization of $\mathcal{I}^{\mathcal{K}}-$ open set.

Theorem 1. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X},\mathcal{T})$. A subset υ of $\mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-open iff each $\mathcal{I}^{\mathcal{K}}$-convergent sequence to $x_0\in\upsilon$ is $\mathcal{I}^{\mathcal{K}}$-eventually in υ.*

P r o o f. Let υ is $\mathcal{I}^{\mathcal{K}}$-open. Then, $\mathcal{X}-\upsilon$ is $\mathcal{I}^{\mathcal{K}}$-closed and $\overline{\mathcal{X}-\upsilon}^{\mathcal{I}^{\mathcal{K}}}=\mathcal{X}-\upsilon$ holds. Let $\tilde{x}=(x_i)\subset\mathcal{X}$ be a sequence s.t. $x_i\overset{\mathcal{I}^{\mathcal{K}}}{\to}x$ and $x\in\upsilon$. Then, $\exists M\in\mathcal{F}(\mathcal{I})$ s.t. the sequence

$$t_i=\begin{cases}x_i, & i\in M,\\ x, & i\notin M\end{cases}$$

is $\mathcal{K}$-convergent to x. Since υ is a neighborhood of x, then we have

$$H=\{i\in\mathbb{N}:x_i\notin\upsilon\}\in\mathcal{K}.$$

If we choose $y_i=t_i$, then

$$\{i\in\mathbb{N}:y_i=x_i\}=\{i\in\mathbb{N}:t_i=x_i\}=M\in\mathcal{F}(\mathcal{I})$$

holds. So, $(y_i)\sim_{\mathcal{I}}(x_i)$ holds and (y_i) is eventually in υ.

Conversely, let $\tilde{x}=(x_i)\subset\mathcal{X}$ is a sequence which is $\mathcal{I}^{\mathcal{K}}$-convergent sequence to a point $x\in\upsilon$ and it is $\mathcal{I}^{\mathcal{K}}$-eventually in υ. Assume that υ is not $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$. So there exists $x_0\in\overline{\mathcal{X}-\upsilon}^{\mathcal{I}^{\mathcal{K}}}$ which $x_0\notin\mathcal{X}-\upsilon$. This means that there exists a sequence $(x_i)\subset\mathcal{X}-\upsilon$ which is $\mathcal{I}^{\mathcal{K}}$-convergence to $x_0\in\upsilon$. So, (x_i) is $\mathcal{I}^{\mathcal{K}}$-eventually in υ.

Therefore, $\exists\tilde{y}=(y_i)\subset\mathcal{X}$ which $\tilde{x}\sim_{\mathcal{I}}\tilde{y}$ and $\tilde{y}$ is $\mathcal{K}$-eventually in υ. This implies that $\tilde{y}$ is $\mathcal{K}$-eventually in υ which is not in case. □

Theorem 2. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X},\mathcal{T})$. A subset $\mathcal{C}\subset\mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-closed iff*

$$\mathcal{C}=\cap\{\mathcal{A}:\mathcal{A}\text{ is }\mathcal{I}^{\mathcal{K}}\text{— closed and }\mathcal{C}\subset\mathcal{A}\}.$$

P r o o f. Let

$$\mathcal{C}=\cap\{\mathcal{A}:\mathcal{A}\text{ is }\mathcal{I}^{\mathcal{K}}\text{— closed and }\mathcal{C}\subset\mathcal{A}\}.$$

Let x be any element of $\mathcal{I}^{\mathcal{K}}$-closure of $\mathcal{C}$. Then there exists $(x_i)\subset\mathcal{C}$ s.t. $x_i\overset{\mathcal{I}^{\mathcal{K}}}{\to}x$. Let $x\notin\mathcal{C}$ so

$$x\notin\cap\{\mathcal{A}:\mathcal{A}\text{ is }\mathcal{I}^{\mathcal{K}}\text{— closed and }\mathcal{C}\subset\mathcal{A}\}.$$

This implies that $\exists\,\mathcal{I}^{\mathcal{K}}$-closed subset F of $\mathcal{X}$ s.t. $x\notin\mathcal{A}$, but $\mathcal{C}$ is $\mathcal{I}^{\mathcal{K}}$-closed and it is a subset of $\mathcal{A}$, which is a contradiction.

