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TRACE IDENTITIES OF FULL MATRIX ALGEBRAS OVER A FIELD OF CHARACTERISTIC ZERO

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Abstract. In this paper we consider the trace identities satisfied in a full matrix algebra of order n. For the case of a field of characteristic zero we prove that all trace identities are consequences of one obtained from the Hamilton-Cayley theorem.

This paper should be regarded as an attempt to find an approach to the problem of determining the identities of a full matrix algebra of order n. As shown by the author in [1] and [2], determining the identities of a matrix algebra of order two is closely related to determining the Lie identities of the Lie algebra $\mathrm{sl}(2, K)$; and determining those specific identities connected with the existence of Kaplansky polynomials, i.e. polynomials whose values on a matrix algebra of order n are scalar matrices, not all of which are zero, is related to the existence of polynomials which are not identities and vanish on $\mathrm{sl}(n, K)$. Starting from this, it appears likely that solving the identity problem for the Lie algebra $\mathrm{sl}(n, K)$ and determining the polynomials which vanish on $\mathrm{sl}(n, K)$ should help in determining the identities of the full matrix algebra. On the other hand, even a cursory comparison of our §4 with the solution of the problem of determining the irreducible representations of the full linear group $GL(V)$ of an n-dimensional linear space V in the tensor product $\bigotimes V^m$ [3] suggests that here also there is some mysterious connection with the identities of the matrix algebra.

The author hopes that an elucidation of this connection will bring closer the solution of the identity problem for the matrix algebra of order n.

In this paper we determine all trace identities for a matrix algebra over a field of characteristic zero. It is intuitively clear what is meant by a trace identity: for example, for the matrix algebra of order one the expression $y - \mathrm{tr}(y)$ vanishes upon substituting any matrix for y; the same is true if in the expression

$$y^2 - y\,\mathrm{tr}\,(y) + \frac{1}{2}\,(\mathrm{tr}\,(y))^2 - \frac{1}{2}\,\mathrm{tr}\,(y^2)$$

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we substitute any matrix of order two for y. The precise definition of a trace identity and also the definitions of a generalized polynomial and verbal ideal will be given in §2.

The material discussed in this paper is arranged as follows.

In §1 we construct a Kaplansky polynomial and show that $\mathrm{tr}(y)$ can be expressed as the quotient of two Kaplansky polynomials. This allows us to obtain from each trace identity an ordinary identity of the matrix algebra. In addition, we construct in this section a trace identity corresponding to the Hamilton-Cayley theorem and show how it implies Amitsur's theorem on the minimal identity of a matrix algebra.

Our main result (a precise formulation is given in §2) is a proof of the fact that there are essentially no trace identities other than the one just mentioned. The proof is given in §§3–5.

In §3 we construct an auxiliary algebra, by means of which we introduce an operation on the set of multilinear generalized polynomials of degree l in the variables $x_1, \cdots, x_l$ converting this set into the group algebra of the symmetric group on $l + 1$ letters. In addition, we construct for this set of multilinear generalized polynomials a symmetric bilinear form which is associative with respect to the new operation. If the characteristic of the ground field is zero, the annihilator of this form consists exactly of all multilinear trace identities of degree l.

Hence in the set of all multilinear generalized polynomials of degree l in $x_1, \cdots, x_l$ ($l = 1, 2, \cdots$) the trace identities form two-sided ideals with respect to the new operation. It turns out that the verbal ideals such that for any natural number l the set of multilinear generalized polynomials of degree l in $x_1, \cdots, x_l$ of this verbal ideal also forms a two-sided ideal with respect to the new operation admits an explicit description in terms of Young tables [4]. This description is given in §4, where, in particular, we prove that any ascending chain of these verbal ideals terminates.

In §5, using the results of §4, we prove that all trace identies of the full matrix algebra of order n over a field of characteristic zero are consequences of one corresponding to the Hamilton-Cayley theorem. In Corollaries 1 and 2 we give a new equivalent formulation of the problem of determining all identities of the matrix algebra of order n, showing, in essence, that for its solution it is necessary to overcome purely technical difficulties. At the end of this section, using §4, we give a simple proof of the fact that the $(p - 1)$th Engel identity does not imply the nilpotency identity $[x_1, \cdots, x_k]$ ($k = 1, 2, \cdots$) if the field has characteristic $p > 3$. This proof also shows that the multilinear trace identities of a matrix algebra over a field of characteristic $p > 0$ cannot be described by the annihilator of the bilinear form mentioned above.

We will employ the following notation: M_n is the full matrix algebra of order n over a field K, and S_l is the group of all permutations on a set of l letters. If $s \in S_l$ and $f(x_1, \cdots, x_l)$ is a multilinear polynomial of degree l, we denote the polynomial

$f(x_{s(1)}, \cdots, x_{s(l)})$ by $s(f)$, and, finally, we denote by $f|_{x_1=a_1, \cdots, x_l=a_l}$ the expression obtained by substituting the elements $a_1, \cdots, a_l$ for the variables $x_1, \cdots, x_l$ in the polynomial $f(x_1, \cdots, x_l)$.

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§1. Kaplansky polynomials. The Hamilton-Cayley polynomial

The aim of this section is to convince the reader, by means of several examples, of the usefulness of studying the trace identities of the matrix algebra M_n. A precise definition of a trace identity will be given in §2. As far as this section is concerned, it will be quite sufficient for the reader to think of a trace identity as an expression containing a formal trace function tr and certain variables which vanishes upon replacing the variables by arbitrary matrices in M_n.

We begin our exposition with a simple lemma, but one which plays an important role in this paper.

Lemma 1. *Suppose $e_1, \cdots, e_{n^2}$ is an arbitrary basis of the full matrix algebra M_n over an arbitrary field, and $e_1^*, \cdots, e_{n^2}^*$ the dual basis with respect to the bilinear form $(x, y) = \operatorname{tr}(xy)$ defined by the trace. Then for any element $a \in M_n$ we have the equality*

$$\sum_{t=1}^{n^2} e_t a e_t^* = \operatorname{tr}(a)\cdot 1, \tag{1}$$

where 1 *denotes the identity matrix.*

Proof. The reader can immediately verify that (1) is true in the special case where $e_1, \cdots, e_{n^2}$ is taken to be the basis consisting of the matrix units e_{ij} $(i, j = 1, \cdots, n)$. Hence it suffices to show that the left-hand side of (1) does not depend on the choice of the basis $e_1, \cdots, e_{n^2}$. If $e'_1, \cdots, e'_{n^2}$ is another basis, where $e'_i = \sum_{j=1}^{n^2} c_{ji} e_j$, then it is well known that $e_i^* = \sum_{j=1}^{n^2} c_{ij} e_j'^*$. Using these two equalities, we obtain

$$\sum_{t=1}^{n^2} e'_t a e_t'^* = \sum_{t=1}^{n^2}\sum_{j=1}^{n^2} c_{jt} e_j a e_t'^* = \sum_{j=1}^{n^2} e_j a \left(\sum_{t=1}^{n^2} c_{jt} e_t'^*\right) = \sum_{j=1}^{n^2} e_j a e_j^*,$$

which shows that the left-hand side of (1) is independent of the choice of basis. The lemma is proved.

As the first application of this lemma we will construct for the matrix algebra M_n over the field K a Kaplansky polynomial, i.e. a polynomial which is not an identity of M_n and assumes scalar matrix values on M_n.

In accordance with the notation used in [2], we denote by A_i a linear transformation acting on the subspace of the linear space of all noncommutative polynomials which is generated by the monomials of degree 1 with respect to each variable occurring in the monomial; by definition, $A_i(ax_ib) = bx_ia$ for each such monomial.

Theorem 1. *Suppose*

$$d = \sum_{s\in S_{n^2}} (-1)^{\varepsilon(s)} y_1 x_{s(1)} y_2 x_{s(2)} \dots y_{n^2} x_{s(n^2)} y_{n^2+1},$$

where the sum extends over all permutations s *on the set* $(1, \dots, n^2)$, *and* $\epsilon(s)$ *is equal to* 0 *or* 1, *depending on the parity of* s. *Then the polynomial*

$$c = \sum_{i=1}^{n^2} x_i y_0 \{(A_i d)|_{x_i=1}\}$$

is a Kaplansky polynomial of the algebra M_n.

Proof. Let $e_1, \dots, e_{n^2}$ be a basis of M_n, and $\mathfrak{F}$ the quotient field of the ring of polynomials in the commutative variables $x_i^{(j)}$ and $y_i^{(j)}$, $j = 1, \dots, n^2$, $i = 0, 1, 2, \dots$, over the field K. To prove the theorem it suffices to show that if in the polynomial c we put

$$x_i = \sum_{j=1}^{n^2} x_i^{(j)} e_j, \quad y_i = \sum_{j=1}^{n^2} y_i^{(j)} e_j,$$

then the resulting expression will be equal to a nonzero scalar matrix. We will prove more, namely that under such a change of variables we have in $\mathfrak{F}\otimes_K M_n$ the equality

$$c = \operatorname{tr}(y_0)\cdot \operatorname{tr}(d)\cdot 1. \tag{2}$$

To prove this equality we must first learn how to compute the trace. If $u = ax_ib$ is a word of degree 1 with respect to each variable occuring in it, then, in view of the well-known property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ of the trace, we have

$$\operatorname{tr}(ax_ib) = \operatorname{tr}(x_iba),$$

or, by the above definition of the operator A_i,

$$\operatorname{tr}(x_j(A_iu)|_{x_i=1}) = \operatorname{tr}(u|_{x_i=x_j}). \tag{3}$$

Consider the elements $x_1, \dots, x_{n^2}$ and $(A_id)|_{x_i=1}$ $(i = 1, \dots, n^2)$. It is obvious that $x_1, \dots, x_{n^2}$ form a basis of the matrix algebra $\mathfrak{F}\otimes_K M_n$ over the field $\mathfrak{F}$; also, in accordance with (3),

$$\operatorname{tr}(x_j(A_i d)|_{x_i=1}) = \operatorname{tr}(d|_{x_i=x_j}).$$

But the polynomial d is such that it vanishes if in place of x_i we substitute x_j for any $j = 1, \dots, i-1, i+1, \dots, n^2$. Hence the elements $(A_id)|_{x_i=1}$ $(i = 1, \dots, n^2)$ form, to within the factor $\operatorname{tr}(d)$, the basis dual to the x_i $(i = 1, \dots, n^2)$ with respect to the bilinear form defined by the trace, and (2) is a direct consequence of Lemma 1. It remains to show that the right-hand side of (2) is not equal to zero. It is clear that

$\operatorname{tr}(y_0) \neq 0$. We must show that $\operatorname{tr}(d) \neq 0$. Since the elements $x_i = \sum_{j=1}^{n^2} x_i^{(j)} e_j$ and $y_i = \sum_{j=1}^{n^2} y_i^{(j)} e_j$ are of "general form" for the algebra M_n, it suffices to show that $\operatorname{tr}(d) \neq 0$ under some substitution of the matrix units e_{st} in place of x_i and y_i. Put

$$x_{i+n(j-1)} = e_{ij} \quad (i, j = 1, \ldots, n), \quad y_1 = e_{11}, \quad y_{n^2+1} = e_{n1},$$

and for $t = 2, \cdots, n^2$ put $y_t = e_{kl}$ if $x_{t-1} = e_{rk}$ and $x_t = e_{ls}$. Under such a substitution only the monomial $y_1 x_1 y_2 x_2 \cdots y_{n^2} x_{n^2} y_{n^2+1} = e_{11}$ in the polynomial d does not vanish; hence $\operatorname{tr}(d) = 1$. The theorem is proved.

We will henceforth assume that the ground field K over which the matrix algebra M_n is considered has characteristic zero, even though this is not essential for some of our subsequent results.

In the course of proving Theorem 1 we obtained the equality (2). Substituting $y_0 = 1$ into both sides of this equality and taking into account that $\operatorname{tr}(1) = n$ in M_n, we obtain

$$c|_{y_0=1} = n \operatorname{tr}(d) \cdot 1;$$

hence we can rewrite (2) as follows:

$$c = \frac{1}{n} \operatorname{tr}(y_0)\{c|_{y_0=1}\}. \tag{4}$$

In this equality c and $c|_{y_0=1}$ are Kaplansky polynomials, and the equality holds upon substituting any matrices in M_n in place of the variables. This equality is not an identity in the usual sense, since it contains the trace. But it demonstrates the possibility of obtaining an ordinary identity of M_n from expressions of the form

$$\sum \alpha_{a_0, a_1, \ldots, a_l} a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_l) = 0, \tag{*}$$

where $a_0, \cdots, a_l$ are words in the variables $x_1, x_2, \cdots$. To do this we multiply (*) by a sufficiently high power of the polynomial $c|_{y_0=1}$ and apply (4) to get rid of the traces.

We now turn to the construction of a trace identity for which it will be proved in §§3–5 that all other trace identities of M_n are consequences of it.

