B. Øksendal, A. Sulem, Partial observation control in an anticipating environment, *Russian Mathematical Surveys*, 2004, Volume 59, Issue 2, 355–375

DOI: 10.1070/RM2004v059n02ABEH000723

Russian Math. Surveys **59**:2 355–375

Uspekhi Mat. Nauk **59**:2 161–184
DOI 10.1070/RM2004v059n02ABEH000723

Partial observation control in an anticipating environment

B. Øksendal and A. Sulem

Abstract. A study is made of a controlled stochastic system whose state $X(t)$ at time t is described by a stochastic differential equation driven by Lévy processes with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The system is assumed to be *anticipating*, in the sense that the coefficients are assumed to be adapted to a filtration $\{\mathcal{G}_t\}_{t\geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0,T]$. The corresponding anticipating stochastic differential equation is interpreted in the sense of *forward integrals*, which naturally generalize semimartingale integrals. The admissible controls are assumed to be adapted to a filtration $\{\mathcal{E}_t\}_{t\in[0,T]}$ such that $\mathcal{E}_t \subseteq \mathcal{F}_t$ for all $t \in [0,T]$. The general problem is to maximize a given performance functional of this system over all admissible controls. This is a *partial observation stochastic control problem in an anticipating environment.* Examples of applications include stochastic volatity models in finance, insider influenced financial markets, and stochastic control of systems with delayed noise effects. Some particular cases in finance, involving optimal portfolios with logarithmic utility, are solved explicitly.

Contents

§ 1. Introduction

Let $B(t) = (B_1(t), \dots, B_m(t))$ and $\eta(t) = (\eta_1(t), \dots, \eta_\ell(t))$ be an m-dimensional Brownian motion and an ℓ-dimensional Lévy process, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$, and let B and η be independent. We assume that $\mathsf{E}[\eta^2(t)] < \infty$ (where E stands for the expectation with respect to P). Let $\widetilde{N}(dt,dz) = (\widetilde{N}_1(dt,dz_1), \dots, \widetilde{N}_\ell(dt,dz_\ell))$, $z = (z_1,\dots,z_\ell)$, be the compensated Poisson random measure corresponding to the process $\eta(t)$, $t \geq 0$.

Let $\{\mathcal{E}_t\}_{t\geq 0}$ and $\{\mathcal{G}_t\}_{t\geq 0}$ be two filtrations such that

$$\mathcal{E}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \quad \text{for any} \quad t \geq 0. \tag{1.1}$$

AMS 2000 *Mathematics Subject Classification.* 93E20, 91B28, 60H07.

Let us consider a controlled stochastic system whose state $X^{(u)}(t) = X(t) = (X_1(t), \ldots, X_n(t))$ at time $t \in [0,T]$ is described by a stochastic differential equation of the form

$$dX(t) = b(t, X(t), u(t), \omega)\, dt + \sigma(t, X(t), u(t), \omega)\, d^- B(t) \\ + \int_{\mathbb{R}^\ell} \theta(t, X(t), u(t), z, \omega)\, \widetilde{N}(d^- t, dz), \qquad X(0) = x \in \mathbb{R}^n,$$

that is,

$$X(t) = X(0) + \int_0^t b(s, X(s), u(s), \omega)\, ds + \int_0^t \sigma(s, X(s), u(s), \omega)\, d^- B(s) \\ + \int_0^t \int_{\mathbb{R}^\ell} \theta(s, X(s^-), u(s^-), z, \omega)\, \widetilde{N}(d^- s, dz), \tag{1.2}$$

where $b\colon [0,T] \times \mathbb{R}^n \times K \times \Omega \to \mathbb{R}^n$, $\sigma\colon [0,T] \times \mathbb{R}^n \times K \times \Omega \to \mathbb{R}^{n\times m}$, and $\theta\colon [0,T] \times \mathbb{R}^n \times K \times \mathbb{R}^\ell \times \Omega \to \mathbb{R}^{n\times\ell}$ are given functions, $K \subset \mathbb{R}^k$ is a given set of *admissible control values*, and our *control process* $u(t) = u(t,\omega) \in K$ is assumed to be *adapted to the filtration* $\{\mathcal{E}_t\}_{t\geq 0}$.

We assume that for any given $x \in \mathbb{R}^n$, $v \in K$, and $z \in \mathbb{R}^\ell$ the random variables

$$b(t, x, v, \cdot\,), \quad \sigma(t, v, v, \cdot\,), \quad \theta(t, x, v, z, \cdot\,) \quad \text{are } \mathcal{G}_t\text{-measurable.} \tag{1.3}$$

In other words, b, σ, and θ are assumed to be adapted to the filtration $\{\mathcal{G}_t\}_{t\geq 0}$. Since $B(t)$ and $\eta(t)$ need not be semimartingales with respect to $\{\mathcal{G}_t\}_{t\geq 0}$, the last two integrals in (1.2) are *anticipating* stochastic integrals. We prefer to interpret these integrals as *forward* integrals (denoted by $d^- B(t)$ and $\widetilde{N}(d^- t, dz)$, respectively), because this is what the integrals would be identical to if we happened to be in a semimartingale context. (See Lemma 2.8b) and Lemma 3.8.)

Let $f\colon [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be given functions and let $\mathcal{A}_\mathcal{E}$ be a given set of *admissible controls* which is contained in the set of $\mathcal{E}_t$-adapted processes $u(t)$ such that the problem (1.2) has a strong $\mathcal{G}_t$-adapted solution $X(t) = X^{(u)}(t)$ and for which the integral

$$J^{(u)}(x) = \mathsf{E}^x \left[\int_0^T f(t, X(t), u(t))\, dt + g(X(T)) \right] \tag{1.4}$$

converges. We consider the following problem of *partial observation control in an anticipating environment.*

Problem 1.1. Find $\Phi(x)$ and $u^* \in \mathcal{A}_\mathcal{E}$ such that

$$\Phi(x) = \sup_{u \in \mathcal{A}_\mathcal{E}} J^{(u)}(x) = J^{(u^*)}(x). \tag{1.5}$$

Problems of this type appear in many situations. We give three examples in mathematical finance.

Example 1.2 (Stochastic volatility models). Suppose that we have a market with one risky investment possibility (for instance, a stock) whose price $S_1(t)$ at time t is described by a stochastic differential equation of the form

$$dS_1(t) = S_1(t^-)\left[\mu(t)\,dt + \sigma(t)\,d^-B(t) + \int_{\mathbb{R}} \theta(t,z)\,\widetilde{N}(d^-t,dz)\right], \qquad (1.6)$$

where B and $\widetilde{N}$ are one-dimensional (for simplicity). The coefficient $\sigma(t) = \sigma(t,\omega)$ need not be $\mathcal{F}_t$-adapted in general stochastic volatility models; this coefficient can be influenced by other noises as well. Thus, the σ-algebra $\mathcal{G}_t$ generated by $\{\sigma(s,\cdot); s \le t\}$ can be bigger than $\mathcal{F}_t$. The same argument can be applied to $\theta(t,z) = \theta(t,z,\omega)$ and to $\mu(t) = \mu(t,\omega)$.

Suppose that the market also has a risk-free investment possibility for which the price $S_0(t)$ at time t is described by the equation

$$dS_0(t) = \rho(t)S_0(t)\,dt, \qquad S_0(0) = 1, \qquad (1.7)$$

where $\rho(t) = \rho(t,\omega)$ is another $\mathcal{G}_t$-adapted process. A *portfolio* $\pi(t) = \pi(t,\omega)$ in this market is an $\mathcal{E}_t$-adapted process giving the *fraction* of the total wealth $X(t)$ of an agent invested in the risky asset at time t. The dynamics of the wealth process $X(t) = X^{(\pi)}(t)$ corresponding to the portfolio π is then found as follows:

$$\begin{aligned} dX(t) = X(t^-)\Big[&(\rho(t) + (\mu(t)-\rho(t))\pi(t))\,dt + \pi(t)\sigma(t)\,d^-B(t) \\ &+ \pi(t)\int_{\mathbb{R}} \theta(t,z)\,\widetilde{N}(d^-t,dz)\Big], \qquad X(0) = x > 0. \end{aligned} \qquad (1.8)$$

The requirement that $\pi(t)$ be $\mathcal{E}_t$-adapted models the situation in which the agent has *partial* information only (less than $\mathcal{F}_t$) at her/his disposal when making portfolio decisions. The optimal portfolio problem for the agent is to find $\Phi(x)$ and $\pi^* \in \mathcal{A}_{\mathcal{E}}$ such that

$$\Phi(x) = \sup_{\pi\in\mathcal{A}_{\mathcal{E}}} \mathsf{E}^x[U(X^{(\pi)}(T))] = \mathsf{E}^x[U(X^{(\pi^*)}(T))], \qquad (1.9)$$

where $U\colon \mathbb{R} \to [-\infty,\infty)$ is a given utility function.