The converse is obvious. □

Theorem 3. *Let $\mathcal{I}$ and $\mathcal{K}$ be ideals of $\mathbb{N}$ and $(\mathcal{X},\mathcal{T})$ be a T.S. A function $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}:\mathcal{P}(\mathcal{X})\to\mathcal{P}(\mathcal{X})$ defined as $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)=\overline{A}^{\mathcal{I}^{\mathcal{K}}}$ is satisfying Kuratowski closure axioms*

$(K1)$ $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\varnothing)=\varnothing$ *and* $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X})=\mathcal{X}$,

$(K2)$ $A \subseteq \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) \quad \forall A \subseteq \mathcal{X}$,
$(K3)$ $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) \quad \forall A \subseteq \mathcal{X}$,
$(K4)$ $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A \cup B) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) \cup \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B) \quad \forall A, B \subseteq \mathcal{X}$.

P r o o f. $(K1)$ and $(K2)$ are clear from the definition of $\mathcal{I}^{\mathcal{K}}$-closure function. By Lemma 4, $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)$ is closed. So, $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)$. Therefore, $(K3)$ holds.

To prove $(K4)$, let $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) \cup \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$. Then, $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)$ or $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$. Without lost of generality assume that $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)$. So, $\exists (x_i) \subset A$ s.t. $x_i \xrightarrow{\mathcal{I}^{\mathcal{K}}} x$. Therefore, $\exists (x_i) \subset A \cup B$ s.t. $x_i \xrightarrow{\mathcal{I}^{\mathcal{K}}} x$. So, $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) \cup \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$.

Conversely, let $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A \cup B)$. Then, there exist a sequence $(x_i) \subset (A \cup B)$ s.t. $x_i \xrightarrow{\mathcal{I}^{\mathcal{K}}} x$. Assume that $x \notin \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)$ and $x \notin \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$. So, neither set A nor set B contains a sequence s.t. $\mathcal{I}^{\mathcal{K}}$-converges to the point x. Consequently, there is not any sequence in the $A \cup B$ which is convergent to x. But $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A \cup B)$ which is a contradiction. Hence,

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A \cup B) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) \cup \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$$

holds. □

Corollary 1. *A subset A of $\mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-closed iff $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A) = A$ and a subset $O \subset \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-open iff $\mathcal{X} - O$ is $\mathcal{I}^{\mathcal{K}}$-closed.*

Theorem 4. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and consider the T.S. $(\mathcal{X}, \mathcal{T})$. Then,*

$$\mathcal{T}_{\mathcal{I}^{\mathcal{K}}} := \{A \subset X : \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - A) = \mathcal{X} - A\}$$

is a topology over the set $\mathcal{X}$.

P r o o f. By $(K1)$, it is clear that $\mathcal{X} \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$ and $\varnothing \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$ hold. Let $A, B \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$ be arbitrary sets. To prove $A \cup B \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$ we must to prove that

$$\mathcal{X} - A \cup B = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - A \cup B)$$

holds. By $(K2)$, we have

$$\mathcal{X} - A \cup B \subset \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - A \cup B).$$

Now, let $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - A \cup B)$ be an arbitrarily element. Then, $\exists (x_i) \subset \mathcal{X} - (A \cup B)$ s.t. it is $\mathcal{I}^{\mathcal{K}}$-convergent to x. This implies that (x_i) is not subset of $A \cup B$. So, (x_i) is neither subset of A nor subset of B. Therefore, $(x_i) \subset \mathcal{X} - A$ or $(x_i) \subset \mathcal{X} - B$ which $\mathcal{I}^{\mathcal{K}}$-converges to point x. So, $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - A)$ or $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - B)$. Since $\mathcal{X} - A$ and $\mathcal{X} - B$ are closed sets, then

$$x \in (\mathcal{X} - A) \cup (\mathcal{X} - B) = \mathcal{X} - A \cup B$$

holds.