An important example. The Hamilton-Cayley theorem asserts that the characteristic polynomial of a matrix y vanishes upon substituting y into it, i.e.

$$\{\operatorname{Det}(\lambda \cdot 1 - y)\}|_{\lambda = y} = 0.$$

But

$$\operatorname{Det}(\lambda \cdot 1 - y) = \sum_{i=0}^{n} (-1)^i \sigma_i (\lambda_1, \ldots, \lambda_n) \lambda^{n-i},$$

where $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of y, and the σ_i are the elementary symmetric functions. It is well known that in the case of a field of characteristic zero each $\sigma_i(\lambda_1, \cdots, \lambda_n)$ can be expressed in terms of polynomials in $\lambda_1^k + \cdots + \lambda_n^k = \mathrm{tr}(y^k)$, where $k = 1, \cdots, n$. Consequently

$$\sigma_i(\lambda_1, \ldots, \lambda_n) = g_i(\mathrm{tr}\ (y), \ldots, \mathrm{tr}\ (y^n)) \tag{5}$$

and the Hamilton-Cayley theorem yields the trace identity

$$\sum_{i=0}^{n} (-1)^i g_i(\mathrm{tr}\ (y), \ldots, \mathrm{tr}\ (y^n))\, y^{n-i} = 0.$$

We call the left-hand side of this equality the Hamilton-Cayley polynomial and denote it by $f_n(y)$. It is possible to obtain a recurrence formula for the calculation of the Hamilton-Cayley polynomial of the matrix algebra M_n:

$$f_1 = y - \mathrm{tr}\ (y), \quad f_n = f_{n-1} \cdot y - \frac{1}{n} \mathrm{tr}\ (f_{n-1} \cdot y).$$

We will not need these formulas, and for our purposes it suffices to know that the Hamilton-Cayley polynomial has the form

$$f_n(y) = y^n + \sum_{\substack{r_0+\ldots+r_t=n \\ r_1^2+\ldots+r_t^2\neq 0}} a_{r_0,r_1,\ldots,r_t} y^{r_0}\, \mathrm{tr}\ (y^{r_1}) \ldots \mathrm{tr}\ (y^{r_t})$$

for certain rational coefficients depending only on n, as can be seen by comparing the degrees of both sides of (5).

We conclude this section with a proof of a theorem of Amitsur, which states that M_n satisfies the standard identity of degree $2n$:

$$P_{2n} = \sum_{s\in S_{2n}} (-1)^{\varepsilon(s)} x_{s(1)} \ldots x_{s(2n)}.$$

We will first show how the identity P_{2n} can be obtained from the identity y^n. The complete linearization of the identity y^n yields the identity $\sum_{s \in S_n} x_{s(1)} \cdots x_{s(n)}$, which we denote by $g(x_1, \cdots, x_n)$. Substitution into this identity of the commutators $x_i = [x_{2i-1}, x_{2i}]$, $i = 1, \cdots, n$, with the subsequent skew-symmetrization of the resulting expression yields the equality

$$\sum_{s\in S_{2n}} (-1)^{\varepsilon(s)} s \Big(g\big|_{\substack{x_1=[x_1,x_2] \\ \cdots\cdots \\ x_n=[x_{2n-1},x_{2n}]}}\Big) = \beta P_{2n},$$

and it can be verified that $\beta \neq 0$. This shows that P_{2n} is a consequence of y^n. Exactly the same argument can be used to obtain P_{2n} from the Hamilton-Cayley identity f_n if we observe that

$$\sum_{s\in S_{2n}} (-1)^{\varepsilon(s)} s\{(a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t))|_{\substack{x_1=[x_1,x_2] \\ \cdots\cdots \\ x_n=[x_{2n-1},x_{2n}]}}\} = 0,$$

where $a_0, \cdots, a_t$ are words in $x_1, \cdots, x_n$, and where the word $a_0 a_1 \cdots a_t$ has degree 1 with respect to each variable x_i $(i = 1, \cdots, n)$. Indeed, if the length of the word a_i is equal to r_i $(i = 1, \cdots, t)$, then the sum on the left-hand side of the equality can be split into summands which are equal, to within a change of variables, to

$$P_{2r_0} \operatorname{tr}(P_{2r_1}) \ldots \operatorname{tr}(P_{2r_t}).$$

But since $tr(AB) = tr(BA)$ for any matrices $A, B \in M_n$, it follows that $tr(P_{2m}) = 0$ for any natural number m. Hence as a result of the indicated process for obtaining P_{2n} from f_n the nonzero terms are obtained only from y^n. In view of what has been said above, this proves Amitsur's theorem.

§2. The algebra of generalized polynomials. Basic concepts

In this section we will be concerned with formalizing the trace identity problem of the matrix algebra M_n and we will state precisely the main result of this paper. For this purpose we will construct the algebra of generalized polynomials, which will play the same role for trace identities as the algebra of noncommutative polynomials plays for ordinary identities.

In order to construct this algebra we need variables $x_1, x_2, \cdots$ and a formal trace function tr( ). We will first construct an auxiliary algebra, which we will denote by G_0. We define the elements of G_0 inductively:

1) Any nonempty word of finite length in the variables $x_1, x_2, \cdots$ is an element of G_0.

2) If A and B are elements of G_0, then AB is an element of G_0.

3) If a is a nonempty word and A and B are elements of G_0, then $tr(AaB)$, $tr(aB)$ and $tr(Aa)$ are elements of G_0.

Item 2) of this definition defines multiplication in G_0 as the operation which attaches the element B to the element A, and this converts G_0 into a semigroup. The reader should look carefully at item 3) and observe that expressions of the form $tr(tr(\cdots))$ are not elements of G_0.

Let G be the semigroup obtained as the factor semigroup of G_0 by the following congruence:

$$\operatorname{tr}(A)B = B\operatorname{tr}(A), \tag{6}$$

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \tag{7}$$

$$\operatorname{tr}(A\operatorname{Sp}(B)) = \operatorname{tr}(A)\operatorname{tr}(B) \tag{8}$$

for any $A, B \in G_0$ such that all expressions occurring in (6)–(8) are defined. In view of the relations (6)–(8), it is clear that G, as a semigroup, is generated by the elements of the forms a and $\operatorname{tr}(b)$, where a and b are arbitrary nonempty words. Hence any element of G has the form

$$a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t), \tag{9}$$

where the a_i are words and $a_1, \cdots, a_t$ are nonempty. The representation of elements of G is obviously not unique; but it is easy to get around this nonuniqueness if as a representative of a class of congruent elements of the form (9) we take the element such that the word $a_0 a_1 \cdots a_t$ is maximal in the sense of the lexicographic ordering.

By the algebra of generalized polynomials over a field K we mean the semigroup algebra over K of the semigroup G, the elements of G being the generalized monomials and the elements of the algebra the generalized polynomials. We will denote this algebra by the letter $\mathfrak{G}$. We can define in a natural way the degree of each generalized monomial with respect to a variable x_i as the number of times that x_i occurs in the representation of the monomial. Thus to each generalized monomial in $x_1, \cdots, x_l$ we can assign a set of numbers $(r_1, \cdots, r_l)$, where r_i is the degree of this monomial with respect to x_i. We call this set of numbers the type of the generalized monomial, and a generalized polynomial which is a linear combination of generalized monomials of the same type $(r_1, \cdots, r_l)$ is called a homogeneous generalized polynomial of type $(r_1, \cdots, r_l)$; for example,

$$x_1 x_2 + x_2 \operatorname{tr}(x_1) + \frac{1}{2} \operatorname{tr}(x_2 x_1)$$

is a homogeneous generalized polynomial of type (1.1), but the polynomial $x_1 + x_2$ is not homogeneous. A homogeneous generalized polynomial $f(x_1, \cdots, x_l)$ of type $(1, \cdots, 1)$ (l ones) is called a multilinear generalized polynomial of degree l.

We define

$$\begin{aligned} &\operatorname{tr}\left(\sum \alpha_{a_0, a_1, \ldots, a_t} a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t)\right) \\ &= \sum \alpha_{a_0, a_1, \ldots, a_t} \operatorname{tr}(a_0) \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t), \end{aligned} \tag{10}$$

whenever the right-hand side is defined. It is easy to see that the right-hand side of (10) is defined if and only if each a_0 is nonempty.

For any generalized polynomial $g(x_1, \cdots, x_l) \in \mathfrak{G}$ and any matrices $B_1, \cdots, B_l \in M_n$ we can construct the expression $g(B_1, \cdots, B_l)$, in which $\operatorname{tr}(B)$ is interpreted as the scalar matrix on whose main diagonal appears the element of the field K equal to the trace of B. Under such an interpretation the equalities (6)–(8) and (10) express well-known properties of the function tr.

Definition 1. A generalized polynomial $g(x_1, \cdots, x_l) \in \mathfrak{G}$ is called a *trace identity of the matrix algebra* M_n if $g(B_1, \cdots, B_l) = 0$ for any matrices $B_1, \cdots, B_l \in M_n$.

Note that the concept of an identity of the algebra M_n is compatible in a natural way with Definition 1, since the subalgebra generated by $x_1, x_2, \cdots$ in the algebra $\mathfrak{G}$ of generalized polynomials is a free associative algebra.

It is clear that the set of trace identities for M_n forms a two-sided ideal of $\mathfrak{G}$. Moreover, if $g(x_1, \cdots, x_l)$ is a trace identity for M_n and the generalized polynomials $h_1, \cdots, h_l$ have the same form as the expression under the trace symbol in the left-hand side of (10), where each a_0 is nonempty, then $g(h_1, \cdots, h_l)$ is also a trace identity for M_n.

Definition 2. A two-sided ideal I of the algebra $\mathfrak{G}$ of generalized polynomials is called a *verbal ideal* if for any generalized polynomials $h_1, \cdots, h_l \in \mathfrak{G}$ having the form

$$\sum \alpha_{a_0, a_1, \ldots, a_t} a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t),$$

where each a_0 is nonempty (the words $a_1, \cdots, a_t$ are nonempty by definition of $\mathfrak{G}$), the fact that $g(x_1, \cdots, x_l) \in I$ implies that $g(h_1, \cdots, h_l) \in I$ for any generalized polynomial g.

In view of this definition and the preceding remark, the set of trace identities forms a verbal ideal of $\mathfrak{G}$.

Definition 3. A generalized polynomial g is called a *consequence of the generalized polynomials* $g_1, \cdots, g_l$ if the verbal ideal generated by $g_1, \cdots, g_l$ (i.e. the smallest verbal ideal containing $g_1, \cdots, g_l$) also contains g. Two sets of generalized polynomials are called *equivalent* if the verbal ideals generated by each set coincide.

Exactly as for the ordinary identities of M_n, we can consider the problem of determining all trace identities of M_n, and, in particular, the question of the existence of a finite set of trace identities of M_n such that all others are consequences of these. The answer to these questions is given by the following theorem:

Theorem 2. *All trace identities of the full matrix algebra M_n over a field of characteristic zero are consequences of the Hamilton-Cayley identity f_n.*

This theorem means that any trace identity of M_n lies in the verbal ideal generated by f_n.

It is evident from the relations (6)–(8) that any verbal ideal is generated by its multilinear generalized polynomials; hence to prove Theorem 2 it suffices to show that all multilinear trace identities follow from f_n.

Let T_l denote the set of multilinear generalized polynomials in the variables $x_1, \cdots, x_l$ of type $(1, \cdots, 1)$ (l ones).

Lemma 2. *The dimension of the linear space T_l is $(l+1)!$.*

Proof. In view of the definition of the algebra of generalized polynomials, it suffices to prove that the semigroup G contains $(l+1)!$ multilinear generalized monomials in $x_1, \cdots, x_l$. As noted above, each element of $G \cap T_l$ can be uniquely represented in the form

$$a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t), \tag{11}$$

where $a_1, \cdots, a_t$ are nonempty words, and where the multilinear word $a_0a_1\cdots a_t > b_0b_1\cdots b_t$ in the sense of the lexicographic ordering if $b_0\mathrm{tr}(b_1)\cdots\mathrm{tr}(b_t)$ is congruent to $a_0\mathrm{tr}(a_1)\cdots\mathrm{tr}(a_t)$ with respect to (6)–(8). If a_0 is nonempty, then the number of generalized monomials of the form $x_ia_0'\mathrm{tr}(a_1)\cdots\mathrm{tr}(a_t)$ (x_i fixed) is equal, by induction, to $l!$; hence the total number of generalized monomials of the form (11), where a_0 is nonempty, is equal to $l(l!)$. If a_0 is empty, then $a_1 = x_la_1'$ and the number of such generalized monomials agrees with the number of generalized monomials

$$a_1'\,\mathrm{tr}\,(a_2)\ldots\,\mathrm{tr}\,(a_t),$$

which is equal, by induction, to $l!$. Therefore the total number of elements of the form (11) is $(l+1)!$.

The lemma is proved.

§3. An auxiliary algebra and bilinear form

In this section we will reduce the problem of determining the multilinear trace identities of degree l to the problem of classfying the two-sided ideals of the group algebra $K[S_{l+1}]$ of the symmetric group S_{l+1}.