Example 1.3 (Insider influenced markets). Let us consider the market (1.6)–(1.7) again. If there are large investors in the market and these investors have *inside* information, this means that they have access to a larger filtration $\mathcal{G}_t \supset \mathcal{F}_t$ when making their decisions. This leads to a price dynamics in which the coefficients $\rho(t)$, $\mu(t)$, $\sigma(t)$, and $\theta(t,z)$ are $\mathcal{G}_t$-measurable and not necessarily $\mathcal{F}_t$-measurable. A partially informed investor in this market faces a problem of the form (1.8)–(1.9) again.

Example 1.4 (Markets with delayed effects from the noise). Suppose that we have a market with no jumps ($\theta = 0$) and with stock prices $S_1(t),\ldots,S_N(t)$ given by

$$d^-S_i(t) = S_i(t)\left[\mu_i(t)\,dt + \sum_{j=1}^{N} \sigma_{ij}(t)\,d^-B_j(t-\delta_i)\right], \qquad 1 \le i \le N. \qquad (1.10)$$

As above, suppose that $B(t) = (B_1(t), \dots, B_N(t))$ is an N-dimensional Brownian motion with filtration $\mathcal{F}_t$. We also assume that $\mu_i(t)$ and $\sigma_{ij}(t)$ are $\mathcal{F}_t$-adapted. However, in this model we admit a *delay* $\delta_i \geq 0$ in the effect of the noise coming from $B(\cdot)$ on the price $S_i(\cdot)$. Moreover, for some of the stocks the effect of the same underlying noise can come later than the effect for others, and hence the numbers δ_i need not be the same.

Integrating (1.10), we obtain

$$\begin{aligned} S_i(t) &= S_i(0) + \int_0^t S_i(s)\mu_i(s)\,ds + \sum_{j=1}^N \int_0^t S_i(s)\sigma_{ij}(s)\,d^-B_j(s-\delta_i) \\ &= S_i(0) + \int_{-\delta_i}^{t-\delta_i} S_i(r+\delta_i)\mu_i(r+\delta_i)\,dr \\ &\quad + \sum_{j=1}^N \int_{-\delta_i}^{t-\delta_i} S_i(r+\delta_i)\sigma_{ij}(r+\delta_i)\,d^-B_j(r). \end{aligned} \tag{1.11}$$

We write

$$\widetilde{S}_i(t) = S_i(t+\delta_i), \qquad -\delta_i \leq t, \quad 1 \leq i \leq N. \tag{1.12}$$

Then (1.11) can be represented as

$$\begin{aligned} \widetilde{S}_i(t) &= S_i(0) + \int_{-\delta_i}^t \widetilde{S}_i(r)\mu_i(r+\delta_i)\,dr + \sum_{j=1}^N \int_{-\delta_i}^t \widetilde{S}_i(r)\sigma_{ij}(r+\delta_i)\,d^-B_j(r) \\ &= \widetilde{S}_i(0) + \int_0^t \widetilde{S}_i(r)\mu_i(r+\delta_i)\,dr + \sum_{j=1}^N \int_0^t \widetilde{S}_i(r)\sigma_{ij}(r+\delta_i)\,d^-B_j(r). \end{aligned} \tag{1.13}$$

Equivalently,

$$d\widetilde{S}_i(t) = \widetilde{S}_i(t)\Big[\widetilde{\mu}_i(t)\,dt + \sum_{j=1}^N \widetilde{\sigma}_{ij}(t)\,d^-B_j(t)\Big], \qquad \widetilde{S}_i(0) = S_i(\delta), \quad 1 \leq i \leq N, \tag{1.14}$$

where $\widetilde{\mu}_i(t) = \mu_i(t+\delta_i)$ and $\widetilde{\sigma}_{ij}(t) = \sigma_{ij}(t+\delta_i)$, $1 \leq i, j \leq N$.

We note that this is a price equation of the same type as that in (1.6) (Example 1.2), where the coefficients $\widetilde{\mu}_i(t)$ and $\widetilde{\sigma}_{ij}(t)$ are adapted to the filtration

$$\mathcal{G}_t := \mathcal{F}_{t+\delta},$$

where

$$\delta = \max(\delta_1, \dots, \delta_N).$$

We can now consider an optimal portfolio problem of the form (1.9) again, where the information available to the agent is modelled by a given filtration $\mathcal{E}_t \subseteq \mathcal{F}_t$.

The purpose of this paper is to give an explicit solution of a problem of the type described in Example 1.2 in the logarithmic utility case, that is, in the case

$$U(x) = \log x, \qquad x > 0. \tag{1.15}$$

For simplicity, we split the discussion into two cases.

(i) The continuous case ($\sigma \neq 0$, $\theta = 0$).

(ii) The pure jump case ($\sigma = 0$, $\theta \neq 0$).

§ 2. The continuous case ($\theta = 0$)

Referring to Examples 1.2 and 1.3, we now study the market $\mathcal{M}(\mathcal{E}, \mathcal{G})$ given by the equations

$$\text{(bond price)} \qquad dS_0(t) = \rho(t) S_0(t)\, dt, \quad S_0(0) = 1, \tag{2.1}$$

$$\text{(stock price)} \qquad dS_1(t) = S_1(t)[\mu(t)\, dt + \sigma(t)\, d^- B(t)], \quad S_1(0) > 0, \tag{2.2}$$

where we assume that $\rho(t)$, $\mu(t)$, and $\sigma(t)$ satisfy the following conditions:

$$\rho(t), \mu(t), \sigma(t) \text{ are } \mathcal{G}_t\text{-adapted (see (1.3))}; \tag{2.3}$$

$$\mathsf{E}\left[\int_0^T \{|\rho(t)| + |\mu(t)| + \sigma^2(t)\}\, dt\right] < \infty; \tag{2.4}$$

$\sigma(t)$ is Malliavin differentiable and the limit $D_{t^+}\sigma(t) = \lim_{s \to t^+} D_s \sigma(t)$ exists for almost all (a.a.) $t \in [0, T]$, where D_s stands for the Malliavin derivative at s (see Definition 2.5 below); (2.5)

the equation (2.2) has a unique $\mathcal{G}_t$-adapted solution $S_1(t)$, $t \in [0, T]$. (2.6)

As above, $\{\mathcal{E}_t\}_{t \in [0,T]}$ and $\{\mathcal{G}_t\}_{t \in [0,T]}$ are given filtrations such that

$$\mathcal{E}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \quad \text{for any} \quad t \in [0, T]. \tag{2.7}$$

Definition 2.1. The set $\mathcal{A}_\mathcal{E}$ of *admissible portfolios* consists of all processes $\pi(t)$ satisfying the following conditions:

$$\pi(t) \quad \text{is } \mathcal{E}_t\text{-adapted}; \tag{2.8}$$

$\pi(t)\sigma(t)$ is *Skorokhod integrable* and *caglad* (that is, left continuous with existing right-hand limit); (2.9)

$$\mathsf{E}\left[\int_0^T |\pi(t) D_{t^+}\sigma(t)|\, dt\right] < \infty; \tag{2.10}$$

$$\mathsf{E}\left[\int_0^T |\mu(t) - \rho(t)| \cdot |\pi(t)|\, dt\right] < \infty. \tag{2.11}$$

Referring to Example 1.2, we study the following partial observation optimal portfolio problem.

