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Adaptive Dimensional Reduction and Divergence Stability

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Abstract. There has been increased interest recently in feed-back methods for reliable, robust, efficient computational methods in mechanics. We will outline the construction of such methods for a class of problems describing bending of plates for certain classes of loads. We emphasize the theoretical performance of this method, such as the selection of basis functions in the dimensional reduction—ensuring optimal convergence rates in the "thin" limit as well as in the order of model increasing. We then discuss the feasibility of efficient solution of the first couple of members of the sequence of models—also with an eye towards reliability and adaptivity.

Key words: Adaptive feedback methods, dimensional reduction, Galerkin methods, plate modelling, divergence stability.

AMS(MOS) subject classifications: 65N30, 65N50, 73C50, 73K25.

1. Introduction

This brief note represents a recap of the talk by the same title given at the international Optimization of Finite Element Approximations (OFEA'95) Conference, June 25–29, 1995 in St.Petersburg, Russia. We have chosen to treat an aspect of our work, an example of dimensional reduction which is described in [14] that is not yet available. Various additional considerations for this elliptic systems' case are given in view of recent results of the authors' published in [17] (but the idea of which already was in [16]). Various other results of theoretical and computational sort for problems of scalar type with quasilinear divergence form monotone operators were also covered in the talk, but are readily available in the literature in [11] through [15], and hence are not covered here. A survey of various methodologies for reducing the dimension was given in [15]. A more detailed version of this work will be published elsewhere (probably to be submitted to *The journal of mathematical physics and numerical mathematics*).

The purpose of our work is making the method of dimensional reduction still more efficient and robust. We continue to be motivated by problems arising from the study of "thin" elastic structures.

Following Kantorovich, Gordeziani, Poniatovskii, Babuška, Vogelius, Schwab, and others, we shall investigate further an energyasymptotic approach where the basic idea—as in [10]—

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is to find a minimizer u_N of the given energy functional in a proper subspace V_N which is characterized by some basis functions $\{\psi_j\}_{j=0}^N$:

$$V_N = \left\{ \sum_{j=0}^{N} c_j(\mathbf{x}')\psi_j\left(x_n/\varepsilon(\mathbf{x}')\right) \right\}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) = (\mathbf{x}', x_n)$ and $\mathbf{x}' \in \omega$, the mid-surface, and $x_n \in [-\varepsilon, \varepsilon]$, and ε denotes the half thickness of the domain. Thus the model of order N of reduced dimension was introduced. The hope is that, for "small" ε, linear combinations from V_N will approximate the exact solution well. See [10] for the choice of $\{\psi_j\}_{j=0}^N$ and related convergence properties (optimal rates) as $\varepsilon \downarrow 0$ or $N \to \infty$. In the next section we introduce the model problem which functions as an example for which we can display some of the details of the underlying general ideas in the ensuing sections. Finally we will comment on solving individual dimensionally reduced models.

Due to the singularities which can stem from the loading, nonsmooth coefficients or the presence of corners and edges, it is necessary (for efficiency and accuracy) to be able to introduce higher order models near such layers only. In [12]–[14], we proposed and analyzed a feedback extension procedure that facilitated this. The generalization which allows different orders N_i in different parts of ω is not treated here beyond that we briefly mention a result from [14] for a scalar, elliptic problem on **adaptivity w.r.t. convergence rate**: Let the approximation quality of the subspace $V_{\mathcal{N}}$ at u be expressed as:

$$\Phi(u, N) = \inf_{N=\|\mathcal{N}\|_\infty \text{fixed}} \ \inf_{v \in V_{\mathcal{N}}} \|u - v\|_{H^1}$$

where $\mathcal{N} = (N_i)_i$. Then it is possible under very strict hypotheses to prove a result as the following, cf. [15]:

Theorem [15]. *Let u and $u_{\mathcal{N}}$ be the exact and dimensionally reduced solutions using above Heuristic with $\alpha_2 = 0$ and, additionally, have gradients bounded uniformly in ε. Let the local regularity of u be given by: $\exists r_i > 1 : \ u|_{\mathbf{I}_i} \in W^{(1,r_i)2}_{(\xi,\eta)}(\mathbf{I}_i \times \mathbf{I}) \setminus W^{(1,r_i+\varepsilon)2}_{(\xi,\eta)}(\mathbf{I}_i \times \mathbf{I})\ \forall \varepsilon > 0$, and $1 \le i \le m$. Then*

$$|u - u_{\mathcal{N}}\| \le C\Phi(u, N)$$

where C does not depend on $\mathcal{N}$.