I. The algebra $\mathcal{G}_n$. Suppose M_n is the full matrix algebra of order n over the complex field C, σ the anti-isomorphism of M_n which is the composition of matrix transposition and complex conjugation, $e_1, \cdots, e_{n^2}$ an orthonormal basis of M_n with respect to the bilinear form $(x, y) = \mathrm{tr}(xy)$ such that $\sigma(e_i) = e_i$ (as the first n vectors we can take the matrix units e_{ii}, and for the others we can take, for example,

$$\frac{1}{\sqrt{2}}(e_{kl}+e_{lk}),\ \sqrt{\frac{-1}{2}}(e_{kl}-e_{lk}),$$

$k > l$). Let $\mathfrak{F}$ be the commutative algebra over C with generators $x_i^{(j)}$, $j = 1, \cdots, n^2$, $i = 1, 2, \cdots$, and defining relations

$$x_i^{(j)}x_i^{(t)}x_i^{(k)} = 0, \quad x_i^{(j)}x_i^{(t)} = \delta_{jt}\,(x_i^{(1)})^2. \tag{12}$$

Let $\widetilde{M}_n$ denote the matrix algebra over the ring $\mathfrak{F}$, i.e. $\widetilde{M}_n = \mathfrak{F}\otimes_C M_n$, and consider the subalgebra $\mathcal{G}_n$ generated by all elements of the form

$$\bar{x}_i = \sum_{j=1}^{n^2} x_i^{(j)}e_j, \quad \mathrm{tr}\,(\bar{x}_{i_1}\bar{x}_{i_2}\,\ldots\,\bar{x}_{i_k}).$$

Here, and from now on, $\mathrm{tr}(B)$ denotes the scalar matrix on whose diagonal appears the element equal to the trace.

Let us mention some properties of the algebra $\mathcal{G}_n$.

1°. Consider the element $\bar{x}_ia\bar{x}_i$, where a is an arbitrary element of $\widetilde{M}_n$. In view of Lemma 1 and the relations (12), we obtain

$$\bar{x}_ia\bar{x}_i = (x_i^{(1)})^2\sum_{j=1}^{n^2} e_jae_j = (x_i^{(1)})^2\,\mathrm{tr}\,(a).$$

If $a = 1$, then

$$\bar{x}_l^2 = n\,(x_l^{(1)})^2 \cdot 1$$

and consequently

$$\bar{x}_l a \bar{x}_l = \frac{\bar{x}_l^2}{n}\,\mathrm{tr}\,(a), \quad \bar{x}_l \bar{x}_l = n \cdot \frac{\bar{x}_l^2}{n} \tag{13}$$

for any element $a \in \mathcal{G}_n$. From (13) we see that

$$[\bar{x}_l^2, \bar{x}_j] = 0, \quad [\bar{x}_l a \bar{x}_l, \bar{x}_j] = 0, \quad \bar{x}_l ab \bar{x}_l = \bar{x}_l ba \bar{x}_l.$$

A direct calculation shows that

$$\bar{x}_l \bar{x}_j \bar{x}_l \bar{x}_j = \frac{\bar{x}_l^2}{n}\,\mathrm{tr}\,(\bar{x}_j)\,\bar{x}_j = \frac{\bar{x}_l^2}{n}\,(x_j^{(1)} + \ldots + x_j^{(n)})\,\bar{x}_j = \frac{\bar{x}_l^2}{n}\,(x_j^{(1)})^2 = \frac{\bar{x}_l^2}{n}\,\frac{\bar{x}_j^2}{n}.$$

From these equalities we obtain

$$\bar{x}_l \bar{x}_j a \bar{x}_l \bar{x}_j = \bar{x}_l a \bar{x}_j \bar{x}_l \bar{x}_j = \bar{x}_l \bar{x}_j \bar{x}_l \bar{x}_j a = \frac{\bar{x}_l^2}{n}\,\frac{\bar{x}_j^2}{n}\,a.$$

On the other hand, it follows from (13) that

$$\bar{x}_l \bar{x}_j a \bar{x}_l \bar{x}_j = \frac{\bar{x}_l^2}{n}\,\mathrm{tr}\,(\bar{x}_j a)\,\bar{x}_j.$$

A comparison of the last two equalities, taking into account that $\bar{x}_i^2/n$ is a nonzero scalar matrix, shows that for any $a \in \mathcal{G}_n$

$$\mathrm{tr}\,(\bar{x}_j a)\,\bar{x}_j = \frac{\bar{x}_j^2}{n}\,a. \tag{14}$$

In view of the above, it is obvious that

$$[\bar{x}_l^2, \bar{x}_j] = 0, \qquad \mathrm{tr}\,(\bar{x}_l^2 a) = \bar{x}_l^2\;\mathrm{tr}\,(a), \qquad \mathrm{tr}(\bar{x}_l^2) = n\bar{x}_l^2. \tag{15}$$

Finally, the first of the defining relations (12) implies that a generalized monomial w in the $\bar{x}_j$ satisfies the equality

$$w = 0, \tag{16}$$

if its degree with respect to some $\bar{x}_i$ is strictly greater than 2. The equality (16) shows that any homogeneous generalized polynomial in the $\bar{x}_j$ of degree at least 3 with respect to some $\bar{x}_i$ is equal to zero. It follows from (13)–(15) that for any homogeneous generalized polynomial $f(x_{i_1}, \cdots, x_{i_s}, x_{j_1}, \cdots, x_{j_t})$ of degree 2 with respect to $x_{i_1}, \cdots, x_{i_s}$ and degree 1 with respect to $x_{j_1}, \cdots, x_{j_t}$ we have

$$f(\bar{x}_{i_1}, \ldots, \bar{x}_{i_s}, \bar{x}_{j_1}, \ldots, \bar{x}_{j_t}) = \frac{\bar{x}_{i_1}^2}{n} \ldots \frac{\bar{x}_{i_s}^2}{n} g(\bar{x}_{j_1}, \ldots, \bar{x}_{j_t}), \tag{17}$$

where $g(x_{j_1}, \cdots, x_{j_t})$ is a generalized polynomial.

2°. The special case of (17) where $t = 0$ is particularly interesting. In this case

$$f = \alpha(f) \frac{\bar{x}_{i_1}^2}{n} \ldots \frac{\bar{x}_{i_s}^2}{n}, \tag{18}$$

where $\alpha(f) \in C$ is the number, uniquely determined by the generalized polynomial f, such that

$$\operatorname{tr}(f(\bar{x}_{i_1}, \ldots, \bar{x}_{i_t})) = n\alpha(f)(x_{i_1}^{(1)})^2 \ldots (x_{i_s}^{(1)})^2. \tag{19}$$

Suppose $u(x_1, \cdots, x_l)$ is a multilinear polynomial. If $\bar{u}$ denotes the matrix $u(\bar{x}_1, \cdots, \bar{x}_l) \in \mathcal{G}_n$, then the element appearing at the intersection of the ith row and the jth column has the form

$$a_{ij} = \sum_{r_1, \ldots, r_l} \beta^{ij}_{r_1, \ldots, r_l} x_1^{(r_1)} \ldots x_l^{(r_l)},$$

where $\beta^{ij}_{r_1, \cdots, r_l} \in C$. A direct calculation of the trace of the matrix $f = \bar{u}\sigma(\bar{u})$, taking into account the equality (which follows from (12))

$$x_1^{(j_1)} \ldots x_l^{(j_l)} x_1^{(t_1)} \ldots x_l^{(t_l)} = \delta^{t_1, \ldots, t_l}_{j_1, \ldots, j_l} (x_1^{(1)})^2 \ldots (x_l^{(1)})^2,$$

where $\delta^{t_1, \cdots, t_l}_{j_1, \cdots, j_l}$ is the Kronecker symbol, yields

$$\operatorname{tr}(f) = \sum |\beta^{ij}_{r_1, \ldots, r_l}|^2 \geqslant 0 \tag{20}$$

and $\operatorname{tr}(f) = 0$ if and only if $\bar{u} = 0$. On the other hand, it is easy to see, by the choice of basis $e_1, \cdots, e_{n^2}$ of M_n, that the algebra $\mathcal{G}_n$ is invariant under the anti-isomorphism σ.

Indeed,

$$\sigma(\bar{x}_i) = \sum_{j=1}^{n^2} x_i^{(j)} \sigma(e_j) = \bar{x}_i,$$

and

$$\sigma(\bar{x}_{i_1} \bar{x}_{i_2} \ldots \bar{x}_{i_t}) = \bar{x}_{i_t} \ldots \bar{x}_{i_2} \bar{x}_{i_1},$$

$$\sigma(\lambda a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t)) = \bar{\lambda}\sigma(a_0) \operatorname{tr}(\sigma(a_1)) \ldots \operatorname{tr}(\sigma(a_t)), \tag{21}$$

where $a_0, \cdots, a_t$ are words in the $\bar{x}_j$. Consequently $\sigma(\bar{u}) \in \mathcal{G}_n$, and the generalized polynomial $f(\bar{x}_1, \cdots, \bar{x}_l) = \bar{u}\sigma(\bar{u})$ is homogeneous of type $(2, \cdots, 2)$ and formula (18) holds. Because of (19), the inequality (20) can be restated as follows:

$$n\alpha(\bar{u}\sigma(\bar{u})) \geqslant 0 \tag{22}$$

for any multilinear generalized polynomial u, and equality is attained if and only if $\bar{u} = 0$. But when does $\bar{u}$ vanish? In view of the defining relations (12) we have

$$u(\bar{x}_1, \ldots, \bar{x}_l) = \sum_{r_1, \ldots, r_l} x_1^{(r_1)} \ldots x_l^{(r_l)} u(e_{r_1}, \ldots, e_{r_l})$$

and $\bar{u} = 0$ if and only if each $u(e_{r_1}, \cdots, e_{r_l}) = 0$. But since u is a multilinear generalized polynomial, these equalities mean that u is a trace identity of M_n.

Thus (22) provides a criterion for testing whether u is a trace identity. Note that in making use of this criterion the algebras M_n and $\widetilde{M}_n$ are not really needed, since the anti-isomorphism σ can be defined formally by the equalities (21) and to compute the function α only the relations (13)–(15) are needed.

We now turn to the construction of an auxiliary algebra which will permit us to completely ignore the algebra $\widetilde{M}_n$ and operate only with generalized polynomials.

II. **The algebra** $\widetilde{\mathfrak{G}}$. Suppose K is a field, and $K(\gamma)$ the quotient field of the polynomial ring $K[\gamma]$ in the one variable γ. Let $\widetilde{\mathfrak{G}}$ denote the factor algebra of the algebra $\mathfrak{G}(\gamma)$ of generalized polynomials over the field $K(\gamma)$ with the following defining relations, where $i, j = 1, 2, \cdots$:

$$x_i a x_i = \frac{x_i^2}{\gamma} \operatorname{tr}(a), \tag{23}$$

$$\operatorname{tr}(x_i a) x_i = \frac{x_i^2}{\gamma} a, \tag{24}$$

$$[x_i^2, x_j] = 0, \quad \operatorname{tr}(x_i^2 a) = x_i^2 \operatorname{tr}(a), \quad \operatorname{tr}(x_i^2) = \gamma x_i^2, \tag{25}$$

$$w = 0; \tag{26}$$

here w is any generalized monomial of degree greater than two with respect to some variable occurring in it, and a is any word.

The structure of the algebra $\widetilde{\mathfrak{G}}$ is descrived by the following lemma.

Lemma 3. a) *The algebra* $\widetilde{\mathfrak{G}}$ *as a linear space is generated by the generalized monomials of degree less than* 3 *with respect to each variable* x_i.

b) *As linear spaces the sets of multilinear polynomials of the algebras* $\mathfrak{G}(\gamma)$ *and* $\widetilde{\mathfrak{G}}$ *are isomorphic.*

c) *Any homogeneous polynomial* $f(x_{i_1}, \cdots, x_{i_s}, x_{j_1}, \cdots, x_{j_t})$ *of degree* 2 *with respect to* $x_{i_1}, \cdots, x_{i_s}$ *and degree* 1 *with respect to* $x_{j_1}, \cdots, x_{j_t}$ *can be represented in the form*

$$f = \frac{x_{i_1}^2}{\gamma} \ldots \frac{x_{i_s}^2}{\gamma} g(x_{j_1}, \ldots, x_{j_t}), \tag{27}$$

where g is a multilinear generalized polynomial.

d) *The representation* (27) *is unique in the sense that if*

$$f = \prod_{k=1}^{s} \frac{x_{i_k}^2}{\gamma} g_1$$

is another such representation, then $g = g_1$ in $\mathcal{G}(\gamma)$.

Proof. The first statement is true because of (26), the second because the defining relations (23)–(26) are of degree at least 2 with respect to some variable. It is obvious that (23)–(25) together with (6)–(8) yield a representation (27). The uniqueness of (27) could be proved directly, it being sufficient to verify that the reduction to the form (27) does not depend on which variable x_{i_k} $(k = 1, \cdots, s)$ one starts with, by proving this for the case where f is a generalized monomial and $s = 2$. We will take another approach.

It is clear that it suffices to prove uniqueness in the case where $K = C$.