Let $\{A_i\}$ be a collection of $\mathcal{I}^{\mathcal{K}}$-open subsets of $\mathcal{X}$. Then, $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - A_i) = \mathcal{X} - A_i$ $\forall i \in \mathbb{N}$. By considering $(K2)$, we have

$$\cap_{i \in \mathbb{N}}(\mathcal{X} - A_i) \subseteq \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}\big(\cap_{n \in \mathbb{N}} (\mathcal{X} - A_i)\big).$$

Let $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}} \cap_{n \in \mathbb{N}} (\mathcal{X} - A_i)$ be an arbitrary element. Then, $\exists (x_i) \subset \cap_{n \in \mathbb{N}}(\mathcal{X} - A_i)$ which is $\mathcal{I}^{\mathcal{K}}$-convergent to x. Then, $(x_i) \subset (\mathcal{X} - A_i)$ $\forall i \in \mathbb{N}$. Since $\mathcal{X} - A_i$ are closed sets, then $x \in \mathcal{X} - A_i$ $\forall i \in \mathbb{N}$. Therefore,

$$x \in \cap_{i \in \mathbb{N}}(\mathcal{X} - A_i).$$

Hence, the set $\mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$ is a topology and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ is a T.S. □

Definition 14. *The T.S. $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ is called as $\mathcal{I}^{\mathcal{K}}$-sequential T.S. For abbreviation we will show it by $\mathcal{I}^{\mathcal{K}}$-seq.-top. An $\mathcal{I}^{\mathcal{K}}$-seq.-top. $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ is said to be $\mathcal{I}^{\mathcal{K}}$-discrete space if $\mathcal{T}_{\mathcal{I}^{\mathcal{K}}} = \mathcal{P}(\mathcal{X})$.*

Theorem 5. *Let $\mathcal{I}$, $\mathcal{K}$, $\mathcal{I}_1$, $\mathcal{K}_1$, $\mathcal{I}_2$ and $\mathcal{K}_2$ stand for ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ represents a T.S. Let $\mathcal{I}_1 \subset \mathcal{I}_2$ and $\mathcal{K}_1 \subset \mathcal{K}_2$. Then,*
1. $\mathcal{T}_{\mathcal{I}^{\mathcal{K}_2}} \prec \mathcal{T}_{\mathcal{I}^{\mathcal{K}_1}}$,
2. $\mathcal{T}_{\mathcal{I}_2^{\mathcal{K}}} \prec \mathcal{T}_{\mathcal{I}_1^{\mathcal{K}}}$.

P r o o f. Let υ be any $\mathcal{I}^{\mathcal{K}_2}$-open subset of $\mathcal{X}$. Then, $\mathcal{X}-\upsilon$ is $\mathcal{I}^{\mathcal{K}_2}$-closed and $\mathrm{cl}_{\mathcal{I}^{\mathcal{K}_2}}(\mathcal{X}-\upsilon) = \mathcal{X}-\upsilon$ hold. To prove υ is $\mathcal{I}^{\mathcal{K}_1}$-open subset of$\mathcal{X}$, we will show that

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}_1}}(\mathcal{X} - \upsilon) \subset \mathcal{X} - \upsilon.$$

Let $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}_1}}(\mathcal{X} - \upsilon)$ be any point. Then, there exists $(x_i) \subset \mathcal{X} - \upsilon$ s.t. $x_i \overset{\mathcal{I}^{\mathcal{K}_1}}{\to} x$. Since $\mathcal{K}_1 \subset \mathcal{K}_2$, then by Proposition 3.6 in [13], $x_i \overset{\mathcal{I}^{\mathcal{K}_2}}{\to} x$. So, $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}_2}}(\mathcal{X} - \upsilon)$. Therefore, $x \in \mathcal{X} - \upsilon$. Hence $\mathcal{X} - \upsilon$ is $\mathcal{I}^{\mathcal{K}_2}$-closed set and υ is $\mathcal{I}^{\mathcal{K}_2}$-open subset of $\mathcal{X}$.