The proof and an explanation of the Heuristic and the regularity is given in [15].

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2. Linear plate models

Consider a homogeneous, linearly elastic material (characterized by E Young's modulus and ν Poisson's ratio) in its reference configuration occupying the region

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon) \subset \mathbf{R}^3$$

where

$$\omega = (0, 1)^2$$

and denote the six faces of the boundary by

$$\Gamma^\varepsilon_{i\pm} \stackrel{\text{def}}{=} \{x \in \partial\Omega^\varepsilon : \text{outward normal } n(x) = \pm e_i\}.$$

Let us consider the classical system of *linear* elasticity

$$\begin{aligned} &\frac{1+\nu}{E}\sigma^\varepsilon_{ij} - \frac{\nu}{E}\sigma^\varepsilon_{kk}\delta_{ij} = \gamma_{ij}, && \text{in } \Omega^\varepsilon \\ &-\partial_j\sigma^\varepsilon_{ij} = f^\varepsilon_i, && \text{in } \Omega^\varepsilon \end{aligned} \tag{1}$$

subject to (partially periodic) boundary conditions

$$\begin{aligned} n^\varepsilon_3\sigma^\varepsilon_{i3} &= g^\varepsilon_i, \quad \text{on } \Gamma^\varepsilon_1 \\ \sigma^\varepsilon_{1i}|_{\Gamma^\varepsilon_{1+}} &= \sigma^\varepsilon_{1i}|_{\Gamma^\varepsilon_{1-}} \quad \text{and} \quad \sigma^\varepsilon_{2i}|_{\Gamma^\varepsilon_{2+}} = \sigma^\varepsilon_{2i}|_{\Gamma^\varepsilon_{2-}} \\ u^\varepsilon_i|_{\Gamma^\varepsilon_{\alpha+}} &= u^\varepsilon_i|_{\Gamma^\varepsilon_{\alpha-}} \end{aligned} \tag{2}$$

using the Einstein summation & range conventions:

- Roman subscripts vary in $\{1,2,3\}$, greek only in $\{1,2\}$.
- σ^ε is the stress tensor.
- u^ε is the displacement vector (as functions of x).
- x is the location in the reference configuration.
- $\gamma(u^\varepsilon)$ symmetric part of the deformation gradient:

$$2\gamma_{ij}(u^\varepsilon) = \partial_i u^\varepsilon_j + \partial_j u^\varepsilon_i.$$

- The body force f and the traction load g are supposed to lie in $(L^2)^3$ and to satisfy the compatibility condition:

$$\int_{\Omega^\varepsilon} f^\varepsilon_i \, dx + \int_{\Gamma^\varepsilon_{3\pm}} g^\varepsilon_i \, d\Gamma = 0, \quad i = 1,2,3.$$

Denote the displacement space:

$$V^\varepsilon = \left\{ v \in \left(H^1(\Omega^\varepsilon)\right)^3 : v|_{\Gamma^\varepsilon_{\alpha+}} = v|_{\Gamma^\varepsilon_{\alpha-}}, \ \alpha = 1,2; \ \int_{\Omega^\varepsilon} v_i \, dx = 0, \quad i = 1,2,3 \right\} \tag{3}$$

and (symmetric) stress space:

$$\Sigma^\varepsilon = \left(L^2(\Omega^\varepsilon)\right)^9_{\text{sym}}.$$

We use the Hellinger-Reissner variational principle to give a mixed formulation:
Find $(u^\varepsilon, \sigma^\varepsilon) \in (V^\varepsilon, \Sigma^\varepsilon)$ such that

$$\begin{aligned} &A(\sigma^\varepsilon, \tau) + B(u^\varepsilon, \tau) = 0, && \forall \tau \in \Sigma^\varepsilon, \\ &B(v, \sigma^\varepsilon) = F(v), && \forall v \in V^\varepsilon, \end{aligned} \tag{4}$$