Suppose $f = \Pi_{k=1}^{s}(x_{i_k}^2/\gamma)g_1$ is another representation, where $g \neq g_1$ in $\mathcal{G}(\gamma)$. Since the coefficients in the generalized polynomial $g - g_1$ are rational functions of γ, there exist infinitely many natural numbers n for which the expression $g - g_1$ makes sense and is unequal to zero when $\gamma = n$. Let $\bar{f}$, $\bar{g}$ and $\bar{g}_1$ denote the elements of $\mathcal{G}_n$ obtained from f, g and g_1 by the replacement $\gamma \to n$, $x_i \to \bar{x}_i$ $(i = 1, 2, \cdots)$. Then since the relations (13)–(15) for computing α are obtained by precisely the same replacement in the defining relations (23)–(25), we have

$$\alpha((\bar{g} - \bar{g}_1)\sigma(\bar{g} - \bar{g}_1)) = \alpha\left(\prod_{k=1}^{s} \frac{x_{i_k}^2}{n} (\bar{g} - \bar{g}_1)\sigma(\bar{g} - \bar{g}_1)\right) = \alpha((\bar{f} - \bar{f})\sigma(\bar{g} - \bar{g}_1)) = 0.$$

But then the criterion (22) shows that $\bar{g} - \bar{g}_1$ is a nontrivial multilinear trace identity of M_n, even when n is greater than the degree t of the generalized polynomial $\bar{g} - \bar{g}_1$. In order to arrive at a contradiction, it remains to prove that if $h(x_1, \cdots, x_l)$ is a multilinear trace identity of M_{l+1}, then $h = 0$. It suffices to find for each multilinear generalized monomial in $x_1, \cdots, x_l$ a set of matrix units in M_{l+1} such that only this monomial does not vanish on it. Suppose this monomial has the form

$$w = a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t),$$

where $a_0 a_1 \cdots a_t = x_1 x_2 \cdots x_l$ and the length of the word a_i is r_i. Put the first r_0 variables equal to $e_{12}, e_{23}, \cdots, e_{r_0(r_0+1)}$, and put the variables corresponding to the a_i $(i = 1, \cdots, t)$ equal successively to $e_{s+1\, s+2}, e_{s+2\, s+3}, \cdots, e_{s+r_i-1\, s+r_i}, e_{s+r_i\, s+1}$, where $s = 1 + \sum_{j=0}^{i-1} r_j$. It can be checked directly that, to within the congruence

(6)–(8), only the monomial w is unequal to zero for the indicated set of matrix units. Hence w occurs with zero coefficient in the trace identity h. Since w is no different from the other generalized monomials occurring in h, it follows that $h = 0$. The lemma is proved.

We now take note of one equality in $\widetilde{\mathfrak{G}}$ which will often be used in what follows. If $a, h, h_1, h_2 \in \widetilde{\mathfrak{G}}$, then from (23) and (6) we obtain

$$x_l a h x_l = \frac{x_l^2}{\gamma} \operatorname{tr}(ah) = \frac{x_l^2}{\gamma} \operatorname{tr}(ha) = x_l h a x_l,$$

and from (22), (6) and (7) we have

$$\operatorname{tr}(h_1 x_l a h_2) x_l = \operatorname{tr}(x_l a h_2 h_1) x_l = \frac{x_l^2}{\gamma} a h_2 h_1 = a \operatorname{tr}(x_l h_2 h_1) x_l = \operatorname{tr}(h_1 x_l h_2) a x_l.$$

From these equalities we obtain that for any multilinear generalized polynomials f and g in $x_1, \cdots, x_l$

$$f|_{x_l = x_l a} \cdot g = f \cdot g|_{x_l = a x_l}.$$

Moreover,

$$f(x_1 a_1, x_2 a_2, \ldots, x_l a_l) g(x_1, \ldots, x_l) = f(x_1, \ldots, x_l) g(a_1 x_1, a_2 x_2, \ldots, a_l x_l), \quad (28)$$

and in exactly the same way it can be shown that

$$f|_{\substack{x_1 = a_1 x_1 \\ \cdots\cdots \\ x_l = a_l x_l}} \cdot g = f \cdot g|_{\substack{x_1 = x_1 a_1 \\ \cdots\cdots \\ x_l = x_l a_l}}. \quad (29)$$

III. **The bilinear form.** The algebra $\mathfrak{G}(y)$ contains as a subring, in a natural way, the algebra $\mathfrak{G}$ of generalized polynomials over the field K. Let $T_l(y)$ and T_l denote the spaces of multilinear polynomials in $x_1, \cdots, x_l$ in $\mathfrak{G}(y)$ and $\mathfrak{G}$, respectively. By Lemma 3 b), these linear spaces can be identified with the linear spans over the fields $K(y)$ and K, respectively, of the multilinear generalized polynomials in $x_1, \cdots, x_l$ in $\mathfrak{G}$. Parts c) and d) of the same lemma show that to any homogeneous generalized polynomial $f(x_1, \cdots, x_l) \in \mathfrak{G}(y)$ of type $(2, \cdots, 2)$ we can uniquely assign $\alpha(f) \in K(y)$ in accordance with the rule

$$f = \alpha(f) \frac{x_1^2}{\gamma} \cdots \frac{x_l^2}{\gamma}.$$

If $f \in \mathfrak{G}$, then it follows from (23)–(25) that $\alpha(f)$ is a polynomial in the variable y. By means of the function α we can define on the space of multilinear generalized polynomials $T_l \subset \mathfrak{G}$ a bilinear form $\mathfrak{b}$, which assumes values in the polynomial ring $K[y]$, by putting, for any $u, v \in T_l$,

$$\mathfrak{b}(u, v) = \gamma \alpha(uv). \quad (30)$$

As an easy exercise in the use of (23)–(25), we invite the reader to verify that

$$\mathfrak{b}(a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t), \prod_{j=1}^{l} \operatorname{tr}(x_j)) = \gamma^{t+1}, \tag{31}$$

where $a_0, \cdots, a_t$ are words and $a_0\operatorname{tr}(a_1)\cdots\operatorname{tr}(a_t) \in T_l$.

Lemma 4. *If $\gamma = n$ in the definition of the bilinear form (30), then $\operatorname{Ann}\mathfrak{b}$ in the linear space T_l coincides with the set of all multilinear trace identities of degree l of the algebra M_n over a field of characteristic zero.*

Proof. First suppose that K is the field of complex numbers. Let ψ denote the homomorphism of the algebra $\mathfrak{G}$ into the algebra $\mathfrak{G}_n$ for which $\psi f(x_1, \cdots, x_l) = f(\bar{x}_1, \cdots, \bar{x}_l)$, and σ the anti-isomorphism of $\mathfrak{G}$ defined by the formulas (21) with $\bar{x}_i$ replaced by x_i. Since when $\gamma = n$ the formulas (23)–(25) and (13)–(15) for computing the function α in $\mathfrak{G}$ and $\mathfrak{G}_n$ agree, we have

$$\alpha(uv)|_{\gamma=n} = \alpha(\psi(u)\psi(v)) \tag{32}$$

for any $u, v \in T_l$. Therefore, in view of (22),

$$\mathfrak{b}(u, \sigma(u))|_{\gamma=n} \geqslant 0$$

and equality is attained if and only if u is a multilinear trace identity of M_n. This proves the lemma when $K = C$.

If K is an arbitrary field of characteristic zero, then in the representation of any specific trace identity of M_n or element of $\operatorname{Ann}\mathfrak{b}$ there appear only finitely many elements of K. Since the field generated by these elements can be embedded in the field of complex numbers, the lemma is true for any field K of characteristic zero. The lemma is proved.

Equalities (32) and (19) show that for any natural number n

$$\mathfrak{b}(u, v)|_{\gamma=n} = \mathfrak{b}(v, u)|_{\gamma=n},$$

and since $\mathfrak{b}(u, v)$ and $\mathfrak{b}(v, u)$ are polynomials for fixed $u, v \in T_l$, we always have $\mathfrak{b}(u, v) = \mathfrak{b}(v, u)$, i.e. $\mathfrak{b}$ is a symmetric bilinear form.

Let $E(\gamma)$ denote some polynomial in $K[\gamma]$, and $\{E(\gamma)\}$ the ideal of $K[\gamma]$ generated by it. To solve the trace identity problem it is necessary that the bilinear form $\mathfrak{b}$ assume values not in the ring $K[\gamma]$, but in the factor ring $K[\gamma]/\{E(\gamma)\}$. We denote the bilinear form defined by (30) and assuming values in $K[\gamma]/\{E(\gamma)\}$ by $\mathfrak{b}_E$. Thus, for example, if $E(\gamma) = \gamma - n$, the bilinear form defined in Lemma 4 should be denoted by $b_{\gamma-n}$.

Lemma 5. *Let $E(\gamma)$ be a fixed polynomial and*

$$\operatorname{Ann}_{T_l}\mathfrak{b}_E = \{f \in T_l \mid \mathfrak{b}_E(f, T_l) = 0 \text{ in } K[\gamma]/\{E(\gamma)\}\}$$

If V denotes the verbal ideal of the algebra $\mathfrak{G}$ generated by $\operatorname{Ann}_{T_l}\mathfrak{b}_E$ for all $l = 1, 2, \cdots$, then $V \cap T_k = \operatorname{Ann}_{T_k}\mathfrak{b}_E$.

Proof. We must prove, essentially, that if $f \in \mathrm{Ann}_{T_l} \mathfrak{b}_E$, $g \in T_k$ $(k > l)$ and g is a consequence of the generalized polynomial f, then $g \in \mathrm{Ann}_{T_k} \mathfrak{b}_E$. Let $V(f)$ denote the verbal ideal of $\mathfrak{G}$ generated by the multilinear generalized polynomial $f(x_1, \cdots, x_l)$. It follows from Definition 2 and equalities (10), (6) and (8) that $V(f)$ as a two-sided ideal is generated by the generalized polynomials $f(a_1, \cdots, a_l)$ where $a_1, \cdots, a_l$ are arbitrary nonempty words; and from the equality

$$bf(a_1, \ldots, a_l) = f(a_1, \ldots, a_l)\,b + \sum_{i=1}^{l} f(a_1, \ldots, [b, a_i], \ldots, a_l),$$

which holds in $\mathfrak{G}$ for any word b, it is evident that $V(f)$ is generated as a right ideal by the indicated generalized polynomials. Consequently $g \in V(f) \cap T_k$ can be represented as a linear combination of generalized polynomials of the form

$$f(a_1, \ldots, a_l)\,h \in T_l,$$

and hence it suffices to prove that $g \in \mathrm{Ann}_{T_k} \mathfrak{b}_E$ for the case $g = f(x_1a_1, \cdots, x_la_l)h$. Applying (28), we obtain for any $v \in T_k$

$$\mathfrak{b}_E(g, v) = \gamma\alpha\Big(f\big|_{\substack{x_1 = x_1a_1 \\ \cdots \\ x_l = x_la_l}} hv\Big) = \gamma\alpha\Big(f \cdot \big(h\,v\big|_{\substack{x_1 = a_1x_1 \\ \cdots \\ x_l = a_lx_l}}\big)\Big)$$

$$= \gamma\alpha\left(f \prod_{j=l+1}^{k} \left(\frac{x_j^2}{\gamma}\right) v_1\right) = \gamma\alpha(fv_1) = \mathfrak{b}(f, v_1) = 0,$$

where v_1 is a multilinear generalized polynomial with coefficients in $K[\gamma]$ satisfying, by Lemma 3 c), the equality

$$h \cdot v\big|_{\substack{x_1 = a_1x_1 \\ \cdots \\ x_l = a_lx_l}} = \prod_{j=l+1}^{k} \left(\frac{x_j^2}{\gamma}\right) v_1.$$

Therefore $g \in \mathrm{Ann}_{T_k} \mathfrak{b}_E$. The lemma is proved.

As indicated in §2, any verbal ideal is generated by the set of its multilinear generalized polynomials. Lemma 5 shows that for certain verbal ideals these multilinear generalized polynomials can be distinguished by the bilinear form $\mathfrak{b}_E$ for some polynomial $E(\gamma)$. Lemma 4 asserts that to such verbal ideals there correspond trace identities of the matrix algebra.

We now turn to the question of which verbal ideals are defined by the bilinear form $\mathfrak{b}_E$.

IV. **Definition of a group algebra structure on the set T_l.** Suppose f and g are the same as in (27); define $\pi(f) = g$.

Lemma 6. *Suppose an operation $\circ$ is defined on the set T_l by means of the rule*

$$f \circ h = \pi\Big(f(x_1, \ldots, x_l) \cdot h\big|_{\substack{x_1 = y_1x_1 \\ \cdots \\ x_l = y_lx_l}}\Big)\Big|_{\substack{y_1 = x_1 \\ \cdots \\ y_l = x_l}} = \pi\Big(f\big|_{\substack{x_1 = x_1y_1 \\ \cdots \\ x_l = x_ly_l}} \cdot h(x_1, \ldots, x_l)\Big)\Big|_{\substack{y_1 = x_1 \\ \cdots \\ y_l = x_l}}, \qquad (33)$$

where $f, h \in T_l$, and, for convenience, $y_1, \cdots, y_l$ denote variables x_i such that $x_1, \cdots, x_l, y_1, \cdots, y_l$ are distinct. Then this operation is well defined, the set of generalized monomials $G \cap T_l$ forms a group with respect to this operation, and the linear space T_l becomes the group algebra of the group $G \cap T_l$. The elements

$$D_i = x_i \prod_{\substack{j=1\\ j\neq i}}^{l} \operatorname{tr}(x_j) \quad (i = 1, \ldots, l),$$

are generators of this group, and the generalized monomial $e = \prod_{j=1}^{l} \operatorname{tr}(x_j)$ is its identity element.