The second one can be proved by using the fact that if $\mathcal{I}_1 \subset \mathcal{I}_2$, then, $x_i \overset{\mathcal{I}_1^{\mathcal{K}}}{\to} x$ implies $x_i \overset{\mathcal{I}_2^{\mathcal{K}}}{\to} x$, it easily can be proved. □

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ represent a T.S. Then, every $\mathcal{I}^*$-open set is $\mathcal{I}^{\mathcal{K}}$-open set.*

P r o o f. If we take $\mathcal{K} = \mathcal{F}in$ then $\mathcal{I}^*$-open set will be $\mathcal{I}^{\mathcal{K}}$-open set. □

Theorem 7. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ represent a T.S. Then, every $\mathcal{I}^{\mathcal{K}}$-open set is $\mathcal{K}$-open set.*

P r o o f. Let υ be an arbitrary $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$. Then, $\mathcal{X} - \upsilon$ is $\mathcal{I}^{\mathcal{K}}$-closed and

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - \upsilon) = \mathcal{X} - \upsilon.$$

To prove υ is $\mathcal{K}$ open, it is sufficient to show that $\mathcal{X} - \upsilon$ is $\mathcal{K}$-closed, i.e,

$$\mathcal{X} - \upsilon = \overline{\mathcal{X} - \upsilon}^{\mathcal{K}}.$$

It is clear that $\mathcal{X} - \upsilon \subset \overline{\mathcal{X} - \upsilon}^{\mathcal{K}}$. Let $x \in \overline{\mathcal{X} - \upsilon}^{\mathcal{K}}$ be an arbitrary element s.t. $\exists (x_i) \subset \mathcal{X} - \upsilon$ satisfying $x_i \overset{\mathcal{K}}{\to} x$.

Then, by Lemma 3.5 in [13] we have $x_i \overset{\mathcal{I}^{\mathcal{K}}}{\to} x$. So, $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{X} - \upsilon) = \mathcal{X} - \upsilon$. Hence, the theorem proved. □

Proposition 2. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T})$ represent a T.S. Then, the following statements are true:*

1. *If $\mathcal{K} \subset \mathcal{I}$, then, each $\mathcal{I}$-open set is $\mathcal{I}^{\mathcal{K}}$-open set.*
2. *If the space $\mathcal{X}$ is a first countable space and the ideal $\mathcal{I}$ has additive property with respect to $\mathcal{K}$ (see Definition 3.10 in [13]), then, each $\mathcal{I}^{\mathcal{K}}$-open set is $\mathcal{I}$-open set.*
3. *If $\mathcal{I} \subset \mathcal{K}$, then every $\mathcal{K}$-open set is $\mathcal{I}^{\mathcal{K}}$-open.*

P r o o f. The proof is obvious from Proposition 3.7 and Theorem 3.11 of [13]. □

5. $\mathcal{I}^{\mathcal{K}}$-continuity of functions

In this section we will define $\mathcal{I}^{\mathcal{K}}-$continuous and sequential $\mathcal{I}^{\mathcal{K}}$-continuous functions. We will prove that in any $\mathcal{I}^{\mathcal{K}}$-sequential T.S. these two concepts coincide. Also, we will state some theorems that give the definition of $\mathcal{I}^{\mathcal{K}}$-continuous function in different words and ways. At the end of this section we will see that the combination of $\mathcal{I}^{\mathcal{K}}$-continuous functions is $\mathcal{I}^{\mathcal{K}}$-continuous.