Problem 2.2. Find $\Phi(x)$ and $\pi^* \in \mathcal{A}_\mathcal{E}$ such that

$$\Phi(x) = \sup_{\pi \in \mathcal{A}_\mathcal{E}} \mathsf{E}^x[\log X^{(\pi)}(T)] = \mathsf{E}^x[\log X^{(\pi^*)}(T)], \tag{2.12}$$

where the process $X^{(\pi)}(t) = X(t)$ is determined by the condition $X(0) = x > 0$ and the equation

$$dX(t) = X(t)[(\rho(t) + (\mu(t) - \rho(t))\pi(t))\, dt + \pi(t)\sigma(t)\, d^- B(t)]. \tag{2.13}$$

The function $\Phi \leq \infty$ is called the *price function* and π^* (if it exists) is called an *optimal portfolio* for Problem 2.2.

Before solving Problem 2.2, we survey some basic mathematical background for the convenience of the reader. We refer the reader to [9], [10], [11] for more details.

Let λ be Lebesgue measure on $[0,T]$ and let $L^2(\lambda^n)$ be the space of all deterministic functions $f\colon [0,T]^n \to \mathbb{R}$ such that

$$\|f\|_{L^2(\lambda^n)} = \int_{[0,T]^n} f^2(x)\,d\lambda(x) = \int_{[0,T]^n} f^2(x_1,\dots,x_n)\,dx_1\cdots dx_n < \infty.$$

If f is a real function on $[0,T]^n$, we define its *symmetrization* $\widetilde{f}$ by

$$\widetilde{f}(t_1,\dots,t_n) = \frac{1}{n!}\sum_{\alpha} f(t_{\alpha_1},\dots,t_{\alpha_n}),$$

where the sum is taken over all permutations α of $\{1,2,\dots,n\}$. We say that f is *symmetric* if $\widetilde{f} = f$ and denote the set of all symmetric functions in $L^2(\lambda^n)$ by $\widetilde{L}^2(\lambda^n)$. We write

$$\mathcal{S}_n = \{(t_1,\dots,t_n) \in [0,T]^n;\ 0 \leq t_1 \leq \cdots \leq t_n \leq T\}.$$

If $f \in L^2(\mathcal{S}_n)$, we define its *n-fold iterated integral* with respect to $B(\cdot)$ by the formula

$$J_n(f) = \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1,\dots,t_n)\,dB(t_1)\cdots dB(t_n),$$

and if $f \in \widetilde{L}^2(\lambda^n)$, then we set

$$I_n(f) := \int_{[0,T]^n} f(t_1,\dots,t_n)\,dB^{\otimes n}(t) := n!\,J_n(f).$$

We can now formulate the *Wiener-Itô chaos expansion theorem.*

Theorem 2.3. *Every $\mathcal{F}_T$-measurable random variable $F \in L^2(\mathsf{P})$ can be represented in the form*

$$F = \mathsf{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) := \sum_{n=0}^{\infty} I_n(f_n)$$

for a unique sequence of deterministic functions $f_n \in \widetilde{L}^2(\lambda^n)$. Moreover, there is the isometry

$$\mathsf{E}[F^2] = (\mathsf{E}[F])^2 + \sum_{n=1}^{\infty} n!\,\|f_n\|^2_{L^2(\lambda^n)}. \tag{2.14}$$

This expansion is useful for the following definition of Skorokhod integrals and Malliavin derivatives.

Let $\phi(t,\omega)\colon [0,T]\times\Omega \to \mathbb{R}$ be a measurable process such that

$$\mathsf{E}[\phi^2(t,\cdot)] < \infty.$$

We assume that

the function $\phi(t,\,\cdot\,)$ is $\mathcal{F}_T$-measurable for any $t \in [0,T]$.

Let

$$\phi(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\,\cdot\,,t))$$

be the chaos expansion of $\phi(t,\,\cdot\,)$ and let $\widetilde{f}(t_1,\ldots,t_n,t_{n+1})$ be the symmetrization of $f(t_1,\ldots,\,t_n,t)$ with respect to the $n+1$ variables $t_1,\ldots,t_n,t_{n+1}=t$.

Definition 2.4. Suppose that

$$\sum_{n=0}^{\infty}(n+1)!\,\|\widetilde{f}_n\|^2_{L^2(\lambda^{n+1})} < \infty. \tag{2.15}$$

Then we define the *Skorokhod integral* of ϕ with respect to $B(\,\cdot\,)$ by the formula

$$\int_0^T \phi(t,\omega)\,\delta B(t) = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n).$$

We note that, by (2.14),

$$\mathsf{E}\Big[\Big(\int_0^T \phi(t,\omega)\,\delta B(t)\Big)^2\Big] = \sum_{n=0}^{\infty}(n+1)!\,\|\widetilde{f}_n\|^2_{L^2(\lambda^{n+1})} < \infty, \tag{2.16}$$

and hence the Skorokhod integral belongs to $L^2(\mathsf{P})$ (it is defined). Moreover,

$$\mathsf{E}\Big[\int_0^T \phi(t,\omega)\,\delta B(t)\Big] = 0. \tag{2.17}$$

One can show that the Skorokhod integral is an extension of the Itô integral, in the sense that if $\phi(t,\omega)$ is $\mathcal{F}_t$-adapted and Skorokhod integrable, then

$$\int_0^T \phi(t,\omega)\,\delta B(t) = \int_0^T \phi(t,\omega)\,dB(t).$$

Definition 2.5. Let $F \in L^2(\mathsf{P})$ be $\mathcal{F}_T$-measurable with the expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n), \qquad f_n \in \widetilde{L}^2(\lambda^n).$$

We say that F is *Malliavin differentiable* and write $F \in \mathbb{D}_{1,2}$ if

$$\|F\|^2_{\mathbb{D}_{1,2}} := (\mathsf{E}[F])^2 + \sum_{n=1}^{\infty} n\,n!\,\|f_n\|^2_{L^2(\lambda^n)} < \infty. \tag{2.18}$$

If $F \in \mathbb{D}_{1,2}$, we define the *Malliavin derivative* of F at $t \in [0,T]$ by

$$D_t F = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(\,\cdot\,,t)),$$

where $I_{n-1}(f_n(\,\cdot\,,t))$ is the $(n-1)$-fold iterated integral of $f(t_1,\dots,t_{n-1},t)$ as a function of the first $n-1$ variables $t_1,\dots,t_{n-1}$.

Since

$$\mathsf{E}\Big[\int_0^T (D_t F)^2\,dt\Big] = \sum_{n=1}^{\infty} n\,n!\,\|f_n\|^2_{L^2(\lambda^n)}, \tag{2.19}$$

we see that if (2.18) holds, then $D_t F$ exists for a.a. $t \in [0,T]$.

The Malliavin derivative D_t satisfies the usual chain rule. For example, the following assertion holds.

Lemma 2.6. *Let $f \in C^1(\mathbb{R})$ be a function with bounded derivatives and let $F \in \mathbb{D}_{1,2}$. Then $f(F) \in \mathbb{D}_{1,2}$ and*

$$D_t(f(F)) = f'(F)\cdot D_t F. \tag{2.20}$$

We now recall the definition of *forward integrals* with respect to $B(\,\cdot\,)$. We refer to [10], [13], and [14] for more information about these integrals.

Definition 2.7. Let $\phi\colon [0,T]\times\Omega \to \mathbb{R}$ be a measurable process which is not necessarily $\mathcal{F}_t$-adapted. Then we define the *forward integral* of ϕ with respect to $B(\,\cdot\,)$ by the formula

$$\int_0^T \phi(t)d^-B(t) = \lim_{\varepsilon\to 0}\int_0^T \phi(t)\frac{B(t+\varepsilon)-B(t)}{\varepsilon}\,dt \tag{2.21}$$

if the limit exists in probability.

By using the stochastic Fubini theorem, we can obtain the following more suggestive description of the forward integral.

Lemma 2.8. a) *Let a function $\phi\colon [0,T]\times\Omega\to\mathbb{R}$ be forward integrable and caglad. Then*

$$\int_0^T \phi(t)\,d^-B(t) = \lim_{\Delta t\to 0}\sum_{j=0}^{N-1}\phi(t_j)(B(t_{j+1})-B(t_j)) \tag{2.22}$$

(*the limit in probability*), *where $0 = t_0 < t_1 < \dots < t_N = T$ is a partition of $[0,T]$ and $\Delta t = t_{j+1}-t_j$ for all $j = 0,\dots,N-1$.*

b) *Suppose in addition that $B(t)$ is a semimartingale with respect to $\mathcal{G}_t$ and that $\phi(t)$ is $\mathcal{G}_t$-measurable for all $t\in[0,T]$. Then*

$$\int_0^T \phi(t)\,d^-B(t) = \int_0^T \phi(t)\,dB(t),$$

where the integral on the right-hand side is the usual (semimartingale) Itô integral.