Proof. Let us first observe that the second equality in (33) is true by (28) and Lemma 3 d). Parts c) and d) of the same lemma guarantee the correctness of the definition (33) in $T_l(y)$. Let us show that the operation $\circ$ defined by $T_l(y)$ is associative, i.e. that for any $f, g, h \in T_l(y)$

$$(f \circ g) \circ h = f \circ (g \circ h). \tag{34}$$

By (33) and (28), the left-hand side of this equality is equal to

$$\begin{aligned} &\pi\Big(f \cdot g\big|_{\substack{x_1=y_1x_1\\ \cdots\cdots\\ x_l=y_lx_l}} \cdot h\big|_{\substack{x_1=z_1y_1\\ \cdots\cdots\\ x_l=z_ly_l}}\Big)\Big|_{\substack{z_1=x_1\\ \cdots\cdots\\ z_l=x_l}} \\ &= \pi\Big(f \cdot g\big|_{\substack{x_1=y_1\\ \cdots\cdots\\ x_l=y_l}} \cdot h\big|_{\substack{x_1=z_1x_1y_1\\ \cdots\cdots\\ x_l=z_lx_ly_l}}\Big)\Big|_{\substack{z_1=x_1\\ \cdots\cdots\\ z_l=x_l}} \\ &= \pi\Big(f\Big(g\big|_{\substack{x_1=y_1\\ \cdots\cdots\\ x_l=y_l}} \cdot h\big|_{\substack{x_1=x_1y_1\\ \cdots\cdots\\ x_l=x_ly_l}}\Big)\Big|_{\substack{x_1=z_1x_1\\ \cdots\cdots\\ x_l=z_lx_l}}\Big)\Big|_{\substack{z_1=x_1\\ \cdots\cdots\\ z_l=x_l}}. \end{aligned}$$

By (33) the last expression is equal to the right-hand side of (34). Thus (34) is proved.

Successive application of (24) yields

$$\prod_{j=1}^{l} \operatorname{tr}(x_j) \circ f = \pi\left(\prod_{j=1}^{l} \operatorname{tr}(x_iy_i) \cdot f\right)\Big|_{\substack{y_1=x_1\\ \cdots\cdots\\ y_l=x_l}} = \Big(f\big|_{\substack{x_1=y_1\\ \cdots\cdots\\ x_l=y_l}}\Big)\Big|_{\substack{y_1=x_1\\ \cdots\cdots\\ y_l=x_l}} = f$$

and therefore $e = \prod_{j=1}^{l} \operatorname{tr}(x_j)$ is a left identity element; in exactly the same way it can be shown that e is a right identity element of the algebra $T_l(y)$. Similarly, we obtain

$$f \circ D_i = \pi(fy_ix_i)\big|_{y_i=x_i}. \tag{35}$$

If $f = a_0 \prod_{j=1}^{t} \operatorname{tr}(a_j)$ is a generalized monomial, then consideration of the two cases $a_0 = a_0' x_i a_0''$ and $a_k = x_i a_k'$, using (23) and (24), respectively, yields

$$\left\{a_0 \prod_{j=1}^{t} \operatorname{tr}(a_j)\right\} \circ D_i = \begin{cases} a_0' \operatorname{tr}(x_i a_0'') \prod_{j=1}^{t} \operatorname{tr}(a_j), \\ a_0 x_i a_k' \prod_{\substack{j=1 \\ j \neq k}}^{t} \operatorname{tr}(a_j), \end{cases} \tag{36}$$

from which it follows that $G \cap T_l$ is invariant under multiplication by D_i. It follows immediately from (36) that

$$D_i \circ D_i = e,$$

$$\left\{g(x_1, \ldots, x_k) \prod_{j=k+1}^{l} \operatorname{tr}(x_j)\right\} \circ D_{k+1} = g(x_1, \ldots, x_k)\, x_{k+1} \prod_{j=k+2}^{l} \operatorname{tr}(x_j),$$
$$\left\{g(x_1, \ldots, x_k) \prod_{j=k+1}^{l} \operatorname{tr}(x_j)\right\} \circ D_{k+1} \circ D_{k+2} \circ \ldots \circ D_{k+t} \circ D_{k+1}$$
$$= g(x_1, \ldots, x_k) \operatorname{tr}(x_{k+1} \ldots x_{k+t}) \prod_{j=k+t+1}^{l} \operatorname{tr}(x_j), \tag{37}$$

where g is a multilinear monomial in $x_1, \cdots, x_k$. The last two equalities show that e can be transformed into any generalized monomial by successive multiplications by elements D_i. Consequently each generalized monomial in $G \cap T_l$ can be represented as a product of elements D_i of the second order with respect to the operation $\circ$. This shows that $G \cap T_l$ is a group and the elements D_i are generators. It is now obvious that T_l is the group algebra of this group and the operation $\circ$ is well defined on T_l. The lemma is proved.

We will now explain how the operation $\circ$ and the symmetric bilinear form $\mathfrak{b}_E$ of Lemma 5 are related.

Lemma 7. *For any elements $f, g, h \in T_l$ we have*

$$\mathfrak{b}_E(f \circ h, g) = \mathfrak{b}_E(f, h \circ g).$$

Proof. To prove the lemma it suffices to assume that $h = D_i$. Using (35), we have

$$\mathfrak{b}_E(f \circ D_i, g) = \gamma\alpha\left(\pi(f y_i x_i)\big|_{y_i = x_i} g\right) = \gamma\alpha\left(\pi(f y_i x_i)\, g\big|_{x_i = y_i}\right)$$
$$= \gamma\alpha\left(f y_i x_i g\big|_{x_i = y_i}\right) = \gamma\alpha\left(f\pi\left(y_i x_i g\big|_{x_i = y_i}\right)\right)$$
$$= \gamma\alpha\left(f\pi(x_i y_i g)\big|_{y_i = x_i}\right) = \mathfrak{b}_E(f, D_i \circ g).$$

The lemma is proved. This lemma has the following corollary.

Corollary. $\mathrm{Ann}_{T_k} b_E$ *is a two-sided ideal of the group algebra* T_k $(k = 1, 2, \ldots)$.

Let S_{l+1} be the group of all permutations, acting on the right, of the set $\{x_0, x_1, \ldots, x_l\}$. Each permutation decomposes uniquely into a product of independent cycles. We denote the cycle $x_{i_1} \to x_{i_2} \to x_{i_3} \to \cdots \to x_{i_k} \to x_{i_1}$ by $(x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_k})$. In addition, if $a = x_{i_1} x_{i_2} \cdots x_{i_k}$, then we denote the cycle $(x_{i_1} x_{i_2} \cdots x_{i_k})$ by (a).

Lemma 8. *The group* $G \cap T_l$ *is isomorphic to the group* S_{l+1} *of all permutations, acting on the right, of a set of* $l + 1$ *letters. The isomorphism is realized by the mapping*

$$\varphi \{a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_l)\} = (x_0 a_0)(a_1) \ldots (a_l).$$

Proof. The mapping ϕ is well defined, since independent cycles commute and $(x_{i_1} x_{i_2} \cdots x_{i_t})$ and $(x_{i_2} \cdots x_{i_1})$ represent the same cycle. By Lemma 2, ϕ is one-to-one. To prove the lemma it suffices to show that for any $g \in G \cap T_l$

$$\varphi(g) \cdot \varphi(D_l) = \varphi(g \circ D_l). \tag{38}$$

But

$$\{(x_0 a_0)(a_1) \ldots (a_l)\} \cdot (x_0 x_l) = \begin{cases} (x_0 a_0')(x_l a_0'')(a_1) \ldots (a_l), & \text{if} \quad a_0 = a_0' x_l a_0'', \\ (x_0 a_0 x_l a_k')(a_1) \ldots (a_{k-1})(a_{k+1}) \ldots (a_l), & \text{if} \quad a_k = x_l a_k', \end{cases}$$

and (38) now follows by comparing this with (36). The lemma is proved.

This lemma implies that the inverse of $g \in G \cap T_l$ with respect to the operation $\circ$ is $\sigma(g)$.

From now on, we will not make any distinction between the group algebras $K[S_{l+1}]$ and T_l.

Let us now summarize the results of this section.

Proposition 1. *On the linear space* T_l *of multilinear generalized polynomials of the algebra* $\mathfrak{G}$ *it is possible to define in accordance with* (33) *an operation* $\circ$ *with respect to which the set* $G \cap T_l$ *becomes a group isomorphic to* S_{l+1}, *and* T_l *becomes the group algebra of this group. In addition, for any polynomial* $E(\gamma)$ *the formula* (30) *defines a symmetric bilinear form* b_E *which is associative with respect to the operation* $\circ$ *and assumes values in the algebra* $K[\gamma]/\{E(\gamma)\}$; *to each such bilinear form there corresponds a verbal ideal* $V \subset \mathfrak{G}$ *such that for all* $l = 1, 2, \cdots$ *we have* $V \cap T_l = \mathrm{Ann}_{T_l} b_E$ *and this set is a two-sided ideal of the algebra* T_l. *If* $E(\gamma) = \gamma - n$, *then the verbal ideal* V *is exactly the set of trace identities of the algebra* M_n.

§4. Classification of bilinear forms and verbal ideals V for which $V \cap T_l$ is a two-sided ideal of T_l

Throughout this section we will assume that the characteristic of the field K is equal to zero. Before reading this section the reader should familiarize himself with

§28 of Chapter IV of the book [4] by Curtis and Reiner, in which are described the simple ideals of the group algebra $K[S_{l+1}]$ over a field of characteristic zero. We mention here without proofs those results of the section indicated which we will need.

By a Young table D of type $(n_1, \cdots, n_k)$ we mean the table in which the number of spaces in the ith row is n_i, and where $n_1 \geq n_2 \geq \cdots \geq n_k$. A Young table D which is arbitrarily filled by the integers from 1 to $n_1 + \cdots + n_k = l + 1$ is called a Young diagram. Let $R(D)$ be the subgroup of S_{l+1} consisting of all elements p which rearrange the numbers in each row of the diagram D, and $C(D)$ the subgroup consisting of the elements q which rearrange the numbers in each column of D.

Proposition 2. *There is a one-to-one correspondence between the set of Young tables D of type $(n_1, \cdots, n_k)$, where $n_1 + \cdots + n_k = l + 1$, and the set of simple two-sided ideals of the group algebra $K[S_{l+1}]$. The ideal corresponding to a Young table D is generated as a two-sided ideal by the element*

$$e(D) = \mathcal{P} \circ Q,$$

where $\mathcal{P} = \Sigma_{p \in R(D)} p$ and $Q = \Sigma_{q \in C(D)} \epsilon(q) q$ (the sums extend over all elements of $R(D)$ and $C(D)$, and $\epsilon(q) = \pm 1$, depending on the parity of q) for some diagram obtained from D. A permutation $s \in S_{l+1}$ can be represented in the form $s = p \circ q$, where $p \in R(D)$ and $q \in C(D)$, if and only if elements belonging to the same row of the diagram D lie in different columns of the diagram Ds; and this representation is unique. For any $x \in K[S_{l+1}]$ we have $\mathcal{P} \circ x \circ Q = \nu \mathcal{P} \circ Q$, where ν is a number depending on x.

We now have at our disposal everything that is necessary to describe all verbal ideals V of the algebra $\mathfrak{G}$ such that, for any natural number l, $V \cap T_l$ is a two-sided ideal of T_l with respect to the operation $\circ$.

It is clear that the set $V \cap T_l$ is a direct sum of simple ideals, each of which is defined by some Young table, but it is not clear

a) whether for a verbal ideal V generated by a set of multilinear generalized polynomials forming a two-sided ideal of T_l it is true that $V \cap T_{l+k}$ is a two sided-ideal of T_{l+k};

b) which Young tables D_1 and D_2 are such that $V_{D_1} \supseteq V_{D_2}$, where V_{D_i} is a verbal ideal generated by a set of multilinear generalized identities forming a simple two-sided ideal of T_{l_i}, corresponding to the Young table D_i; and

c) whether ascending chains of verbal ideals of the form under investigation terminate.

In this section we will answer all of these questions.

The following lemma provides an answer to the first question.

Lemma 9. *Suppose a verbal ideal V is generated by a set of multilinear generalized polynomials of degree l forming a two-sided ideal of T_l with respect to the operation $\circ$. Suppose $f \in T_l$ generates the two-sided ideal $V \cap T_l$. Then $V \cap T_{l+k}$ is a two-sided ideal of T_{l+k} for any $k \geq 0$, and*

$$f(x_1, \ldots, x_l) \prod_{i=1}^{k} \operatorname{tr}(x_{l+i})$$

generates this two-sided ideal.