Definition 15. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces. A function f, from $\mathcal{X}$ to $\mathcal{Y}$ is said to be*

(*i*) *$\mathcal{I}^{\mathcal{K}}$-continuous which provides that inverse image of any $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-open in $\mathcal{X}$.*

(*ii*) *Sequentially $\mathcal{I}^{\mathcal{K}}$-continuous which provides that $f(x_i) \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} f(x)$ $\forall (x_i) \subset \mathcal{X}$ with $x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$.*

Theorem 8. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces; and f, from $\mathcal{X}$ to $\mathcal{Y}$ be a function. Then, f is $\mathcal{I}^{\mathcal{K}}$-continuous iff it is sequentially $\mathcal{I}^{\mathcal{K}}$-continuous.*

P r o o f. Let f be an $\mathcal{I}^{\mathcal{K}}$-continuous function. Then, inverse image of any $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-open subset in $\mathcal{X}$. Let $(x_i) \subset \mathcal{X}$ be a sequence with $x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$. Then, there exists $M \in \mathcal{F}(\mathcal{I})$ s.t. the following sequence

$$t_i := \begin{cases} x_i, & i \in M, \\ x, & i \notin M \end{cases}$$

is $\mathcal{K}$-convergent to x. That is, for each neighborhood υ of x we have

$$\{i \in \mathbb{N} : \ t_i \in \upsilon\} \in \mathcal{F}(\mathcal{K}).$$

Let $\mathcal{V}$ be any $\mathcal{I}^{\mathcal{K}}$-open neighborhood of $f(x)$. Then, $f^{-1}(\mathcal{V})$ is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$ which contains the point x. So, it is a neighborhood of x. Therefore,

$$\{i \in \mathbb{N} : \ t_i \in f^{-1}(\mathcal{V})\} \in \mathcal{F}(\mathcal{K}),$$

implies that $\{i \in \mathbb{N} : \ f(t_i) \in \mathcal{V}\} \in \mathcal{F}(\mathcal{K})$. Hence, the sequence

$$f(t_i) := \begin{cases} f(x_i), & i \in M, \\ f(x), & i \notin M \end{cases}$$

is $\mathcal{K}$-convergent to $f(x)$. So, $f(x_i) \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} f(x)$. Hence, f is sequentially $\mathcal{I}^{\mathcal{K}}$-continuous function.

Conversely, let the function f be sequentially $\mathcal{I}^{\mathcal{K}}$-continuous and υ is any $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$. Assume that $f^{-1}(\upsilon)$ is not $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$. Then, $\mathcal{X} - f^{-1}(\upsilon)$ is not $\mathcal{I}^{\mathcal{K}}$-closed subset of $\mathcal{X}$. So,

$$\exists (x_i) \subset \mathcal{X} - f^{-1}(\upsilon) \quad s.t. \quad x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x \quad \text{and} \quad x \notin \mathcal{X} - f^{-1}(\upsilon),$$

i.e. $x_i \notin f^{-1}(\upsilon)$ $\forall n$ and $x_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} x$ which means $x \in f^{-1}(\upsilon)$. Since f is $\mathcal{I}^{\mathcal{K}}$-sequentially continuous function then $f(x_i) \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} f(x)$. So, $f(x) \in \upsilon$ and $f(x_i) \notin \upsilon$ $\forall n$. This is a contradiction. □

Lemma 5. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces and f, from $\mathcal{X}$ to $\mathcal{Y}$ be an $\mathcal{I}^{\mathcal{K}}$-continuous function. If $(y_i) \subset \mathcal{Y}$ be a sequence s.t. $y_i \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} y$, then $f^{-1}(y_i) \stackrel{\mathcal{I}^{\mathcal{K}}}{\to} f^{-1}(y)$.*