where the forms are:

$$\begin{aligned} A(\sigma^\varepsilon,\tau) &= \int_{\Omega^\varepsilon} \frac{1+\nu}{E}\sigma^\varepsilon_{ij}\tau_{ij} - \frac{\nu}{E}\sigma^\varepsilon_{ii}\tau_{jj}\,dx, \\ B(u^\varepsilon,\tau) &= \int_{\Omega^\varepsilon} \gamma_{ij}(u^\varepsilon)\tau_{ij}\,dx, \quad \text{and} \qquad (5)\\ F(v) &= \int_{\Omega^\varepsilon} f^\varepsilon_i v_i\,dx + \int_{\Gamma^\varepsilon_3} g^\varepsilon_i v_i\,d\Gamma. \qquad (6) \end{aligned}$$

We may formulate (4) in a symmetric way:
Find $(u^\varepsilon,\sigma^\varepsilon) \in (V^\varepsilon,\Sigma^\varepsilon)$ such that

$$\mathcal{A}\Big((u^\varepsilon,\sigma^\varepsilon);(v,\tau)\Big) = F(v), \quad \forall (v,\tau) \in V^\varepsilon \times \Sigma^\varepsilon, \tag{7}$$

where we define

$$\mathcal{A}\Big((u^\varepsilon,\sigma^\varepsilon);(v,\tau)\Big) \stackrel{\text{def}}{=} A(\sigma^\varepsilon,\tau) + B(u^\varepsilon,\tau) + B(v,\sigma^\varepsilon). \tag{8}$$

Use $\|(v,\tau)\|_{X^\varepsilon} = \|\gamma(v)\|_{L^2} + \|\tau\|_{L^2}$ as a norm on $X^\varepsilon = V^\varepsilon \times \Sigma^\varepsilon$ (by Korn's inequality). The saddle-point problem (4) is well-posed since:

$$\inf_{(u,\sigma)\in X^\varepsilon} \sup_{(v,\tau)\in X^\varepsilon} \frac{|\mathcal{A}((u,\sigma);(v,\tau))|}{\|(u,\sigma)\|_{X^\varepsilon}\,\|(v,\tau)\|_{X^\varepsilon}} \geq C > 0, \tag{9}$$

where C only depends on E,ν (and not on ε) (by [26]). We then define a *reduced model* of order N by restricting all forms in (4) to spaces

$$\begin{aligned} V^\varepsilon_N &= \left\{ \sum_{j=0}^{N} v_j(x_1,x_2)\ell_j(x_3) : v_j \in \Big(H^1(\omega)\Big)^3 \right\} \cap V^\varepsilon, \\ \Sigma^\varepsilon_N &= \left\{ \sum_{j=0}^{N} \tau_j(x_1,x_2)\ell_j(x_3) : \tau_j \in \Big(L^2(\omega)\Big)^9 \right\} \cap \Sigma^\varepsilon, \end{aligned} \tag{10}$$

where ℓ_j denotes the j-th Legendre polynomial (with point value $+1$ at $x_3 = +1$) that is,
Find $(u^\varepsilon_N,\sigma^\varepsilon_N) \in (V^\varepsilon_N,\Sigma^\varepsilon_N)$ such that

$$\begin{aligned} &A(\sigma^\varepsilon_N,\tau) + B(u^\varepsilon_N,\tau) = 0, \quad \forall \tau \in \Sigma^\varepsilon_N, \\ &B(v,\sigma^\varepsilon_N) = F(v), \qquad\qquad\quad\;\; \forall v \in V^\varepsilon_N. \end{aligned} \tag{11}$$

Due to (9), we have

$$\|(u^\varepsilon,\sigma^\varepsilon) - (u^\varepsilon_N,\sigma^\varepsilon_N)\|_{X^\varepsilon} \leq C \inf_{(v,\tau)\in V^\varepsilon_N\times\Sigma^\varepsilon_N} \|(u^\varepsilon,\sigma^\varepsilon) - (v,\tau)\|_{X^\varepsilon} \tag{12}$$

which leads to quasi-optimal estimates with respect to $N \nearrow \infty$.