Proof. We first establish that $V \cap T_{l+k}$ is a two-sided ideal. It suffices to verify that $V \cap T_{l+k}$ is stable under multiplication by the generators D_i. In the proof of Lemma 5 it was shown that the linear space $V \cap T_{l+k}$ is generated by the elements

$$g_1 = g(a_1, \ldots, a_l)\, a_{l+1} \prod_{j=1}^{r} \operatorname{tr}(b_j),$$

where $g(x_1, \cdots, x_l) \in V \cap T_l$, and a_i and b_j are words. Using (35), (28), (29) and (36), we obtain

$$g_1 \circ D_l = \pi\Big(g(a_1, \ldots, a_l)\, a_{l+1} \prod_{j=1}^{r} \operatorname{tr}(b_j)\, y_l x_l\Big)\Big|_{y_l = x_l}$$

$$= \begin{cases} g(a_1, \ldots, a_l)\, a_{l+1} x_l b_s' \prod\limits_{\substack{j=1 \\ j \neq s}}^{r} \operatorname{tr}(b_j), & \text{if} \quad b_s = x_l b_s', \\ g(a_1, \ldots, a_l)\, a_{l+1}' \operatorname{tr}(x_l a_{l+1}'') \prod\limits_{j=1}^{r} \operatorname{tr}(b_j), & \text{if} \quad a_{l+1} = a_{l+1}' x_l a_{l+1}'', \\ (g(x_1, \ldots, x_l) \circ D_s)\big|_{\substack{x_j = a_j (j \neq s) \\ x_s = a_{l+1} x_l a_s''}} \cdot a_s' \prod\limits_{j=1}^{r} \operatorname{tr}(b_j), & \text{if} \quad s \leqslant l,\ a_s = a_s' x_l a_s'', \end{cases}$$

which implies that $V \cap T_{l+k}$ is a right ideal. Similarly, it can be shown that $V \cap T_{l+k}$ is a left ideal.

Let us prove the second part of the lemma. Taking into account (37) and what has been said above, it suffices to show that

$$g_1 = g(a_1, \ldots, a_l) \prod_{j=1}^{k-r+1} \operatorname{tr}(x_{i_j}),$$

where $g(x_1, \cdots, x_l) \in V \cap T_l$ lies in the two-sided ideal generated by

$$f_1 = f(x_1, \ldots, x_l) \prod_{j=1}^{k} \operatorname{tr}(x_{l+j}).$$

It is well known that if $\delta = (y_{11} \cdots y_{1r})(y_{21} \cdots y_{2s}) \cdots (y_{m1} \cdots y_{mt})$ is the decomposition of $\delta \in S_n$ into independent cycles, then for any permutation $\delta_1 \in S_n$

$$\delta_1^{-1} \delta \delta_1 = (\delta_1(y_{11}) \ldots \delta_1(y_{1r}))(\delta_1(y_{21}) \ldots \delta_1(y_{2s})) \ldots (\delta_1(y_{m1}) \ldots \delta_1(y_{mt})).$$

Hence by a suitable conjugation the element g_1 can be transformed into the element

$$g_2 = g(a_1' x_1, \ldots, a_l' x_l) \prod_{j=k+l}^{l+r} \operatorname{tr}(x_j).$$

It is obvious that

$$g_3 = g(x_1, \ldots, x_l) \prod_{j=1}^{k} \operatorname{tr}(x_{l+j})$$

lies in the two-sided ideal generated by f_1. If we put

$$s = x_1 a_1' \ldots x_l a_l' \prod_{j=k+l}^{l+r} \operatorname{tr}(x_j),$$

then, using (24), (28) and (29), we obtain

$$g_3 \circ s = \pi(g(x_1 y_1, \ldots, x_l y_l) \prod_{j=1}^{k} \operatorname{tr}(x_{l+j} y_{l+j})\, s)\Big|_{\substack{y_1 = x_1 \\ \cdots\cdots \\ y_{l+k} = x_{l+k}}}$$

$$= \pi(g(x_1, \ldots, x_l)\, y_1 x_1 a_1' y_2 x_2 a_2' \ldots y_l x_l a_l' \prod_{j=l+k}^{l+r} \operatorname{tr}(x_j))\Big|_{\substack{y_1 = x_1 \\ \cdots\cdots \\ y_l = x_l}}.$$

$$= \pi(g(a_1' x_1, \ldots, a_l' x_l) \prod_{j=l+k}^{l+r} \operatorname{tr}(x_j)\, y_1 x_1 \ldots y_l x_l)\Big|_{\substack{y_1 = x_1 \\ \cdots\cdots \\ y_l = x_l}} = g_2 \circ t,$$

where $t = x_1 \cdots x_l \operatorname{tr}(x_{l+1}) \cdots \operatorname{tr}(x_{l+k})$. Thus $g_2 \circ t$ lies in the two-sided ideal generated by f_1, and hence the elements $g_2 = (g_2 \circ t) \circ t^{-1}$ and g_1 lie in the same ideal. The lemma is proved.

Lemma 10. *Suppose the Young table D_1 of type $(n_1, \cdots, n_k)$ is a subtable of the Young table D_2 of type $(n_1', \cdots, n_r')$, i.e. $n_i \le n_i'$ for all i, and let $l_1 = \Sigma_1^k n_i$ and $l_2 = \Sigma_1^r n_i'$. Let V_{D_i} $(i = 1, 2)$ denote a verbal ideal of the algebra $\mathcal{G}$ generated by a set of multilinear polynomials of degree l_i forming a simple two-sided ideal of T_{l_i}, corresponding to the Young table D_i. Then $V_{D_1} \supseteq V_{D_2}$.*

Proof. Fill in the table D_2 with the integers from 1 to l_2 in such a way that the integers from 1 to l_1 lie in the table D_1. If $e(D_i) = \mathcal{P}_i \circ Q_i$ $(i = 1, 2)$ as in Proposition 2, then $e(D_i)$ generates the two-sided ideal $V_{D_i} \cap T_{l_i}$. We may view the permutation group S_{l_1+1}, acting on $\{x_0, \cdots, x_{l_1}\}$, as being embedded in a natural way in the group S_{l_2+1}, acting on $\{x_0, \cdots, x_{l_2}\}$, by identifying S_{l_1+1} with the subgroup of S_{l_2+1} which fixes the elements $x_{l_1+1}, \cdots, x_{l_2}$; and then $R(D_2) \supseteq R(D_1)$ and $C(D_2) \supseteq C(D_1)$. Carrying this over to the group rings, we may view T_{l_1} as embedded in T_{l_2}. In the sense of this embedding, the element $e(D_1)$, by Lemma 9, generates $V_{D_1} \cap T_{l_2}$ as a two-sided ideal. On the other hand,

$$\mathcal{P}_2 \circ e(D_1) \circ Q_2 = (\mathcal{P}_2 \circ \mathcal{P}_1) \circ (Q_1 \circ Q_2) = |\mathcal{P}_1|\,|Q_1|\,(\mathcal{P}_2 \circ Q_2) = |\mathcal{P}_1|\,|Q_1|\,e(D_2),$$

where $|\mathcal{P}_1|$ is the order of the group $R(D_1)$, and $|Q_1|$ the order of $C(D_1)$. Consequently the generator $e(D_2)$ of the ideal $V_{D_2} \cap T_{l_2}$ lies in the ideal $V_{D_1} \cap T_{l_2}$, and therefore $V_{D_1} \cap T_{l_2} \supseteq V_{D_2} \cap T_{l_2}$. It now follows from the definition of V_{D_1} and V_{D_2} that $V_{D_1} \supseteq V_{D_2}$. The lemma is proved.

We now turn to the bilinear form $\mathfrak{b}_E$ of § 3, which is defined on $T_l = K[S_{l+1}]$ and assumes the values in the ring $K_E = K[\gamma]/\{E(\gamma)\}$.

Let τ_t denote the sum of those permutations of S_{l+1} which decompose into exactly t independent cycles, and let $d = \sum_{s \in S_{l+1}} \mathfrak{b}_E(s, 1)s$, where the sum extends over permutations $s \in S_{l+1}$. In view of (31), $d = \sum_{t=1}^{l+1} \gamma^t \tau_t$ is an element of the center of the algebra $T_l(\gamma)$. If $g = \sum_{\delta \in S_{l+1}} \beta_\delta \delta$, then let $\sigma(g)$ denote the element $\sum_{\delta \in S_{l+1}} \beta_\delta \delta^{-1}$. By the associativity of the bilinear form $\mathfrak{b}_E$ we obtain

$$d \circ g = \sum_s \sum_\delta \beta_\delta \mathfrak{b}_E(s, 1)\, s\delta = \sum_\delta \sum_{r = s\delta} \beta_\delta \mathfrak{b}_E(r, \delta^{-1})\, r$$

$$= \sum_r \mathfrak{b}_E\left(r, \sum_\delta \beta_\delta \delta^{-1}\right) r = \sum_{s \in S_{l+1}} \mathfrak{b}_E(s, \sigma(g))\, s,$$

from which it follows that $g \in \mathrm{Ann}_{T_l} \mathfrak{b}_E$ if and only if $d \circ \sigma(g) = 0$ in $K_E[S_{l+1}]$. But since $0 = \sigma(d \circ \sigma(g)) = g \circ d = d \circ g$, we have proved

Lemma 11. *A generalized polynomial g belongs to $\mathrm{Ann}_{T_l} \mathfrak{b}_E$ if and only if $d \circ g = 0$ in $K_E[S_{l+1}]$.*

If 1_D is the identity element of the simple ideal corresponding to the Young table D, then from the fact that d is an element of the center of the algebra $T_l(\gamma)$ we obtain

$$d \circ 1_D = D(\gamma)\, 1_D;$$

moreover, for any element a of this ideal

$$d \circ a = d \circ (1_D \circ a) = D(\gamma)\, a, \tag{39}$$

where $D(\gamma)$ is a polynomial depending only on the Young table D. If we denote by χ_D the character of the irreducible representation of the group S_{l+1} corresponding to D, and by m_D the dimension of this representation, then

$$D(\gamma) = \sum_{t=1}^{l+1} \gamma^t \frac{\chi_D(\tau_t)}{m_D}. \tag{40}$$

Lemma 12. *Let $E(\gamma)$ be a polynomial, $\mathfrak{b}_E$ the bilinear form of § 3, and V the verbal ideal corresponding to this bilinear form. As in Lemma 10, let V_D be a verbal ideal of the algebra $\mathfrak{G}$ generated by a set of multilinear polynomials of degree l forming a two-sided ideal of T_l, corresponding to the Young table D. Then $V \supseteq V_D$ if and only if $E(\gamma)$ divides $D(\gamma)$.*

Proof. If a is an arbitrary element of $V_D \cap T_l$, then, by Lemma 11, $a \in \mathrm{Ann}_{T_l} \mathfrak{b}_E$ if and only if $d \circ a = 0$ in the ring $K_E[S_{l+1}]$. It follows from (39) that this is equivalent to the fact that $E(\gamma) | D(\gamma)$. The lemma is proved.

Corollary. *If V_{D_1} and V_{D_2} are as in Lemma* 10, *and $D_1(\gamma)$ and $D_2(\gamma)$ are the polynomials defined by* (40), *then*

$$D_1(\gamma)\,|\,D_2(\gamma).$$

Proof. Put $E(\gamma) = D_1(\gamma)$, and let V be the verbal ideal of the algebra $\mathfrak{G}$ corresponding to the bilinear form $\mathfrak{b}_E$. Then, by Lemma 12, $V \supseteq V_{D_1}$, and by Lemma 10, $V_{D_1} \supseteq V_{D_2}$. Since $V \supseteq V_{D_2}$, it follows from Lemma 12 that $E(\gamma) = D_1(\gamma)|D_2(\gamma)$. The corollary is proved.

Lemma 12 and its corollary demonstrate the necessity of finding an explicit formula for the polynomial $D(\gamma)$, more precisely, the decomposition of $D(\gamma)$ into prime factors.

Lemma 13. *Let D be a rectangular Young table, the length of whose rows is l and the length of whose columns is k. Then the sum of the roots of the polynomial $D(\gamma)$ is equal to*

$$l\binom{k}{2} - k\binom{l}{2} \quad \left(\binom{1}{2} = 0\right).$$

Proof. The coefficient of the leading term of $D(\gamma)$ in (40) is equal to $\chi_D(1)/m_D = 1$. Hence the sum of the roots of $D(\gamma)$, taken with a minus sign, is equal to

$$\nu = \frac{\chi_D(\tau_{lk-1})}{m_D}.$$

It is obvious that τ_{lk-1} is the sum of all transpositions $t_{ij} \in S_{lk}$, and ν can be computed from the formula

$$\nu e(D) = \tau_{lk-1} e(D) = \mathcal{P} \circ \tau_{lk-1} \circ Q,$$

where $e(D)$, $\mathcal{P}$ and Q are as in Proposition 2 for some Young diagram corresponding to the table D. We first compute $\mathcal{P} \circ t_{ij} \circ Q$, where t_{ij} is the transposition of the indices i and j. Proposition 2 implies that this element is equal to $n\mathcal{P} \circ Q$. It is clear that n is equal to the coefficient of 1 in the expression $\mathcal{P} \circ t_{ij} \circ Q$; hence $n = \Sigma\epsilon(s'')$, where the sum extends over all pairs (s', s''), where $s' \in R(D)$ and $s'' \in C(D)$, such that $s' \circ t_{ij} \circ s'' = 1$, i.e. $t_{ij} = (s')^{-1} \circ (s'')^{-1}$. Then, by Proposition 2, the elements of a row of the diagram D lie in different columns of the diagram Dt_{ij}. But this is possible if and only if the indices i and j lie either in one row or in one column of D. In the first case $\mathcal{P} \circ t_{ij} = \mathcal{P}$; in the second $t_{ij} \circ Q = -Q$. In view of the uniqueness of the representation $t_{ij} = (s')^{-1} \circ (s'')^{-1}$ we have

$$\mathcal{P} \circ t_{ij} \circ Q = \begin{cases} \mathcal{P} \circ Q, & \text{if } t_{ij} \in R(D), \\ -\mathcal{P} \circ Q, & \text{if } t_{ij} \in C(D), \\ 0, & \text{if } t_{ij} \notin R(D) \cup C(D). \end{cases}$$

Hence $\nu = \nu_1 - \nu_2$, where ν_1 is the number of transpositions in $R(D)$, and ν_2 the number of transpositions in $C(D)$. Obviously $\nu_1 = k\binom{l}{2}$ and $\nu_2 = l\binom{k}{2}$, from which the required formula follows. The lemma is proved.

Fill in the Young table D of type $(n_1, \cdots, n_k)$ with integers by placing $i - j$ in the jth space of the ith row $(1 \leq j \leq n_i)$. For example,

0	−1	−2	−3	−4	−5
1	0	−1	−2	−3	−4
2	1	0	−1		
3	2				

The polynomial

$$\prod (\gamma - (i - j)), \tag{41}$$

where the product extends over all spaces of the table D, is well known and is called the *graph* of D.

Lemma 14. *The polynomial* $D(\gamma)$ *defined by* (40) *is equal to the graph of the Young table* D.

Proof. We first observe that the coefficient of the leading term of $D(\gamma)$ is equal to $\chi_D(1)/m_D = 1$. We will prove the lemma by induction on the number of spaces in the Young table D. The basis for the induction is a table with one space, in which case (40) and (41) agree. Suppose the lemma has been proved for tables with less than n spaces. Let D be a table of type $(n_1, \cdots, n_k)$. If D is not rectangular, we can obtain from it two distinct Young tables D' and D'' by removing the spaces (k, n_k) and (r, n_r), respectively, where $1 \leq r < k$. Let D_0 denote the Young table obtained by removing from D these two spaces simultaneously. By the induction assumption, we may assume that $D'(\gamma)$, $D''(\gamma)$ and $D_0(\gamma)$ agree with the graphs of the corresponding Young tables. From the corollary of Lemma 12 we obtain

$$D'(\gamma) = D_0(\gamma)(\gamma - (r - n_r)), \quad D''(\gamma) = D_0(\gamma)(\gamma - (k - n_k))$$

$$D'(\gamma) \mid D(\gamma), \quad D''(\gamma) \mid D(\gamma).$$

Since $r < k$ and $n_k < n_r$, it follows that $n_r - r > n_k - k$ and these roots are distinct; hence

$$D(\gamma) = D_0(\gamma)(\gamma - (r - n_r))(\gamma - (k - n_k)),$$

and the lemma is proved for a nonrectangular table D.

If D is a rectangular table with row length l and column length k, we obtain only the equality $D(\gamma) = D'(\gamma)(\gamma - \xi)$, where D' is the Young table obtained from D by removing the space (k, l) and ξ is some integer. By the induction assumption, $D'(\gamma)$ and the graph of D' agree; hence they have the same root sum, which we denote by Σ'. Then $\Sigma' + \xi$ is the sum of the roots of $D(\gamma)$ and, by Lemma 13, $\Sigma' + \xi = l\binom{k}{2} - k\binom{l}{2}$.

It can be shown directly that the sum of the roots of the graph of D is also equal to $l\binom{k}{2} - k\binom{l}{2}$, which implies that $\xi = k - l$. The lemma is proved.

Corollary. *If $E(y) = D(y)$ and $\mathfrak{b}_E$, V and V_D are as in Lemma 12, then $V = V_D$. Moreover, if V_{D_1} is a verbal ideal of the algebra $\mathfrak{G}$ such that $V_{D_1} \cap T_{l_1}$ is a simple two-sided ideal of T_{l_1} corresponding to the Young table D_1, and V_{D_1} as a verbal ideal is generated by the set $V_{D_1} \cap T_{l_1}$, then $V_D \supseteq V_{D_1}$ if and only if D is a subtable of D_1.*

Proof. By Lemma 12 we have $V \supseteq V_D$, and $V \supseteq V_{D_1}$ if and only if $E(y) = D(y)|D_1(y)$. It is evident from (41) that $D(y)|D_1(y)$ if and only if D is a subtable of D_1. But by Lemma 10, if D is a subtable of D_1, then $V_D \supseteq V_{D_1}$. Consequently $V \supseteq V_{D_1}$ if and only if $V_D \supseteq V_{D_1}$. Both statements of the corollary now follow from the corollary to Lemma 7 of §3, which implies that V is generated by all verbal ideals of the form V_{D_1} contained in V. The corollary is proved.

This corollary answers the second question raised at the beginning of this section.

The following theorem summarizes the results of Lemmas 9–14.

Theorem 3. *To each Young table D of type $(n_1, \cdots, n_k)$ with $l + 1 = \Sigma_1^k n_i$ spaces, whose graph $D(y)$ is given by (41), it is possible to assign a bilinear form $\mathfrak{b}_D$ and a verbal ideal V_D corresponding to $\mathfrak{b}_D$ such that for any natural number k we have $V_D \cap T_k = \operatorname{Ann}_{T_k} \mathfrak{b}_D$, and this set forms a two-sided ideal of T_k with respect to the operation $\circ$; V_D as a verbal ideal of the algebra $\mathfrak{G}$ is generated by the multilinear generalized polynomials $V_D \cap T_l$ forming a simple two-sided ideal of the group algebra $T_l = K[S_{l+1}]$, corresponding to the Young table D. The correspondence between the Young tables D and the verbal ideals V_D of $\mathfrak{G}$ is one-to-one, and $V_{D_1} \supseteq V_{D_2}$ if and only if D_1 is a subtable of D_2.*

If $E(y)$ is a polynomial and $\mathfrak{b}_E$ the bilinear form of §3, then the verbal ideal V corresponding to this bilinear form contains the verbal ideal V_D corresponding to the Young table D if and only if $E(y)|D(y)$; in particular, V is nonzero if and only if all of the roots of $E(y)$ are integers.

Finally, if V is a verbal ideal of the algebra $\mathfrak{G}$ such that $V \cap T_k$ is a two-sided ideal of the algebra T_k with respect to the operation $\circ$ for any natural number k, then there exist a finite number of Young tables $D_1, \cdots, D_n$ such that the verbal ideal generated by $\{V_{D_1}, \cdots, V_{D_n}\}$ is equal to V.

All parts of this theorem have already been proved except for the last, which is equivalent to the following theorem.

Theorem 4. *Any ascending chain of verbal ideals $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ of the algebra $\mathfrak{G}$ such that $V_i \cap T_k$ is a two-sided ideal of the algebra T_k with respect to* *the operation* $\circ$ *for any natural numbers* i *and* k *terminates, i.e. for some* N *we have* $V_N = V_{N+1} = \cdots$.

Proof. As indicated in the proof of the corollary to Lemma 14, each verbal ideal V_i is generated by all verbal ideals V_D, corresponding to Young tables D, which are contained in V_i. Hence it suffices to prove the theorem in the case where V_i is generated by verbal ideals $\{V_{D_1}, \cdots, V_{D_i}\}$ corresponding to Young tables $D_1, \cdots, D_i$. Suppose that the chain $V_1 \subsetneq V_2 \subseteq \cdots$ does not terminate. Then we may assume that $V_k \neq V_{k+p}$ for any natural numbers k and p. By Theorem 3, this means that in the sequence of Young tables $D_1, D_2, \cdots$ we have that D_k is not a subtable of D_{k+p} for any natural numbers k and p. We must show that this leads to a contradiction.

Consider the topological abelian group H whose elements are all functions defined on the set of integers Z and taking values in Z and whose neighborhoods of zero have the form $U_n = \{f \in H | f(m) = 0,\ m = 0, \pm 1, \cdots, \pm n\}$. To each table D_i we can assign the function $g_i \in H$ for which $g_i(m)$ is equal to the multiplicity of the root m of the polynomial $D_i(\gamma)$. From the definition of a Young table and the explicit form (41) of $D_i(\gamma)$ it is clear that

$$g_i(m) \geqslant g_i(m+1) \text{ when } m \geqslant 0, \quad g_i(m-1) \leqslant g_i(m) \text{ when } m \leqslant 0; \tag{42}$$

moreover, for a sufficiently large number $N_i = N(D_i)$ and $|m| > N_i$ we have $g_i(m) = 0$. The condition that D_k is not a subtable of D_{k+p} for any natural numbers k and p can be restated as follows: for any natural numbers k and p there exists $j \in Z$ such that

$$g_k(j) > g_{k+p}(j). \tag{43}$$

Let $r = \max(n, l)$, where n is the length of the first row and l the length of the first column of D_1. Then $g_k(0) \leq r$, or else D_1 would be a subtable of D_k.

Consequently, by (42), the set of functions $\{g_i | i = 1, 2, \cdots\}$ is bounded by the number r, i.e. $g_i(m) \leq r$ for any i and m. We leave it to the reader to verify that the set $A_r = \{f \in H | f(m) \leq r,\ m = 0, \pm 1, \cdots\}$ is compact, and hence the sequence of functions $\{g_i\}$ has a limit point g. Since congruence in the sense of the given topology implies the pointwise convergence of functions, we obtain from (42) for the limit function that $g(0) \leq r$, $g(m) \geq g(m+1) \geq 0$ when $m \geq 0$, and $0 \leq g(m-1) \leq g(m)$ when $m \leq 0$. But then there exists a natural number M such that when $|m| > M$ we have $g(m) = m_0 = \text{const} \geq 0$. Since g is the limit point of $\{g_i\}$, there exists a function g_{i_0} for which $g - g_{i_0} \in U_m$. Choose a natural number $M_1 > M$ such that when $|m| > M_1$ we have $g_i(m) = 0$, and choose g_{i_1} such that $i_1 > i_0$ and $g - g_{i_1} \in U_{M_1}$. Then

$$g_{i_0}(m) = \begin{cases} g(m), & |m| \leqslant M, \\ \leqslant m_0, & M < |m| \leqslant M_1, \\ 0, & M_1 < |m|, \end{cases}$$

$$g_{i_1}(m)=\begin{cases} g(m), & |m|\leqslant M, \\ m_0, & M<|m|\leqslant M_1, \\ \geqslant 0, & M_1<|m|, \end{cases}$$

which implies that $g_{i_0}(m) \le g_{i_1}(m)$ for all $m \in Z$, contradicting (43). The theorem is proved.

§5. Proof of Theorem 2 and other consequences of Theorem 3

As in §4, we identify the algebra T_l under the operation $\circ$ with the group algebra $K[S_{l+1}]$ by means of the isomorphism ϕ of Lemma 8.

Proof of Theorem 2. If $E(\gamma) = \gamma - n$, then by Lemma 4 the set of all trace identities of the algebra M_n over a field of characteristic zero forms a verbal ideal V corresponding to the bilinear form $\mathfrak{b}_E$. By Proposition 1, the set of multilinear trace identities $V \cap T_k = \mathrm{Ann}_{T_k} \mathfrak{b}_E$ forms a two-sided ideal of T_k with respect to the operation $\circ$ for any natural number k; hence, by Theorem 3, V as a verbal ideal is generated by all verbal ideals V_D contained in it, where D is a Young table. The same theorem implies that $V \supseteq V_D$ if and only if $E(\gamma) = \gamma - n$ divides $D(\gamma)$. It is evident from the formula (41) for the graph $D(\gamma)$ of the Young table D that $\gamma - n | D(\gamma)$ if and only if the length of the first column of this table is at least $n+1$, i.e. D contains a Young subtable $D^{(n)}$ of type $(1, \cdots 1)(n+1$ ones) with an $(n+1)$th space. Thus the set of all trace identities of M_n is equal to the verbal ideal $V_{D(n)}$, which by Theorem 3 is generated by the multilinear polynomials $V_{D(n)} \cap T_n$ forming a simple two-sided ideal of T_n corresponding to the Young table $D^{(n)}$. It is well known that the two-sided ideal of $T_n = K[S_{n+1}]$ corresponding to the Young table of type $(1, \cdots, 1)$ $(n+1$ ones) is one-dimensional. By Proposition 2, it is generated by the element $f = \sum_{s \in S_{n+1}} \epsilon(s)s$, where the sum extends over all permutations s, and $\epsilon(s) = \pm 1$ depending on the parity of s.

Hence the multilinear generalized polynomial in $x_1, \cdots, x_n$

$$f = \varphi^{-1}(f) = (-1)^n \sum (-1)^t a_0 \operatorname{tr}(a_1) \ldots \operatorname{tr}(a_t), \tag{44}$$

where the sum extends over all generalized monomials in T_n, generates the verbal ideal $V_{D(n)}$. It remains to show that this verbal ideal is generated by the Hamilton-Cayley identity f_n. This follows immediately from the fact that the degree of f_n is n and its complete linearization is a nontrivial multilinear generalized polynomial in $x_1, \cdots, x_n$, and, since $V_{D(n)} \cap T_n$ is one-dimensional, this polynomial must agree with (44) to within a factor. Theorem 2 is proved.