P r o o f. Let f be an $\mathcal{I}^{\mathcal{K}}$-continuous function. Let $y_i \overset{\mathcal{I}^{\mathcal{K}}}{\to} y$ then $\exists M \in \mathcal{F}(\mathcal{I})$ s.t. the sequence

$$s_n = \begin{cases} y_i, & i \in M, \\ y, & i \notin M \end{cases}$$

is $\mathcal{K}$-convergent to y. So, for each neighborhood υ of $\mathcal{Y}$,

$$\{i \in \mathbb{N} : y_i \in \upsilon\} \in \mathcal{F}(\mathcal{K}).$$

Since f is $\mathcal{I}^{\mathcal{K}}$-continuous function, then inverse image of any $\mathcal{I}^{\mathcal{K}}-$ open set in $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-open in $\mathcal{X}$, $f^{-1}(\upsilon)$ is open neighborhood of x in $\mathcal{X}$. Then

$$\{i \in \mathbb{N} : f^{-1}(y_i) \in f^{-1}(\upsilon)\} \in \mathcal{F}(\mathcal{K}).$$

Therefore,

$$f^{-1}(s_n) = \begin{cases} f^{-1}(y_i), & i \in M, \\ f^{-1}(y), & i \notin M, \end{cases}$$

is $\mathcal{K}$-convergent to $f^{-1}(y)$ and hence $f^{-1}(y_i) \overset{\mathcal{I}^{\mathcal{K}}}{\to} f^{-1}(y)$. □

Theorem 9. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces. Then the function f, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-continuous iff*

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$$

holds $\forall B \subset \mathcal{Y}$.

P r o o f. Assume that function f, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-continuous function. Let

$$x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B)).$$

Then, $\exists (x_i) \subset f^{-1}(B)$ s.t. $x_i \overset{\mathcal{I}^{K}}{\to} x$. Since f is $\mathcal{I}^{\mathcal{K}}$-continuous so,

$$f(x_i) \overset{\mathcal{I}^{\mathcal{K}}}{\to} f(x).$$

In another hand $(x_i) \subset B$, so $f(x) \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$ and $x \in f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B))$.

Now, let $x \in f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B))$, i.e. $f(x) \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$. Therefore, $\exists (y_i) \subset B$ s.t. $x_i \overset{\mathcal{I}^{\mathcal{K}}}{\to} x$. Then, by Lemma 5 there exists $(x_i) = (f^{-1}(y_i) \subset f^{-1}(B)$ s.t. $x_i \overset{\mathcal{I}^{\mathcal{K}}}{\to} x$, where $x = f^{-1}(y)$ holds. So, $x \in \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B))$. Hence,

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B).$$

Conversely, let

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B), \quad \forall B \in \mathcal{P}(\mathcal{Y}).$$

Let υ be $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$ then

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{Y} - B) = \mathcal{Y} - B.$$

Let $B = \mathcal{Y} - \upsilon$, then

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(\mathcal{Y} - \upsilon)) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\mathcal{Y} - \upsilon)) = f^{-1}(\mathcal{Y} - \upsilon).$$

This shows that $f^{-1}(\mathcal{Y} - \upsilon)$ is $\mathcal{I}^{\mathcal{K}}$-closed. Hence, the following equality

$$f^{-1}(\mathcal{Y} - \upsilon) = \mathcal{X} - f^{-1}(\upsilon)$$

implies that $\mathcal{X} - f^{-1}(\upsilon)$ is $\mathcal{I}^{\mathcal{K}}$-closed. Therefore $f^{-1}(\upsilon)$ is $\mathcal{I}^{\mathcal{K}}$-open set. □

Corollary 2. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces. A function f, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-continuous iff*

$$\mathrm{int}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B) = f^{-1}(\mathrm{int}_{\mathcal{I}^{\mathcal{K}}}(B) \quad \forall B \subset \mathcal{Y}.$$

Definition 16. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces and f, from $\mathcal{X}$ to $\mathcal{Y}$ be a function. The function f is $\mathcal{I}^{\mathcal{K}}$-continuous at a point $x \in \mathcal{X}$ if inverse image of any neighborhood of $f(x)$ is a neighborhood of x in $\mathcal{X}$.*

Corollary 3. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces. Then, the function f, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-continuous iff it is $\mathcal{I}^{\mathcal{K}}$-continuous at every point $x \in \mathcal{X}$.*