Proof. This well-known result follows by the same argument as in [1], (2.2), and Corollary 2.5.

A proof of the following basic relation between the forward integral and the Skorokhod integral can be found in [1], Lemma 2.2.

Lemma 2.9. *Suppose that* $\phi\colon [0,T]\times\Omega\to\mathbb{R}$ *is Skorokhod integrable and caglad. Moreover, assume that the limit*

$$D_{t^+}\phi(t) := \lim_{s\to t^+} D_s\phi(t)$$

exists for a.a. $t\in[0,T]$ *and*

$$\int_0^T |D_{t^+}\phi(t)|\,dt < \infty.$$

Then the forward integral of ϕ *exists and*

$$\int_0^T \phi(t)\,d^-B(t) = \int_0^T \phi(t)\,\delta B(t) + \int_0^T D_{t^+}\phi(t)\,dt. \tag{2.23}$$

Since the Skorokhod integrals have zero expectation (see (2.17)), we can derive the following result from Lemma 2.9.

Corollary 2.10. *Let* ϕ *be as in Lemma* 2.9. *Then*

$$\mathsf{E}\Big[\int_0^T \phi(t)\,d^-B(t)\Big] = \mathsf{E}\Big[\int_0^T D_{t^+}\phi(t)\,dt\Big] \tag{2.24}$$

if the expectations exist.

We also need the following Itô formula for forward integrals.

Theorem 2.11 [14]. *Let* $X(t)$ *be a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \alpha(s)\,ds + \int_0^t \gamma(s)\,d^-B(s).$$

Let $f\in C^{1,2}(\mathbb{R}^2)$, *and let*

$$Y(t) = f(t, X(t)).$$

Then

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t \frac{\partial f}{\partial s}(s,X(s))\,ds + \int_0^t \frac{\partial f}{\partial x}(s,X(s))\,d^-X(s) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X(s))\gamma^2(s)\,ds, \end{aligned} \tag{2.25}$$

where

$$d^-X(s) = \alpha(s)\,ds + \gamma(s)\,d^-B(s).$$

We now proceed to the solution of Problem 2.2.

Applying Theorem 2.11 to the forward equation (2.13), we obtain a (unique) solution

$$\begin{aligned} X^{(\pi)}(T) = x\exp\Big\{&\int_0^T \Big[\rho(t) + (\mu(t)-\rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\Big]\,dt \\ &+ \int_0^T \pi(t)\sigma(t)\,d^-B(t)\Big\}. \end{aligned} \tag{2.26}$$

Hence, using (2.24), we see that

$$\begin{aligned}
&\mathsf{E}[\log X^{(\pi)}(T)] - \log x \\
&\quad = \mathsf{E}\Big[\int_0^T \Big\{\rho(t) + (\mu(t)-\rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t)\Big\}\,dt + \int_0^T \pi(t)\sigma(t)\,d^-B(t)\Big] \\
&\quad = \mathsf{E}\Big[\int_0^T \Big\{\rho(t) + (\mu(t)-\rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\sigma^2(t) + D_{t^+}(\pi(t)\sigma(t))\Big\}\,dt\Big].
\end{aligned} \tag{2.27}$$

Since $\pi(t)$ is $\mathcal{E}_t$-measurable and $\mathcal{E}_t \subseteq \mathcal{F}_t$, we have

$$D_s\pi(t) = 0 \quad \text{for any} \quad s > t. \tag{2.28}$$

Therefore, by the chain rule for the Malliavin derivative,

$$D_{t^+}(\pi(t)\sigma(t)) = \sigma(t)D_{t^+}\pi(t) + \pi(t)D_{t^+}\sigma(t) = \pi(t)D_{t^+}\sigma(t).$$

Substituting this formula into (2.27), we obtain

$$\mathsf{E}[\log X^{(\pi)}(T)] - \log x = \mathsf{E}\Big[\int_0^T \Big\{\rho(s) + \beta(s)\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\Big\}\,ds\Big], \tag{2.29}$$

where

$$\beta(s) := \mu(s) - \rho(s) + D_{s^+}\sigma(s). \tag{2.30}$$

Equation (2.29) can also be written as

$$\mathsf{E}[\log X^{(\pi)}(T)] - \log x = \mathsf{E}\Big[\int_0^T \Big\{\widehat{\rho}(s) + \widehat{\beta}(s)\pi(s) - \frac{1}{2}\widehat{\sigma}^2(s)\pi^2(s)\Big\}\,ds\Big], \tag{2.31}$$

where

$$\widehat{\rho}(s) = \mathsf{E}[\rho(s)|\mathcal{E}_s]$$

and similarly for $\widehat{\sigma}$, $\widehat{\beta}$, $\widehat{\sigma}^2$. We can now maximise pointwise for each s with respect to π under the integral sign. We obtain

$$\pi^*\widehat{\sigma}^2(s) = \widehat{\beta}(s).$$

Summarizing the above we get the following result:

Theorem 2.12. a) *Suppose that* $\sigma(t) \neq 0$ *for a.a.* (t,ω) *and*

$$\mathsf{E}\Big[\int_0^T \frac{\beta^2(s)}{\sigma^2(s)}\,ds\Big] < \infty, \tag{2.32}$$

where $\beta(s)$ *is defined in* (2.30). *Then the price function* Φ *of Problem* 2.2 *is*

$$\begin{aligned}
\Phi(x) &= \log x + \mathsf{E}\Big[\int_0^T \Big\{\rho(s) + \frac{\beta(s)\mathsf{E}[\beta(s)\mid\mathcal{E}_s]}{\mathsf{E}[\sigma^2(s)\mid\mathcal{E}_s]} - \frac{\sigma^2(s)}{2}\Big(\frac{\mathsf{E}[\beta(s)\mid\mathcal{E}_s]}{\mathsf{E}[\sigma^2(s)\mid\mathcal{E}_s]}\Big)^2\Big\}\,ds\Big] \\
&= \log x + \mathsf{E}\Big[\int_0^T \Big\{\rho(s) + \frac{\widehat{\beta}^2(s)}{2\widehat{\sigma}^2(s)}\Big\}\,ds\Big] < \infty.
\end{aligned} \tag{2.33}$$

b) *Suppose that* $\sigma(t) \neq 0$ *for a.a.* (t,ω) *and that*

$$\widehat{\pi}(s) := \frac{\mathsf{E}[\beta(s)\mid\mathcal{E}_s]}{\mathsf{E}[\sigma^2(s)\mid\mathcal{E}_s]} \in \mathcal{A}_{\mathcal{E}}. \tag{2.34}$$

Then $\pi^*(s) := \widehat{\pi}(s)$ *is an optimal control for Problem* 2.2.

c) *Suppose that there is an optimal portfolio* $\pi^* \in \mathcal{A}_{\mathcal{E}}$ *for Problem* 2.2. *Then*

$$\pi^*(s)\mathsf{E}[\sigma^2(s)\mid\mathcal{E}_s] = \mathsf{E}[\beta(s)\mid\mathcal{E}_s]. \tag{2.35}$$

Corollary 2.13. a) *Suppose that*

$$\sigma(s) \text{ is } \mathcal{F}_s\text{-measurable for all } s \in [0,T]. \tag{2.36}$$