We now indicate some corollaries of Theorem 2 concerning the ordinary identities of the algebra M_n.

Let W_{l+1} denote the linear space of full cycles, i.e. the linear space generated in $K[S_{l+1}]$ by those permutations s which are cycles of length $l+1$.

Corollary 1. *Let $I(f)$ be the two-sided ideal of the group algebra $K[S_{l+1}] = T_l$ generated by the element*

$$f=\sum_{\substack{s(x_l)=x_l\\ l>n}}\varepsilon(s)s,$$

where the sum extends over all permutations in S_{l+1} *fixing the elements* $x_{n+1},\cdots,x_l$ *(rearranging the elements* $x_0,\cdots,x_n$*), and* $\epsilon(s)=\pm 1$ *depending on the parity of* s*. Then the set of multilinear identities of degree* l *is equal to* $I(f)\cap W_{l+1}$.

Proof. As we noted in §2, the algebra $\mathfrak{G}$ of generalized polynomials contains in a natural way the algebra F of noncommutative polynomials in the variables $x_1, x_2, \cdots$. It is obvious that a polynomial $g\in F$ is an identity of M_n if it is a trace identity, i.e. if $g\in V_{D(n)}$. Consequently, to determine the multilinear identities in $x_1,\cdots,x_l$ we must determine all $g\in V_{D(n)}\cap T_l$ having the form

$$g(x_1,\ \ldots,\ x_l)=\sum\beta_{i_1,\ldots,i_l}x_{i_1}\ \ldots\ x_{i_l}. \tag{45}$$

From the definition of the isomorphism ϕ of Lemma 8 it is clear that $\phi(g)$ belongs to the space W_{l+1} of full cycles. Since we have identified T_l and $K[S_{l+1}]$, the elements of the form (45) comprise the whole space W_{l+1} of full cycles. The lemma now follows from the fact that $V_{D(n)}\cap T_n$, as we have seen in the proof of Theorem 2, is generated by the element $\Sigma_{s\in S_{n+1}}\epsilon(s)s$, and from Lemma 9, which implies that $V_{D(n)}\cap T_l=I(f)$. The corollary is proved.

The result of Corollary 1 can be sharpened. Let $D^{(n)}$ be as in the proof of Theorem 2. Let ϕ_1 denote the automorphism of the algebra $\mathfrak{G}$ of generalized polynomials such that

$$\varphi_1\{a_0\ \mathrm{tr}\ (a_1)\ \ldots\ \mathrm{tr}\ (a_t)\}=(-1)^t a_0\,\mathrm{tr}\ (a_1)\ \ldots\ \mathrm{tr}\ (a_t).$$

It is easy to verify that ϕ_1 carries verbal ideals into verbal ideals and that ϕ_1 induces an automorphism in the group ring $T_l=K[S_{l+1}]$ for which $\phi_1(s)=\epsilon(s)s$. In view of Proposition 2, the simple ideal of T_l corresponding to the Young table D maps into the simple ideal corresponding to the Young table D^* obtained by transposing D. Hence the verbal ideal V_D corresponding to the Young table D maps into the verbal ideal V_{D^*}. The automorphism ϕ_1 acts on the space W_{l+1} of full cycles as ± 1, depending on the parity of the cycles. Therefore

$$V_{D^{(n)}}\cap W_{l+1}=V_{(D^{(n)})^*}\cap W_{l+1}$$

and by Corollary 1 the multilinear identities of M_n (hence all identities of M_n) lie in the verbal ideal $V=V_{D(n)}\cap V_{(D(n))^*}$ and are equal to $V\cap W_{l+1}$.

The verbal ideal V possesses the property that $V\cap T_k$ is a two-sided ideal of T_k with respect to the operation $\circ$ for all natural numbers k. By Theorem 3, V is generated by all verbal ideals V_D contained in it for some Young table D. But $V\supseteq V_D$ if and only if D contains $D^{(n)}$ and $(D^{(n)})^*$ as subtables, and hence also the Young table $\mathfrak{M}_n$

0	-1	-2	...	$-n$
1				
2				
...				
n				

with a $(2n+1)$th space. Consequently $V = V_{\mathfrak{M}_n}$, and the multilinear identities of degree l of the algebra M_n are exactly $V_{\mathfrak{M}_n} \cap W_{l+1}$. In the case where $l = 2n$, it follows from Amitsur's theorem that $V_{\mathfrak{M}_n} \cap W_{2n+1}$ is one-dimensional and is generated by the standard identity $P_{2n}(x_1, \cdots, x_{2n})$. On the other hand, $P_{2n} \in V_{\mathfrak{M}_n} \cap T_{2n}$, the simple two-sided ideal corresponding to the Young table $\mathfrak{M}_n$, and hence P_{2n} generates $V_{\mathfrak{M}_n} \cap T_{2n}$ as a two-sided ideal of the algebra T_{2n}. In view of Lemma 9, we obtain

Corollary 2. *If $I(P_{2n})$ is the two-sided ideal of the group algebra $K[S_{l+1}] = T_l$ generated by the element $\phi(P_{2n}x_{2n+1}\cdots x_l) \in W_{l+1}$, then the set of all multilinear identities of degree l is exactly $W_{l+1} \cap I(P_{2n})$.*

Proof. Lemma 9 implies that the two-sided ideal $V_{\mathfrak{M}_n} \cap T_l$ is generated by the element

$$P_{2n} \prod_{j=2n+1}^{l} \operatorname{tr}(x_j);$$

the fact that this element generates the ideal $I(P_{2n})$ follows from (37). The corollary is proved.

This corollary shows that in some generalized sense all identities of the matrix algebra M_n follow from the standard identity of degree $2n$.

We must observe, however, that in spite of the fact that we have a description of the ideals $I(f)$ and $I(P_{2n})$ in terms of Young tables, Corollaries 1 and 2 do not give explicit formulas for the identities of matrix algebras, but give essentially another formulation of the problem of finding the identities of the full matrix algebra M_n over a field of characteristic zero.

The following result was obtained in [2].

Corollary 3. *If A_i is the linear transformation of §1, defined on the set of multilinear polynomials, then the set of multilinear identities of the matrix algebra M_n is invariant under A_i.*

Proof. Corollary 1 implies that the set of multilinear identities of degree l of the algebra M_n is $V_{D(n)} \cap W_{l+1} \subset T_l$. Since $V_{D(n)} \cap T_l$ and W_{l+1} are invariant under a linear transformation defined by conjugation by an element D_i of the group $S_{l+1} = G \cap T_l$, it follows that $f \in V_{D(n)} \cap W_{l+1}$ if and only if $D_i \circ f \circ D_i \in V_{D(n)} \cap W_{l+1}$. Formula (36) shows that in this case $D_i \circ f \circ D_i = A_i(f)$. The corollary is proved.

If the characteristic of the field K is not equal to zero, then neither Lemma 4 nor Theorem 3 is true. However, the construction of Lemma 5 makes it possible to construct an interesting example pertaining to identities of Lie algebras.

Theorem 5. *Suppose the field K has characteristic $p > 3$ and $E(y) = \prod_{i=0}^{p-1}(y - i)$. Suppose also, as in Lemma 5, that $\mathfrak{b}_E$ is the bilinear form corresponding to $E(y)$ and that V is the verbal ideal corresponding to this bilinear form. Then*

$$[x, \underbrace{y, \ldots, y}_{p-1 \text{ times}}] \in V$$

and for any natural number k we have $[x_1, \cdots, x_k] \notin V$, i.e. the $(p-1)$th Engel identity does not imply the nilpotency of a Lie algebra.

Proof. Since $[x_1, \cdots, x_k]$ is a multilinear polynomial, it suffices to show, in view of the equality $\mathrm{Ann}_{T_k} \mathfrak{b}_E = V \cap T_k$, that

$$[x_1, \ldots, x_k] \notin \mathrm{Ann}_{T_k}\mathfrak{b}_E.$$

From the definition of $\mathfrak{b}_E$ and also (23) and (7), we have

$$\mathfrak{b}_E([x_1, \ldots, x_k], x_k \ldots x_1) = \gamma\alpha([x_1, \ldots, x_k] x_k \ldots x_1)$$
$$= \gamma\alpha\left([x_1, \ldots, x_{k-1}]\frac{x_k^2}{\gamma}\gamma x_{k-1}, \ldots, x_1\right) = \gamma^2\alpha([x_1, \ldots, x_{k-1}] x_{k-1} \ldots x_1)$$
$$= \gamma^{k-1}\alpha([x_1, x_2] x_2 x_1) = (\gamma^2 - 1)\gamma^{k-1},$$

and since $E(y)$ does not divide this polynomial when $p > 3$, it follows that $[x_1, \cdots, x_k] \notin V$.

The identity $[x, y, \cdots, y]$ is equivalent to the multilinear identity

$$\sum_{s \in S_p} x_{s(1)} \ldots x_{s(p)},$$

where the sum extends over all permutations s of the set $\{1, \cdots, p\}$. In view of Lemma 8, which identifies T_p under the operation $\circ$ with $K[S_{p+1}]$ by means of the isomorphism ϕ, this multilinear polynomial is equal to τ_1, the sum of all permutations which are cycles of length $p + 1$. By Lemma 11, which is also true for a field of characteristic $p > 0$, in order to prove that $\tau_1 \in V \cap T_p = \mathrm{Ann}_{T_p}\mathfrak{b}_E$ it suffices to verify that $d \circ \tau_1 = 0$ in the group ring $K_E[S_{p+1}]$, where $K_E = K[y]/\{E(y)\}$.

Let us first show to what the element $d \circ \tau_1$ is equal when the characteristic of K is zero. In this case

$$d = d \circ 1 = d \circ \left(\sum_D 1_D\right) = \sum_D D(\gamma) 1_D,$$

where the sum extends over all Young tables D with graph $D(y)$, and

$$\tau_1 = \tau_1 \circ \left(\sum_D 1_D\right) = \sum_D \frac{\chi_D(\tau_1)}{m_D} 1_D.$$

But it is clear from (40) that $\chi_D(\tau_1)/m_D$ is the coefficient of the first power of the variable γ in the polynomial $D(\gamma)$, and from the explicit formula (41) for the graph it follows trivially that $\chi_D(\tau_1)/m_D = 0$ if in the second row of the Young table D there is more than one space. Therefore,

$$d \circ \tau_1 = \sum D(\gamma) \frac{\chi_D(\tau_1)}{m_D} 1_D,$$

where the sum extends over tables D of the form

0	-1	-2	...	$-k$
1				
2				
...				
$p-k$				

(46)

For such tables, $\chi_D(s) = (-1)^{p-k}$, where s is a cycle of length $p+1$ (see, for example, [5]). Therefore $\chi_D(\tau_1) = (-1)^{p-k}p!$. On the other hand, the explicit formula for 1_D [5] shows that

$$\frac{(p+1)!}{m_D} 1_D = \sum_{s \in S_{p+1}} \beta_s s,$$

where the β_s are integral coefficients. Consequently

$$\frac{\chi_D(\tau_1)}{m_D} 1_D = \frac{(-1)^{p-k}}{p+1} \sum_{s \in S_{p+1}} \beta_s s$$

for any D, and this element has meaning in the case of characteristic $p > 0$, i.e.

$$\frac{\chi_D(\tau_1)}{m_D} 1_D \in K[S_{p+1}].$$

It remains to observe that the polynomials $D(\gamma)$, where the D have the form (46), are divisible by $E(\gamma) = \prod_{i=0}^{p-1}(\gamma - i)$ in the case of a field of characteristic $p > 0$. Consequently $d \circ \tau_1 = 0$ in $K_E[S_{p+1}]$. The theorem is proved.

In conclusion, we propose that the reader independently try to lower the estimate given by Higman [6] for the nilpotency class of algebras satisfying the identity $y^n = 0$: *in any algebra over a field of characteristic zero satisfying the identity* $y^n = 0$ *we have the identity* $x_1 \cdots x_{n^2} = 0$. Hint: in the algebra of generalized polynomials consider the verbal ideal $V_{D(n)} + V_{(D(n))^*}$ and apply Theorem 3.

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BIBLIOGRAPHY

1. Ju. P. Razmyslov, *Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero*, Algebra i Logika 12 (1973), 83–113 = Algebra and Logic 12 (1973), 47–63.

2. Ju. P. Razmyslov, *On a problem of Kaplansky*, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 483–501 = Math. USSR Izv. 7 (1973), 479–496.

3. F. D. Murnaghan, *The theory of group representations*, Johns Hopkins Press, Baltimore, Md., 1938; reprint, Dover, New York, 1963. MR 31 #258.

4. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962. MR 26 #2519.

5. Gilbert de B. Robinson, *Representation theory of the symmetric group*, Math. Expositions, no. 12, Univ. of Toronto Press, Toronto, 1961. MR 23 #A3182.

6. Graham Higman, *On a conjecture of Nagata*, Proc. Cambridge Philos. Soc. 52 (1956), 1–4. MR 17, 453.

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