Definition 17. *Let $\mathcal{I}$ and $\mathcal{K}$ stand for the ideals of $\mathbb{N}$ and $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ represent $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces and f, from $\mathcal{X}$ to $\mathcal{Y}$ be a function, f is said to be $\mathcal{I}^{\mathcal{K}}$-closure preserving if*

$$f(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f(A) \quad \forall A \subset \mathcal{X}.$$

Theorem 10. *The function f, from $\mathcal{X}$ to $\mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-continuous iff it is $\mathcal{I}^{\mathcal{K}}$-closure preserving.*

P r o o f. Let $f : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{I}^{\mathcal{K}}$-continuous function. Then, for any subset B of $\mathcal{Y}$

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(B) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(B)$$

holds. Consider a set $A \subset \mathcal{X}$ s.t. $f(A)$ is subset of $\mathcal{Y}$. So,

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(f(A)) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f(A))$$

holds and it implies that $f(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f(A))$ $\forall A \subset \mathcal{X}$ holds.

Conversely, let f be $\mathcal{I}^{\mathcal{K}}$-closure preserving function, then

$$f(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f(A)) \quad \forall A \subset \mathcal{X}.$$

Let υ be any subset of $\mathcal{Y}$, then $f^{-1}(\upsilon)$ is subset of $\mathcal{X}$ and

$$f(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(\upsilon))) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f(f^{-1}(\upsilon) = \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\upsilon)$$

holds. So

$$\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(f^{-1}(\upsilon) = f^{-1}(\mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(\upsilon)$$

and by Theorem 9 the function f is $\mathcal{I}^{\mathcal{K}}$-continuous. □

Theorem 11. *Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be $\mathcal{I}^{\mathcal{K}}$-seq.-top. spaces. Let f, from $\mathcal{X}$ to $\mathcal{Y}$ and g, from $\mathcal{Y}$ to $\mathcal{Z}$ be $\mathcal{I}^{\mathcal{K}}$-continuous functions. Then $g \circ f : \mathcal{X} \to \mathcal{Z}$ is $\mathcal{I}^{\mathcal{K}}$-continuous functions.*

P r o o f. Let υ be any $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Z}$. Since g is $\mathcal{I}^{\mathcal{K}}$-continuous function then $g^{-1}(\upsilon)$ is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$ and because f is $\mathcal{I}^{\mathcal{K}}$-continuous function therefore $f^{-1}(g^{-1}(\upsilon))$ is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$ hence $(g \circ f)^{-1}(\upsilon)$ is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$. □

6. Subspace of $\mathcal{I}^{\mathcal{K}}$-seq.-top. space

In this section subspaces of the $\mathcal{I}^{\mathcal{K}}$-seq.-top. space and its properties under an $\mathcal{I}^{\mathcal{K}}$-continuous function will be discussed.

Definition 18. *Let $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ be an $\mathcal{I}^{\mathcal{K}}$-seq.-top. space and $\mathcal{Y} \subset \mathcal{X}$. Then*

$$C_Y : \mathcal{P}(\mathcal{Y}) \to \mathcal{P}(\mathcal{Y}), \quad C_Y(A) = \mathcal{Y} \cap \mathrm{cl}_{\mathcal{I}^{\mathcal{K}}}(A)$$

is a Kuratowsky operator. Define a T.S. as $(\mathcal{Y}, \mathcal{T}^{\mathcal{Y}}_{\mathcal{I}^{\mathcal{K}}})$, where

$$\mathcal{T}^{\mathcal{Y}}_{\mathcal{I}^{\mathcal{K}}} = \{U \cap \mathcal{Y}, Y \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}\} \subset \mathcal{P}(\mathcal{Y}).$$

This T.S. is called $\mathcal{I}^{\mathcal{K}}$-subspace of $\mathcal{X}$.