Then

$$D_{s+}\sigma(s) = 0 \quad \text{for any} \quad s \in [0,T], \tag{2.37}$$

and hence

$$\beta(s) = \mu(s) - \rho(s). \tag{2.38}$$

Under the assumptions of Theorem 2.12, *this gives*

$$\pi^*(s) = \frac{\mathsf{E}[\mu(s) - \rho(s) \mid \mathcal{E}_s]}{\sigma^2(s)} \tag{2.39}$$

with the corresponding price function

$$\Phi(x) = \log x + \mathsf{E}\Big[\int_0^T \Big\{\rho(s) + \frac{\beta(s)\mathsf{E}[\beta(s) \mid \mathcal{E}_s]}{\sigma^2(s)} - \frac{1}{2}\frac{(\mathsf{E}[\beta(s) \mid \mathcal{E}_s])^2}{\sigma^2(s)}\Big\}\, ds\Big]. \tag{2.40}$$

b) *In particular, if*

$$\mathcal{E}_t = \mathcal{F}_t = \mathcal{G}_t \quad \text{for any} \quad t \in [0,T], \tag{2.41}$$

then there follows the well-known result

$$\pi^*(s) = \frac{\mu(s) - \rho(s)}{\sigma^2(s)} \tag{2.42}$$

and

$$\Phi(x) = \log x + \mathsf{E}\Big[\int_0^T \Big\{\rho(s) + \frac{1}{2}\Big(\frac{\mu(s) - \rho(s)}{\sigma(s)}\Big)^2\Big\}\, ds\Big] \tag{2.43}$$

provided that

$$\mathsf{E}\Big[\int_0^T \Big(\frac{\mu(s) - \rho(s)}{\sigma(s)}\Big)^2 ds\Big] < \infty. \tag{2.44}$$

Example 2.14 (Delayed noise effect). Suppose that $\mathcal{E}_t = \mathcal{F}_t$ and $\mathcal{G}_t = \mathcal{F}_{t+\delta}$ for some $\delta > 0$. Let $\mu(s)$ and $\rho(s)$ be bounded and $\mathcal{F}_{s+\delta}$-measurable. We take

$$\sigma(s) = \exp(B(s+\delta)), \qquad s \in [0,T].$$

(See Example 1.4.) Then $D_{s+}\sigma(s) = \sigma(s)$, and hence the corresponding optimal portfolio is, by Theorem 2.12,

$$\pi^*_\delta(s) = \frac{\mathsf{E}[\mu(s) - \rho(s) + \sigma(s) \mid \mathcal{F}_s]}{\mathsf{E}[\sigma^2(s) \mid \mathcal{F}_s]} \quad \text{for} \quad \delta > 0. \tag{2.45}$$

On the other hand, if $\mathcal{E}_t = \mathcal{F}_t = \mathcal{G}_t$ (corresponding to $\delta = 0$), then $D_{s^+}\sigma(s) = 0$, and we know by Corollary 2.13 that the optimal portfolio is

$$\pi_0^*(s) = \frac{\mu(s) - \rho(s)}{\sigma^2(s)} \,. \tag{2.46}$$

Comparing (2.45) and (2.46), we see that, perhaps surprisingly,

$$\lim_{\delta \to 0^+} \pi_\delta^*(s) \neq \pi_0^*(s). \tag{2.47}$$

Similarly, if the corresponding price functions are denoted by $\Phi_\delta(s)$ and $\Phi_0(x)$, respectively, then

$$\lim_{\delta \to 0^+} \Phi_\delta(x) = \log x + \mathsf{E}\Big[\int_0^T \Big\{\rho(s) + \frac{1}{2}\Big(\frac{\mu(s) - \rho(s)}{\sigma(s)} + 1\Big)^2\Big\}\, ds\Big] \neq \Phi_0(x). \tag{2.48}$$

We conclude that any positive delay δ in the information, no matter how small, has a substantial effect on the optimal control and the price function.

§ 3. The pure jump case ($\sigma = 0$)

Referring to Example 1.2, we now consider the market $\mathcal{N}(\mathcal{E}, \mathcal{G})$ given by the conditions

$$\text{(bond price)}\;\; dS_0(t) = \rho(t) S_0(t)\, dt, \;\; S_0(0) = 1, \tag{3.1}$$

$$\text{(stock price)}\;\; dS_1(t) = S_1(t^-)\Big[\mu(t)\, dt + \int_{\mathbb{R}} \theta(t,z)\, \widetilde{N}(d^-t, dz)\Big], \;\; S_1(0) > 0, \tag{3.2}$$

where we assume that $\rho(t)$, $\mu(t)$, and $\theta(t,z)$ satisfy the following conditions:

$$\rho(t),\ \mu(t), \text{ and } \theta(t,z) \text{ are } \mathcal{G}_t\text{-measurable for any } t \in [0,T] \text{ and } z \in \mathbb{R}; \tag{3.3}$$

$\theta(t,z)$ is bounded and Malliavin differentiable and the limit
$D_{t^+,z}\theta(t,z) := \lim_{s \to t^+} D_{s,z}\sigma(t,z)$ exists for a.a. t, z and is bounded,
where $D_{s,z}$ is the Malliavin derivative at s, z (see Definition 3.5); (3.4)

$$\mathsf{E}\Big[\int_0^T \Big\{|\rho(s)| + |\mu(s)| + \int_{\mathbb{R}} (|\theta(s,z)| + |D_{s^+,z}\theta(s,z)|)\, \nu(dz)\Big\}\, ds\Big] < \infty,$$

where ν is the Lévy measure of $\eta(\,\cdot\,)$, and therefore

$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)\, dt; \tag{3.5}$$

(3.2) has a unique $\mathcal{G}_t$-adapted solution $S_1(t)$, $t \in [0,T]$. (3.6)

As above, $\{\mathcal{E}_t\}_{t\in[0,T]}$ and $\{\mathcal{G}_t\}_{t\in[0,T]}$ are given filtrations such that

$$\mathcal{E}_t \subseteq \mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F} \quad \text{for any} \quad t \in [0,T]. \tag{3.7}$$

Definition 3.1. The set $\mathcal{A}_{\mathcal{E}}$ of admissible portfolios consists of all processes $\pi(t)$ satisfying the following conditions:

$$\pi(t) \text{ is } \mathcal{E}_t\text{-adapted}; \tag{3.8}$$

$$\begin{aligned}&\pi(t)\theta(t,z) \text{ is Skorokhod integrable with respect to } \widetilde{N}(\,\cdot\,,\cdot\,)\\ &\text{(see Definition 3.4) and caglad;}\end{aligned} \tag{3.9}$$

$$\begin{aligned}&\pi(t)\theta(t,z) > -1+\varepsilon \text{ for a.a. } t,z \text{ (where } \varepsilon>0 \text{ may depend on } \pi)\\ &\text{and } \mathsf{E}\Big[\int_0^T\int_{\mathbb{R}} |\log(1+\pi(s)\theta(s,z))|\,\nu(dz)\,dt\Big] < \infty;\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\pi(t)(\theta(t,z)+D_{t^+,z}\theta(t,z)) > -1+\varepsilon \text{ for a.a. } t,\ z\\ &\text{(where } \varepsilon>0 \text{ can depend on } \pi\text{), and}\\ &\mathsf{E}\Big[\int_0^T\int_{\mathbb{R}} |\log(1+\pi(t)(\theta(t,z)+D_{t^+,z}\theta(t,z)))|\,\nu(dz)\,dt\Big] < \infty.\end{aligned} \tag{3.11}$$

Problem 3.2. Find $\Phi(x)$ and $\pi^* \in \mathcal{A}_{\mathcal{E}}$ such that

$$\Phi(x) = \sup_{\pi\in\mathcal{A}_{\mathcal{E}}} \mathsf{E}^x[\log X^{(\pi)}(T)] = \mathsf{E}^x[\log X^{(\pi^*)}(T)], \tag{3.12}$$

where $X^{(\pi)}(t) = X(t)$ is determined by $X(0) = x > 0$ and the equation

$$dX(t) = X(t^-)\Big[(\rho(t)+(\mu(t)-\rho(t))\pi(t))\,dt + \pi(t)\int_{\mathbb{R}} \theta(t,z)\,\widetilde{N}(d^-t,dz)\Big]. \tag{3.13}$$

The function $\Phi \le \infty$ is called the *price function* and π^* (if it exists) is called an *optimal portfolio* for Problem 3.2.

Before studying Problem 3.2 more closely, we survey some mathematical background concerning the Malliavin calculus and anticipating calculus for jump diffusions. For the proofs and the details, we refer to [3]. See also [4], where a similar approach was used to find an optimal portfolio for an investor having inside information.

We first recall the chaos expansion in terms of iterated integrals with respect to the compensated Poisson random measure $\widetilde{N}(dt,dz)$, introduced in [5]. (See also [8].)