Actually we may wish to vary N over the components of the displacement vector and stress tensor. Let then

$$\mathbf{N} = (N_1,N_2,N_3) \quad \text{where } N_1 = N_2 = N_3 \pm 1,$$

where we still reserve the right to adjust the third exponent up or down according to some particular criterion.

3. Scaling of the third variable x_3

We scale $x_3 \rightsquigarrow x_3/\varepsilon$; we let

$$\begin{aligned} u_\alpha(x_1, x_2, x_3/\varepsilon) &= u^\varepsilon_\alpha(x_1, x_2, x_3), \\ u_3(x_1, x_2, x_3/\varepsilon) &= \varepsilon u^\varepsilon_3(x_1, x_2, x_3) \end{aligned}$$

for $x \in \Omega^\varepsilon$—similar scalings are done for σ, f, and g (à la Ciarlet & Destuynder). This yields a similar problem over the domain

$$\Omega = \omega \times (-1, 1).$$

Modifying the spaces to

$$V = \left\{ v \in \left(H^1(\Omega)\right)^3 : v|_{\Gamma_{\alpha+}} = v|_{\Gamma_{\alpha-}},\ \alpha = 1, 2;\ \int_\Omega v_i = 0,\ i = 1, 2, 3 \right\}$$

and

$$\Sigma = \left(L^2(\Omega)\right)^9_{\mathrm{sym}},$$

we now pose (4) over $X = V \times \Sigma$ in stead and the form A becomes parameter dependent:

$$A = A_0 + \varepsilon^2 A_2 + \varepsilon^4 A_4,$$

where

$$\begin{aligned} A_0(\sigma, \tau) &= \int_\Omega \frac{1+\nu}{E} \sigma_{\alpha\beta}\tau_{\alpha\beta} - \frac{\nu}{E} \sigma_{\alpha\alpha}\tau_{\beta\beta}\, dx, \\ A_2(\sigma, \tau) &= \int_\Omega \frac{1+\nu}{E} \sigma_{\alpha 3}\tau_{\alpha 3} - \frac{\nu}{E} (\sigma_{\mu\mu}\tau_{33} + \sigma_{33}\tau_{\mu\mu})\, dx, \\ A_4(\sigma, \tau) &= \int_\Omega \frac{1}{E} \sigma_{33}\tau_{33}\, dx. \end{aligned} \tag{13}$$

We may formally expand

$$(u, \sigma) = \sum_{i=0}^{\infty} (u^i, \sigma^i)\varepsilon^{2i}. \tag{14}$$

and match to see that the coefficient functions must satisfy

$$\begin{aligned} &A_0(\sigma^0, \tau) + B(u^0, \tau) = 0, \quad &\forall \tau \in \Sigma, \\ &B(v, \sigma^0) = F(v), &\forall v \in V \end{aligned} \tag{15}$$

for the reduced solution (with the $\cdot^\varepsilon$ dropped in the definition of F) and we get for the higher orders $j \geq 1$:

$$\begin{aligned} &A_0(\sigma^j, \tau) + B(u^j, \tau) = -A_2(\sigma^{j-1}, \tau) - A_4(\sigma^{j-2}, \tau), \quad &\forall \tau \in \Sigma, \\ &B(v, \sigma^j) = 0, &\forall v \in V \end{aligned}$$

(with $\sigma^{-1} \stackrel{\text{def}}{=} 0$). Let $\int_\Omega u^j = 0$, $j \geq 1$ for uniqueness. Please note the striking resemblance with Stokes' type problems for the individual models. Assuming that

- f is the restriction to $\overline{\Omega}$ of a 1-periodic function $F \in (C^\infty(\mathbf{R}^2 \times [-1,1])^3$ and
- g is the restriction to $\overline{\Gamma_3}$ of a 1-periodic function G with $G(\cdot,\cdot,\pm 1) \in (C^\infty(\mathbf{R}^2))^3$.

Paumier was able to show existence, uniqueness, and periodicity of the coefficient functions (u^j, σ^j), see Theorem 1 in [18].