Lemma 6. *Let $\mathcal{Y}$ be an $\mathcal{I}^{\mathcal{K}}$-subspace of $\mathcal{I}^{\mathcal{K}}$-seq.-top. space $\mathcal{X}$. If set A is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$ and $\mathcal{Y}$ is an $\mathcal{I}^{\mathcal{K}}$-subset of $\mathcal{X}$. Then A is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$.*

P r o o f. Let A be $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$. Then $\exists U \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$ s.t. $A = \mathcal{Y} \cap U$. Since $\mathcal{Y}$ is an $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$. Then $A \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$. ☐

Proposition 3. *Let $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ and $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ be $\mathcal{I}^{\mathcal{K}}$-sequential spaces, $f : \mathcal{X} \to \mathcal{Y}$ be $\mathcal{I}^{\mathcal{K}}$-continuous function and $A \subset \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-subspace of $\mathcal{X}$. Then $f_{/A} : A \to \mathcal{Y}$, the restriction f over A is $\mathcal{I}^{\mathcal{K}}$-continuous function.*

P r o o f. Let U be an $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{Y}$. Since f is $\mathcal{I}^{\mathcal{K}}$-continuous function then $f^{-1}(U)$ is $\mathcal{I}^{\mathcal{K}}$-open subset of $\mathcal{X}$. That is $f^{-1}(U) \in \mathcal{T}_{\mathcal{I}^{\mathcal{K}}}$.
In other hand $f^{-1}_{/A}(U) = A \cap f^{-1}(U)$. So $f^{-1}_{/A}(U)$ is $\mathcal{I}^{\mathcal{K}}$-open subset of subspace A. Hence $f_{/A}$ is $\mathcal{I}^{\mathcal{K}}$-continuous function. ☐

Lemma 7. *If A is $\mathcal{I}^{\mathcal{K}}$-subspace of $\mathcal{I}^{\mathcal{K}}$-sequential T.S. $\mathcal{X}$. Then the inclusion map $j : A \to \mathcal{X}$ is $\mathcal{I}^{\mathcal{K}}$-continuous.*

P r o o f. If U is $\mathcal{I}^{\mathcal{K}}$-open in $\mathcal{X}$ then $j^{-1}(U) = U \cap A$ is $\mathcal{I}^{\mathcal{K}}$-open in subspace $\mathcal{Y}$ hence j is $\mathcal{I}^{\mathcal{K}}$-continuous. ☐

Proposition 4. *Let $(\mathcal{X}, \mathcal{T}_{\mathcal{I}^{\mathcal{K}}})$ and $(\mathcal{Y}, \mathcal{T}'_{\mathcal{I}^{\mathcal{K}}})$ be $\mathcal{I}^{\mathcal{K}}$-sequential spaces, $B \subset \mathcal{Y}$ be subspace of $\mathcal{Y}$ and $f : \mathcal{X} \to B$ be $\mathcal{I}^{\mathcal{K}}$-continuous function. Then, $h : \mathcal{X} \to \mathcal{Y}$ obtained by expanding the range of f is $\mathcal{I}^{\mathcal{K}}$-continuous.*

P r o o f. To show $h : \mathcal{X} \to \mathcal{Y}$ is $\mathcal{I}^{\mathcal{K}}$-continuous function, if B as subspace of $\mathcal{Y}$ then note that h is the composition of the map $f : \mathcal{X} \to B$ and $j : B \to \mathcal{Y}$. ☐

7. Conclusion

In this article we defined the notion of $\mathcal{I}^{\mathcal{K}}$-closed (resp. $\mathcal{I}^{\mathcal{K}}$-open) set in a T.S. $(\mathcal{X}, \mathcal{T})$ and established some important results concerning this notion. Furthermore, we defined the $\mathcal{I}^{\mathcal{K}}$-seq.-top., which is a generalized form of the $\mathcal{I}^*$-sequential space. We also talked about $\mathcal{I}^{\mathcal{K}}$-continuity of functions and saw that in $\mathcal{I}^{\mathcal{K}}$-seq.-top. space the notion of continuity and sequential continuity are the same. And in the last section of the paper, subspace of $\mathcal{I}^{\mathcal{K}}$-sequential space have been studied and some important results established.

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