Let λ be the Lebesgue measure on $[0,T]$ and let $L^2((\lambda\times\nu)^n)$ be the space of all deterministic functions $f\colon ([0,T]\times\mathbb{R})^n \to \mathbb{R}$ such that

$$\|f\|^2_{L^2((\lambda\times\nu)^n)} := \int_{([0,T]\times\mathbb{R})^n} f^2(t_1,z_1,\dots,t_n,z_n)\,dt_1\,\nu(dz_1)\cdots dt_n\,\nu(dz_n) < \infty.$$

If f is a real function on $([0,T]\times\mathbb{R})^n$, we define its *symmetrization* $\widetilde{f}$ with respect to the variables $(t_1,z_1),\dots,(t_n,z_n)$ by

$$\widetilde{f}(t_1,z_1,\dots,t_n,z_n) = \frac{1}{n!}\sum_\sigma f(t_{\sigma_1},z_{\sigma_1},\dots,t_{\sigma_n},z_{\sigma_n}),$$

where the sum is taken over all permutations σ of $\{1,\dots,n\}$. We say that f is *symmetric* if $\widetilde{f} = f$. Let $\widetilde{L}^2((\lambda\times\nu)^n)$ be the set of all symmetric functions in $L^2((\lambda\times\nu)^n)$. We write

$$G_n = \{(t_1,z_1,\dots,t_n,z_n);\ 0\le t_1\le\cdots\le t_n\le T \text{ and } z_i\in\mathbb{R},\ i=1,\dots,n\}.$$

Let $L^2(G_n)$ be the set of functions $g\colon G_n\to\mathbb{R}$ such that

$$\|g\|^2_{L^2(G_n)} := \int_{G_n} g^2(t_1,z_1,\dots,t_n,z_n)\,dt_1\,\nu(dz_1)\cdots dt_n\,\nu(dz_n) < \infty.$$

We note that

$$\|f\|^2_{L^2((\lambda\times\nu)^n)} = n!\,\|f\|^2_{L^2(G_n)}, \qquad f\in\widetilde{L}^2((\lambda\times\nu)^n).$$

For $f\in L^2(G_n)$ we define its *n-fold iterated integral* with respect to $\widetilde{N}(\,\cdot\,,\cdot\,)$ by

$$J_n(f) = \int_0^T\int_{\mathbb{R}}\cdots\int_0^{t_2}\int_{\mathbb{R}} f(t_1,z_1,\dots,t_n,z_n)\widetilde{N}(dt_1,dz_1)\cdots\widetilde{N}(dt_n,dz_n)$$

and for $f\in\widetilde{L}^2((\lambda\times\nu)^n)$ we set

$$I_n(f) := \int_{([0,T]\times\mathbb{R})^n} f(t_1,z_1,\dots,t_n,z_n)\,\widetilde{N}^{\otimes n}(dt,dz) := n!\,J_n(f).$$

Then we have the following chaos expansion theorem.

Theorem 3.3 ([5], [8]). *Every $\mathcal{F}_T$-measurable random variable $F\in L^2(\mathsf{P})$ can be represented as*

$$F = \mathsf{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) := \sum_{n=0}^{\infty} I_n(f_n) \tag{3.14}$$

for a unique sequence of deterministic functions $f_n\in\widetilde{L}^2((\lambda\times\nu)^n)$. Moreover, there is the isometry

$$\mathsf{E}[F^2] = (\mathsf{E}[F])^2 + \sum_{n=1}^{\infty} n!\,\|f_n\|^2_{L^2((\lambda\times\nu)^n)}. \tag{3.15}$$

Using this expansion theorem, we can now define Skorokhod integration and Malliavin differentiation as follows.

Definition 3.4. Let $\phi(t,z,\omega)\colon [0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ be a random field such that

$$\mathsf{E}[\phi^2(t,z)] < \infty.$$

Suppose that

$$\phi(t,z,\,\cdot\,) \text{ is } \mathcal{F}_T\text{-measurable for any } (t,z)\in[0,T]\times\mathbb{R}.$$

Let

$$\phi(t,z)=\sum_{n=0}^{\infty} I_n(f_n(\,\cdot\,,t,z))$$

be the chaos expansion of $\phi(t,z,\cdot)$, as given by Theorem 3.3. Let $\widetilde{f}_n(t_1,z_1,\dots,t_n,z_n,t_{n+1},z_{n+1})$ be the symmetrization of $f_n(t_1,z_1,\dots,t_n,z_n,t,z)$ regarded as a function of the $n+1$ variables $(t_1,z_1),\dots,(t_n,z_n),(t_{n+1},z_{n+1})=(t,z)$. Suppose that

$$\sum_{n=0}^{\infty}(n+1)!\,\|\widetilde{f}_n\|^2_{L^2((\lambda\times\nu)^n)}<\infty. \tag{3.16}$$

Then the *Skorokhod integral* of ϕ with respect to $\widetilde{N}$ is defined by

$$\int_0^T\int_{\mathbb{R}}\phi(t,z)\,\widetilde{N}(\delta t,dz)=\sum_{n=0}^{\infty}I_{n+1}(\widetilde{f}_n). \tag{3.17}$$

We note that

$$\mathsf{E}\Big[\Big(\int_0^T\int_{\mathbb{R}}\phi(t,z)\,\widetilde{N}(\delta t,dz)\Big)^2\Big]=\sum_{n=0}^{\infty}(n+1)!\,\|\widetilde{f}_n\|^2_{L^2((\lambda\times\nu)^n)}, \tag{3.18}$$

and thus the Skorokhod integral of ϕ belongs to $L^2(\mathsf{P})$ if it exists. Moreover,

$$\mathsf{E}\Big[\int_0^T\int_{\mathbb{R}}\phi(t,z)\,\widetilde{N}(\delta t,dz)\Big]=0. \tag{3.19}$$

The Skorokhod integral with respect to a Poisson random measure was first constructed by Kabanov ([6], [7]). It is an extension of the Itô integral in the sense that if $\phi(t,z)$ is assumed to be $\mathcal{F}_t$-measurable for any $(t,z)\in[0,T]\times\mathbb{R}$, then the two integrals coincide:

$$\int_0^T\int_{\mathbb{R}}\phi(t,z)\,\widetilde{N}(\delta t,dz)=\int_0^T\int_{\mathbb{R}}\phi(t,z)\,\widetilde{N}(dt,dz). \tag{3.20}$$

(See also [3], Proposition 3.2.)

Definition 3.5. Let $F\in L^2(\mathsf{P})$ be $\mathcal{F}_T$-measurable and have the expansion

$$F=\sum_{n=0}^{\infty}I_n(f_n),\qquad f_n\in\widetilde{L}^2((\lambda\times\nu)^n).$$

Let $F\in\mathbb{D}_{1,2}$, that is,

$$\|F\|^2_{\mathbb{D}_{1,2}}:=(\mathsf{E}[F])^2+\sum_{n=1}^{\infty}n\,n!\,\|f_n\|^2_{L^2((\lambda\times\nu)^n)}<\infty. \tag{3.21}$$

Then we define the *Malliavin derivative* (or the *stochastic derivative*) of F at $(t,z) \in [0,T] \times \mathbb{R}$ by

$$D_{t,z}F := \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(\,\cdot\,,t,z)), \tag{3.22}$$

where the expression $I_{n-1}(f_n(\,\cdot\,,t,z))$ means that we consider the $(n-1)$-fold iterated integral with respect to the first $n-1$ variable pairs $(t_1,z_1),\dots,(t_{n-1},z_{n-1})$ and let $(t_n,z_n) = (t,z)$.

Using the isometry

$$\mathsf{E}\left[\int_0^T \int_{\mathbb{R}} (D_{t,z}F)^2\, \nu(dz)\, dt\right] = \sum_{n=1}^{\infty} n\, n!\, \|f_n\|^2_{L^2((\lambda\times\nu)^n)} < \infty, \tag{3.23}$$

we see that if (3.22) holds, then $D_{t,z}F$ exists for a.a. $(t,z) \in [0,T]\times\mathbb{R}$ with respect to $\lambda \times \nu$.

In the pure jump case, the Malliavin derivative $D = D_{t,z}$ is a *difference* operator, in the sense that it satisfies the product rule

$$D(F\cdot G) = F\cdot DG + G\cdot DF + DF\cdot DG \tag{3.24}$$

when both F and G are Malliavin differentiable random variables ([3], Lemma 3.9, and [8]).

This implies the following result.