We note that one can avoid boundary layers under such periodicity assumptions. Paumier was furthermore able to show convergence of the asymptotic expansion series:

Theorem 1. *Assume that f and g are trigonometric polynomials in $2\pi\,(k_1,k_2)\cdot(x_1,x_2)$, $|k| \leq K$ and that the the coefficients for f belong to $C^\infty([-1,1])$. Let m, $q \in \mathbf{N}$. Then there exists a constant $Q > 0$ such that the expansion (14) converges to (u,σ) in $H^m((-1,1);H^q(\omega))$ provided $\varepsilon K < Q$.*

The proof along with a discussion of the necessity of the finiteness of the Fourier expansion may be found in [18].

The terms in the asymptotic expansion in (14) can be verified to be polynomials in x_3.

4. The selection of basis functions

The first term—the reduced solution :

u^0_α are linear polynomials in x_3,
u^0_3 is constant w.r.t. x_3.

See [7]. Thus,

$$\begin{aligned} (u^0_\alpha, 0) &\in V_1 \quad \text{and} \\ (0,0,u^0_3) &\in V_0. \end{aligned}$$

[18] and [20] show that

$$\begin{aligned} (u^1_\alpha; 0) &\in V_3 \quad \text{and} \\ (0,0,u^1_3) &\in V_2. \end{aligned}$$

By induction, one may check that

$$\begin{aligned} (u^j_\alpha; 0) &\in V_{2j+1} \quad \text{and} \\ (0,0,u^j_3) &\in V_{2j}. \end{aligned}$$

See also [25]. Hence we should actually redefine V_N to accommodate this variation in order over the three displacement components. Denoting by (u^n, σ^n): the partial sum in (14) up to n, and using the following three tools: (12)—provided courtesy of the *inf-sup* optimality guarantee, periodicity, and the hypotheses mentioned before Theorem 7.1 sufficient for the $(n+1)^{\text{st}}$ term in the asymptotic expansion to exist, we can get that

$$\begin{aligned} \|(u,\sigma) - (u_N,\sigma_N)\|_X &\leq C \inf_{(v_N,\tau_N)} \|(u,\sigma) - (v_N,\tau_N)\|_X \\ &\leq C\|(u,\sigma) - (u^n,\sigma^n)\|_X \qquad (16) \\ &\leq C\varepsilon^{2n+2}. \end{aligned}$$

Thus we can get

Theorem 2. *Assume that f is the restriction to $\overline{\Omega}$ of a 1-periodic function $F \in (C^\infty(\mathbf{R}^2 \times [-1,1])^3$ and that g is the restriction to $\overline{\Gamma_3}$ of a 1-periodic function G with $G(\cdot,\cdot,\pm 1) \in (C^\infty(\mathbf{R}^2))^3$. Then there exists C depending on N, f, and g but not on ε so that*

$$\|(u,\sigma) - (u_N,\sigma_N)\|_X \le C\varepsilon^{2\mathrm{int}(N/2)+2}.$$

Here $n = \mathrm{int}(N/2)$ denotes the *integer* value of $N/2$. The challenge of general loads remains.

5. Divergence stability

In our paper [17] we have outlined how it is possible to define finite element spaces with optimal stability and convergence rate qualities for saddle-point problems of semi-elliptic type. This typically arises in situations with constraints of first-order type (such as a divergence constraint like that of Kirchhoff-Love). We now outline a thumb-nail sketch of how the connection comes about.

First let us introduce a few of the first individual plate models that result from a dimensional reduction as described above:

The zero'th order, $\mathbf{N} = (0,0,1)$, approximation consists of the Kirchhoff-Love-Germain-Lagrange fourth order, biharmonic model with very classical ties to Stokes' system.

First let $\mathbf{N} = (1,1,0)$:

We begin with the strong formulation of the system: seek $-(v_1^{(1,1,0)}, v_2^{(1,1,0)})/\varepsilon$, the rotation of fibers, normal to the mid-plane, here denoted φ_ε, and the transverse displacement of the mid-plane ϖ_ε so that

$$\begin{aligned} -2\varepsilon^2 D\{(1-\nu)\Delta\varphi_\varepsilon + (1+\nu)\nabla\nabla\cdot\varphi_\varepsilon\} - \kappa\mu(\nabla\varpi_\varepsilon - \varphi_\varepsilon) &= 0, \\ -\kappa\mu/(4\varepsilon^2)\,\nabla\cdot(\nabla\varpi_\varepsilon - \varphi_\varepsilon) &= g \end{aligned} \tag{17}$$

where $D = E/[12(1-\nu^2)]$ is the flexural rigidity of the plate scaled by $8\varepsilon^3$ and κ is a shear-correction (=fudge-) factor. μ (and λ) are the two Lamé coefficients given by E and ν. In the limit $\varepsilon \searrow 0$, the Kirchhoff-Love constraint, $\varphi_\varepsilon = \nabla\varpi_\varepsilon$, is enforced. We then go on to a variational formulation. Then, seeking φ_ε and ϖ_ε and, following [6], introducing the Helmholtz decomposition:

$$\frac{1}{4\varepsilon^2}(\nabla\varpi_\varepsilon - \varphi_\varepsilon) = \nabla r_\varepsilon + \nabla\times p_\varepsilon,$$

one gets the elliptic system (dropping $\cdot_\varepsilon$ subscripts):

$$\begin{aligned} &(\nabla r, \nabla\mu) = (g,\mu) && \text{for all } \mu \in V_1, \\ &(\nabla\varphi, \nabla q) - (\nabla\times p, \psi) = (\nabla r, \psi) && \text{for all } \psi \in V, \\ &-(\varphi, \nabla\times q) - \varepsilon^2(\nabla\times p, \nabla\times q) = 0 && \text{for all } q \in V_1, \\ &(\nabla\varpi, \nabla s) = (\varphi + \varepsilon^2\nabla r, \nabla s) && \text{for all } s \in V_1, \end{aligned} \tag{18}$$

letting V_1 denote a one-component variant of V corresponding to one of the first two components. Here we are cheating in the sense that we did not specify boundary conditions (and appropriate subspaces) as well as replacing the fourth order elasticity tensor by an identity tensor. This system is of the Reissner-Mindlin type, cf. [4], [6], [24]. If one 'rotates' the variable φ, one obtains a singularly perturbed Stokes system. Optimal order error estimates are now possible due to the correct inf-sup condition being satisfied (from divergence-stability) and the fact that we may construct the discrete spaces so that a Helmholts decomposition still holds. A price is paid in terms of setting all up in higher-order polynomial spaces, cf. [17]. The details for this model will be treated elsewhere.

Finally consider $\mathbf{N} = (1, 1, 2)$:

We state the strong formulation of the system: seek φ_ε, ϖ_ε, and y_ε so that

$$\begin{aligned} -2\varepsilon^2 D\{(1-\nu)\Delta\varphi_\varepsilon + (1-\nu)/(1-2\nu)\nabla\nabla\cdot\varphi_\varepsilon\} & \\ -\mu(\nabla\varpi_\varepsilon - \varphi_\varepsilon) + \lambda\nabla y_\varepsilon &= 0, \\ -\kappa\mu/(4\varepsilon^2)\ \nabla\cdot(\nabla\varpi_\varepsilon - \varphi_\varepsilon) &= g, \\ -\mu/(20\varepsilon^2)\ \Delta y_\varepsilon - \lambda/(4\varepsilon^2)\ \nabla\cdot\varphi_\varepsilon + 3/(4\varepsilon^4)\ (2\mu+\lambda)y_\varepsilon &= g. \end{aligned} \tag{19}$$

We have for some time utilized the freedom in spaces of higher-order polynomials to mimic exactly these kinds of constructions on the discrete level. We refer the interested reader to [17] for further details. It will also be the subject of further work.

6. Possible future directions

Among joint works in progress, I would like to mention *Using Lagrange multipliers for joint structures in connection with dimensional reduction* [w. Dorr] and *Using optimal p-stable Stokes-like elements to solve some models resulting from dimensional reduction* [w. Janik]. Other topics we are considering are *Mathematical modelling of constitutive laws*, *Using dimensional reduction for efficient computation of stress intensity factors*, and *Finding robust, efficient solvers for individual models*. Obviously, this area is rich in challenges.

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