Lemma 3.6. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be continuous and let F be a Malliavin differentiable random variable. Then for $D = D_{t,z}$*

$$D(f(F)) = f(F+DF) - f(F). \tag{3.25}$$

Proof. By (3.24) we see that

$$D(F^2) = 2F\cdot DF + DF\cdot DF = (F+DF)^2 - F^2,$$

and by induction

$$D(F^n) = (F+DF)^n - F^n.$$

Hence, the formula (3.25) holds for any polynomial f, and hence for any compactly supported continuous function f (by the Weierstrass approximation theorem). The result then follows by a limit argument which uses the closedness of $D_{t,z}$.

We now turn to the definition of the *forward integral* with respect to $\widetilde{N}(\,\cdot\,,\cdot\,)$. (Compare with Definition 2.7.)

Definition 3.7 [3]. The *forward integral* of a random field $\phi(t,z) = \phi(t,z,\omega)$ with respect to $\widetilde{N}(\cdot,\cdot)$ is defined by

$$\int_0^T \int_{\mathbb{R}} \phi(t,z,\omega)\, \widetilde{N}(d^-t,dz) = \lim_{m\to\infty} \int_0^T \int_{K_m} \phi(t,z)\, \widetilde{N}(dt,dz) \tag{3.26}$$

if the limit exists in probability. Here $\{K_m\}_{m=1}^{\infty}$ is an increasing sequence of compact sets in $\mathbb{R}\setminus\{0\}$ such that

$$\mathbb{R}\setminus\{0\} = \bigcup_{m=1}^{\infty} K_m \quad \text{and} \quad \nu(K_m) < \infty \quad \text{for any } m. \tag{3.27}$$

Just as in the continuous case (Lemma 2.8), we have the following assertion.

Lemma 3.8. *Suppose that $t \to \phi(t,z,\omega)$ is caglad for a.a. z,ω with respect to $\nu \times P$ and that $\phi(t,z,\cdot)$ is $\mathcal{G}_t$-measurable for any $t \in [0,T]$ and a.a. z with respect to ν. Moreover, assume that $\eta(t)$ is a semimartingale with respect to $\mathcal{G}_t$. If ϕ is forward integrable with respect to $\widetilde{N}$, then*

$$\int_0^T \int_{\mathbb{R}} \phi(t,z,\omega)\, \widetilde{N}(d^-t,dz) = \int_0^T \int_{\mathbb{R}} \phi(t,z,\omega)\, \widetilde{N}(dt,dz), \tag{3.28}$$

where the integral on the right-hand side is the ordinary semimartingale integral.

Here is a relation between forward integrals and Skorokhod integrals (compare with Lemma 2.9).

Lemma 3.9 ([3], Lemma 4.3). *If the forward integral of ϕ exists in $L^2(\mathsf{P})$, then*

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \phi(t,z)\, \widetilde{N}(d^-t,dz) &= \int_0^T \int_{\mathbb{R}} D_{t^+,z}\phi(t,z)\,\nu(dz)\,dt \\ &\quad + \int_0^T \int_{\mathbb{R}} (\phi(t,z) + D_{t^+,z}\phi(t,z))\, \widetilde{N}(\delta t,dz)\end{aligned} \tag{3.29}$$

provided that the limit

$$D_{t^+,z}\phi(t,z) = \lim_{s\to t^+} D_{s,z}\phi(t,z)$$

exists and is integrable with respect to $\lambda \times \nu$.

Lemma 3.10. *Let ϕ be as in Lemma 3.9. Then*

$$\mathsf{E}\Big[\int_0^T \int_{\mathbb{R}} \phi(t,z)\, \widetilde{N}(d^-t,dz)\Big] = \mathsf{E}\Big[\int_0^T \int_{\mathbb{R}} D_{t^+,z}\phi(t,z)\,\nu(dz)\,dt\Big] \tag{3.30}$$

if the integrals exist.

Finally, we state an Itô formula for forward integrals with respect to $\widetilde{N}(\cdot,\cdot)$ (compare with Theorem 2.11).

Theorem 3.11 [3]. *Let $X(t)$ be a process of the form*

$$X(t) = x + \int_0^t \alpha(s)\,ds + \int_0^t \int_{\mathbb{R}} \theta(s,z)\, \widetilde{N}(d^-s,dz) \tag{3.31}$$

and let $f \in C^2(\mathbb{R})$. Then

$$\begin{aligned} f(X(t)) &= f(x) + \int_0^t f'(X(s))\alpha(s)\,ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \{f(X(s^-)+\theta(s,z)) - f(X(s^-)) - f'(X(s^-))\theta(s,z)\}\,\nu(dz)\,dt \\ &\quad + \int_0^t \int_{\mathbb{R}} \{f(X(s^-)+\theta(s,z)) - f(X(s^-))\}\, \widetilde{N}(d^-s,dz) \end{aligned} \tag{3.32}$$

provided that at least one of the integrals converges.

We now have the necessary mathematical machinery to solve Problem 3.2. First, applying the Itô formula for forward integrals (Theorem 3.11), we see that the solution of the equation (3.13) is given by

$$\begin{aligned} X(t) = x \exp\Bigg[\int_0^t \Big\{\rho(s) + (\mu(s) - \rho(s))\pi(s) \\ + \int_{\mathbb{R}} [\log(1 + \pi(s)\theta(s,z)) - \pi(s)\theta(s,z)]\,\nu(dz)\Big\}\,ds \\ + \int_0^t \int_{\mathbb{R}} \log(1 + \pi(s)\theta(s,z))\,\widetilde{N}(d^- s, dz)\Bigg]. \end{aligned} \tag{3.33}$$

(See, for instance, Example 1.2.2 in [12].) Hence, using Lemma 3.10, we obtain

$$\begin{aligned} \mathsf{E}\Big[\log \frac{X(T)}{x}\Big] &= \mathsf{E}\Bigg[\int_0^T \Big\{\rho(s) + (\mu(s) - \rho(s))\pi(s) \\ &\quad + \int_{\mathbb{R}} [\log(1 + \pi(s)\theta(s,z)) - \pi(s)\theta(s,z)]\,\nu(dz)\Big\}\,ds \\ &\quad + \int_0^T \int_{\mathbb{R}} \log(1 + \pi(s)\theta(s,z))\,\widetilde{N}(d^- s, dz)\Bigg] \qquad (3.34) \\ &= \mathsf{E}\Bigg[\int_0^T \Big\{\rho(s) + (\mu(s) - \rho(s))\pi(s) \\ &\quad + \int_{\mathbb{R}} [\log(1 + \pi(s)\theta(s,z)) - \pi(s)\theta(s,z) \\ &\quad + D_{s^+,z} \log(1 + \pi(s)\theta(s,z))]\,\nu(dz)\Big\}\,ds\Bigg] =: F(\pi). \qquad (3.35) \end{aligned}$$

By Lemma 3.6,

$$\begin{aligned} &D_{s^+,z} \log(1 + \pi(s)\theta(s,z)) \\ &\quad = \log\big(1 + \pi(s)\theta(s,z) + D_{s^+,z}(\pi(s)\theta(s,z))\big) - \log\big(1 + \pi(s)\theta(s,z)\big) \\ &\quad = \log\big(1 + \pi(s)(\theta(s,z) + D_{s^+,z}\theta(s,z))\big) - \log\big(1 + \pi(s)\theta(s,z)\big) \\ &\quad = \log\Big(1 + \frac{\pi(s) D_{s^+,z}\theta(s,z)}{1 + \pi(s)\theta(s,z)}\Big). \end{aligned} \tag{3.36}$$

By substituting this expression into (3.35), we see that

$$\begin{aligned} F(\pi) = \mathsf{E}\Bigg[\int_0^T \Big\{\rho(s) + (\mu(s) - \rho(s))\pi(s) \\ + \int_{\mathbb{R}} [\log(1 + \pi(s)(\theta(s,z) + D_{s^+,z}\theta(s,z))) - \pi(s)\theta(s,z)]\,\nu(ds)\Big\}\,ds\Bigg]. \end{aligned} \tag{3.37}$$

We want to maximize the function

$$\pi \to F(\pi), \qquad \pi \in \mathcal{A}_{\mathcal{E}}.$$

Suppose that an optimal portfolio $\pi^* \in \mathcal{A}_{\mathcal{E}}$ exists. Then for all bounded $\eta \in \mathcal{A}_{\mathcal{E}}$ there exists a $\delta > 0$ such that $\pi^* + r\eta \in \mathcal{A}_{\mathcal{E}}$ for $r \in (-\delta, \delta)$ and the function

$$f(r) := F(\pi^* + r\eta), \qquad r \in (-\delta, \delta),$$

is maximal for $r = 0$. Therefore,

$$\begin{aligned} 0 = f'(0) = \mathsf{E}\bigg[\int_0^T \bigg\{ &(\mu(s) - \rho(s))\eta(s) \\ &+ \int_{\mathbb{R}} [(1 + \pi^*(s)\widetilde{\theta}(s,z))^{-1}\widetilde{\theta}(s,z)\eta(s) - \theta(s,z)\eta(s)]\,\nu(dz)\bigg\}\,ds\bigg], \end{aligned} \tag{3.38}$$

where we used the notation

$$\widetilde{\theta}(s,z) = \theta(s,z) + D_{s^+,z}\theta(s,z). \tag{3.39}$$

Hence,

$$\int_0^T \mathsf{E}\bigg[\bigg\{\mu(s) - \rho(s) + \int_{\mathbb{R}} [(1 + \pi^*(s)\widetilde{\theta}(s,z))^{-1}\widetilde{\theta}(s,z) - \theta(s,z)]\,\nu(dz)\bigg\}\eta(s)\bigg]\,ds = 0.$$

Since for each s the random variables $\eta(s)$, $\eta \in \mathcal{A}_{\mathcal{E}}$, generate the whole σ-algebra $\mathcal{E}_s$, we conclude that

$$\mathsf{E}\bigg[\bigg\{\mu(s) - \rho(s) + \int_{\mathbb{R}} [(1 + \pi^*(s)\widetilde{\theta}(s,z))^{-1}\widetilde{\theta}(s,z) - \theta(s,z)]\,\nu(dz)\bigg\} \,\bigg|\, \mathcal{E}_s\bigg] = 0 \tag{3.40}$$

for any $s \in [0,T]$. This proves part a) of the following result.

Theorem 3.12. a) *Suppose that there exists an optimal portfolio* $\pi^* \in \mathcal{A}_{\mathcal{E}}$ *for Problem* 3.2. *Then* $y = \pi^*(s)$ *satisfies the equation*

$$\begin{aligned} \mathsf{E}\bigg[\int_{\mathbb{R}} &\frac{\theta(s,z) + D_{s^+,z}\theta(s,z)}{1 + y(\theta(s,z) + D_{s^+,z}\theta(s,z))}\,\nu(dz) \,\bigg|\, \mathcal{E}_s\bigg] \\ &= \mathsf{E}\bigg[\bigg\{\rho(s) - \mu(s) + \int_{\mathbb{R}} \theta(s,z)\,\nu(dz)\bigg\} \,\bigg|\, \mathcal{E}_s\bigg], \qquad s \in [0,T]. \end{aligned} \tag{3.41}$$

b) *Suppose that*

$$\theta(s,z) + D_{s^+,z}\theta(s,z) \geq 0 \quad \textit{for a.a. } s,\ z \tag{3.42}$$

and that for all s *there is a solution*

$$y =: \widehat{\pi}(s)$$

of the equation (3.41). *Suppose that*

$$\widehat{\pi}(s) \in \mathcal{A}_{\mathcal{E}}.$$

Then $\widehat{\pi}$ *is an optimal portfolio for Problem* 3.2.

Proof of the assertion b). If (3.42) holds, then the function $F(\pi)$ given by (3.36) is concave.

Example 3.13 (Poisson process). Suppose that $\eta(t)$ is a compensated Poisson process. Then the Lévy measure $\nu(dz)$ is the point mass at $z = 1$, and (3.40) becomes

$$\mathsf{E}\left[\frac{\widetilde{\theta}(s)}{1 + y\widetilde{\theta}(s)} \;\middle|\; \mathcal{E}_s\right] = \mathsf{E}[\rho(s) - \mu(s) + \theta(s,1) \mid \mathcal{E}_s], \tag{3.43}$$

where

$$\widetilde{\theta}(s) = \theta(s,1) + D_{s^+,1}\theta(s,1). \tag{3.44}$$

We assume in addition that

$$\widetilde{\theta}(s) \quad \text{is } \mathcal{E}_s\text{-measurable.} \tag{3.45}$$

Then (3.43) has the solution

$$y = \widehat{\pi}(s) = \pi^*(s) = (\mathsf{E}[\rho(s) - \mu(s) + \theta(s,1) \mid \mathcal{E}_s])^{-1} - (\widetilde{\theta}(s))^{-1} \tag{3.46}$$

provided that

$$\mathsf{E}[\rho(s) - \mu(s) + \theta(s,1) \mid \mathcal{E}_s] \neq 0 \quad \text{and} \quad \widetilde{\theta}(s) \neq 0, \quad s \in [0,T]. \tag{3.47}$$

Corollary 3.14 (complete information case). *Suppose that*

$$\mathcal{E}_t = \mathcal{F}_t = \mathcal{G}_t \quad \textit{for any} \quad t \in [0,T]$$

and that there is an optimal portfolio $\pi^* \in \mathcal{A}_\mathcal{E}$ *for Problem* 3.2. *Then* $y = \pi^*(s)$ *solves the equation*

$$\int_\mathbb{R} \frac{\theta(s,z)}{1 + y\theta(s,z)}\, \nu(dz) = \rho(s) - \mu(s) + \int_\mathbb{R} \theta(s,z)\, \nu(dz). \tag{3.48}$$

In the special case of Markovian coefficients, this result could have been obtained by dynamic programming.

Bibliography

[1] F. Biagini and B. Øksendal, *A general stochastic calculus approach to insider trading*, preprint no. 17/2002, Dept. of Mathematics, Univ. of Oslo, Oslo 2002.

[2] F. Delbaen and W. Schachermayer, "A general version of the fundamental theorem of asset pricing", *Math. Ann.* **300** (1994), 463–520.

[3] G. Di Nunno, T. Meyer-Brandis, B. Øksendal, and F. Proske, *Malliavin calculus and anticipative Itô formulae for Lévy processes*, preprint no. 16/2003, Dept. of Mathematics, Univ. of Oslo, Oslo 2003.

[4] G. Di Nunno, T. Meyer-Brandis, B. Øksendal, and F. Proske, *Optimal portfolio for an insider in a market driven by Lévy processes*, preprint no. 36/2003, Dept. of Mathematics, Univ. of Oslo, Oslo 2003.

[5] K. Itô, "Spectral type of the shift transformation of differential processes with stationary increments", *Trans. Amer. Math. Soc.* **81** (1956), 253–263.

[6] Yu. M. Kabanov, "A generalized Ito formula for an extended stochastic integral with respect to a Poisson random measure", *Uspekhi Mat. Nauk* **29**:4 (1974), 167–168. (Russian)

[7] Yu. M. Kabanov, "On extended stochastic integrals", *Teor. Veroyatnost. i Primenen.* **20** (1975), 725–737; English transl., *Theory Probab. Appl.* **20** (1976), 710–722.

[8] A. Løkka, *Martingale representations and functionals of Lévy processes*, preprint no. 21/2001, Dept. of Mathematics, Univ. of Oslo, Oslo 2001.
[9] D. Nualart, *The Malliavin calculus and related topics*, Springer-Verlag, New York 1995.
[10] D. Nualart and É. Pardoux, "Stochastic calculus with anticipating integrands", *Probab. Theory Related Fields* **78** (1988), 535–581.
[11] B. Øksendal, "An introduction to Malliavin calculus with applications to economics", Working Paper 3/1996, Norwegian School of Economics and Business Administration, Oslo 1996.
[12] B. Øksendal and A. Sulem, *Applied stochastic control of jump diffusions*, Springer-Verlag, Berlin 2004 (to appear).
[13] F. Russo and P. Vallois, "Forward, backward and symmetric stochastic integration", *Probab. Theory Related Fields* **97** (1993), 403–421.
[14] F. Russo and P. Vallois, "Stochastic calculus with respect to continuous finite quadratic variation processes", *Stochastics Stochastics Rep.* **70** (2000), 1–40.

Center of Mathematics for Applications, University of Oslo,
Norwegian School of Economics and Business Administration;
INRIA, France

Received 20/JUN/03
Translated by IPS(DoM)

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX