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Lattices of topologies of algebraic systems

V. I. Arnautov, K. M. Filippov

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Abstract. The article is based on the report, which was delivered on a plenary sitting on the 5th International Algebraic Conference in the Ukraine (Odessa, July, 2005) and is a survey of the results on the lattices of topologies of different algebraic systems.

1. Introduction

A natural question whether a given algebraic system can be equipped with a topology having the desired properties, had arisen during the study of algebraic systems equipped with a topology (topological groups, topological rings, topological modules over topological rings etc.). In particular, it was a question whether an infinite group (infinite ring) admits a non-discrete separated topology such that the group (ring) operations are continuous in it.

It was academician A.A.Markov, who had begun the investigations on that subject.

Now every abelian group and every countable nilpotent group is known to admit a non-discrete group topology (see [23] and [33]).

Examples of groups admitting no separated non-discrete topology are built in [27], [29].

Every module over a ring equipped with the discrete topology is known to admit a non-discrete separated module topology (see [8]). An

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example of a topological ring and of a module over it admitting the only module topology, i.e. the antidiscrete topology, is built ibidem.

The question on the non-discrete topologization of infinite rings was investigated in a series of works, containing both nontrivial topologizations of some rings (see [1], [22], [26]), and the attempts to construct a topology in general rings (see [2], [3], [4], [5], [7], [20], [21], [31], [32]).

The problem on the non-discrete topologization is resolved positively for countable rings (see [3]) and infinite associative-commutative rings ([4], [5], [20], [21]).

As to the general case, it is known (see [6]), that for every cardinal greater or equal to the continuum there exists a ring of that cardinality admitting the only separated ring topology on it, which is the discrete topology.

A survey on the ring topologization can be also found in [10].

After the question on the existence of non-discrete topologies in different algebraic systems was resolved there was a series of works devoted to the question on the number of topologies ([17], [24] [33] ,[34]), and to the question on the existence of long chains of topologies ([17], [24], [33], [34]). These works might be considered to be the initial attempts of the investigation of the lattice of all topologies on an algebraic system such that their operations are continuous.

Although the question on the existence of long chains of topologies was investigated, the question on the existence of "neighbouring" topologies, i.e. such topologies that $\tau_1 \prec \tau_0$ (see Definition 2.1), was remaining out of field of sight of the researchers.

The survey is devoted namely to that question.

2. The lattice of topologies of an algebraic system

Let A be an algebraic system. If Ω is the set of all topologies such that algebraic operations are continuous in it then Ω is known to be a complete lattice under inclusion.

We write $\mathbf{1}_\Omega$ for the largest, $\mathbf{0}_\Omega$ for the least elements of Ω.

We write $\tau \vee \mu$ for the supremum and $\tau \wedge \mu$ for the infimum of the set $\{\tau, \mu\}$.

It is easily seen that the element $\mathbf{0}_\Omega$ is always the antidiscrete topology.

The topology $\mathbf{1}_\Omega$ needs not to be the discrete topology, but it is if the signature of the algebra A is equipped with the discrete topology or is equipped with no topology. It takes place, in particular, when A is a semigroup, group, ring, skew-field, module over a discrete ring etc.

We always mean by Ω the above mentioned lattice or its sublattice (specified when necessary).

2.1 Definition. Let $\mathcal{L}$ be a lattice, $\tau_1, \tau_0 \in \mathcal{L}$. We say that τ_1 precedes τ_0 and write $\tau_1 \prec \tau_0$, if $\{\tau \in \mathcal{L} \mid \tau_1 \leqslant \tau \leqslant \tau_0\} = \{\tau_1, \tau_0\}$.

Let $n \in \mathbb{N}$. Element $\mu \in \mathcal{L}$ is said to be an n-coatom, if n is the minimal of the naturals k such that $\mu = \tau_k \prec \tau_{k-1} \prec, \dots, \prec \tau_1 \prec \mathbf{1}_{\mathcal{L}}$ for some $\tau_k, \tau_{k-1}, \dots, \tau_1 \in \mathcal{L}$.

1-coatoms are also called coatoms.

2.2 Remark. If the lattice $\mathcal{L}$ is modular, $\tau_0, \tau_* \in \mathcal{L}$, $\tau_0 \not\leqslant \tau_*$ and τ_* is a coatom then it follows from the modularity of the lattice that $(\tau_0 \wedge \tau_*) \prec \tau_0$.

2.3 Definition. Let $\mathcal{L}$ be a complete lattice and $I \subseteq \mathcal{L}$. We say that the elements of the set I are strongly incomparable, if for every nonvoid set $J \subseteq I$ and every element $\tau \in I \setminus J$ elements τ and $\inf J$ are incomparable.

2.4 Remark. If a complete lattice $\mathcal{L}$ contains a countable set I such that its elements are strongly incomparable then $\mathcal{L}$ contains a sublattice which is isomorphic to the real line equipped with the natural order.[1]

2.5 Remark. If a modular lattice $\mathcal{L}$ contains n coatoms which are strongly incomparable, then it contains also k-coatoms for every $k \leqslant n$. If $\mathcal{L}$ contains an infinite set S of strongly incomparable coatoms, then $\mathcal{L}$ contains an infinite decreasing unrefinable chain[2] which is anti-isomorphic to the lattice of naturals equipped with the natural order.[3]

2.6 Remark. If Ω is a complete sublattice of the lattice of topologies of an algebraic system which is a group, a ring or a module over a discrete ring, and $\mathbf{1}_\Omega$ is the discrete topology then, since the set $\Omega \setminus \{\mathbf{1}_\Omega\}$ is inductive, Ω contains coatoms. Moreover, every unrefinable chain containing more than one element and containing the discrete topology contains also a coatom.

2.7 Problem. Let Ω be the lattice mentioned in Remark 2.6. Does every infinite unrefinable chain which contains $\mathbf{1}_\Omega$ contain n-coatoms for every (some) $n \geqslant 2$.

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[1]Indeed, we index I with rationals, i.e. assume $I = \{i_q \mid q \in \mathbb{Q}\}$. Now we put in correspondence to every real r an element $m_r \in \mathcal{L}$, where $m_r = \inf\{i_q \mid q > r\}$. One can easily check that $M = \{m_r \mid r \in \mathbb{R}\}$ is the desired set.

[2] A chain $S \subseteq \Omega$ is said to be unrefinable provided it contains every element of Ω which is situated between some two elements of S and is comparable with every element of S.

[3]indeed, if $\{\tau_i \mid i \in \mathbb{N}\} \subseteq S$ then the set $\mathcal{N} = \{\inf\{\tau_1, \dots, \tau_n\} \mid n \in \mathbb{N}\}$ is the desired one.

2.8 Theorem. If A is an algebraic system which is a group, a ring or a module over a topological ring;

Ω is a complete sublattice of the lattice of all group topologies, or of the lattice of all ring topologies, or of the lattice of all module topologies, respectively, and $\tau_0 \in \Omega$;

B is a dense algebraic subsystem in (A, τ_0);

Ω_B is equal to the set of all $\tau \mid_B$ [4] where $\tau \in \Omega$, and $\tau_1 \in \Omega$ such that $\tau_1 \prec \tau_0$ then $\tau \mid_B \prec \tau_0 \mid_B$ in $\Omega \mid_B$.

2.9 Theorem. If A, Ω, τ_0, B and Ω_B satisfy the condition of Theorem 2.8, and for every $\tau \in \Omega_B$ the topology $[\tau]_{(A,\tau_0)}$ belongs to Ω where the topology $[\tau]_{(A,\tau_0)}$ is defined by the set $\{[U]_{(A,\tau_0)} \mid U$ is a neighbourhood of the neutral element in $(B, \tau)\}$ which is assumed to be a fundamental system of neighbourhoods of the neutral element in $(A, [\tau]_{(A,\tau_0)})$, then for every topology $\tau_1 \in \Omega_B$ such that $\tau_1 \prec \tau_0 \mid_B$ it holds that $[\tau_1]_{(A,\tau_0)} \in \Omega$, $[\tau_1]_{(A,\tau_0)} \prec \tau_0$, and $[\tau_1]_{(A,\tau_0)} \mid_B = \tau_1$.

2.10 Remark. Let A be a group, ring, module over a ring R and Ω be one of the following lattices:

If A is a group then Ω is the lattice of all group topologies τ on A, such that either A is a thin set in (A, τ), or (A, τ) admits a fundamental system of neighbourhoods of zero consisting of normal divisors;

If A is a ring then Ω is the lattice of all ring topologies τ on A such that either A is bounded from the left in (A, τ), or A is bounded from the right in (A, τ), or A is bounded in (A, τ), or (A, τ) admits a fundamental system of neighbourhoods of zero consisting of left, (right, two-sided) ideals;

If A is a module over a topological ring (R, τ_R) and Ω is the lattice of all topologies τ on A such that τ is either a (R, τ_R)-module topology, or τ is an (R, τ_R)-module topology such that (A, τ) admits fundamental system of neighbourhoods of zero consisting of subgroups, or τ is an (R, τ_R)-module topology such that (A, τ) admits fundamental system of neighbourhoods of zero consisting of submodules.

If $\tau_0 \in \Omega$ and B is a dense algebraic subsystem in (A, τ_0) then it is easily seen that for every topology $\tau \in \Omega_B$ the topology $[\widehat{\tau} \mid_B]_{(A,\tau_0)}$ belongs to Ω, i.e. it holds the condition of Theorem 2.9.

3. The lattice of module topologies

3.1 Theorem. Let (R, τ) be a topological ring, M be some R-module. Then the following assertions are valid:

[4] We write $\mu \mid_M$ for the topology induced by the topology μ on the set M.

1) The set of all (R, τ)-module topologies is a complete sublattice of the modular lattice of all group topologies of the group $(M, +)$ and hence is also a modular lattice.

2) The set of all linear[5] (R, τ)-module topologies is a complete sublattice of the lattice of all (R, τ)-module topologies.

3) The set of all (R, τ)-module topologies such that M admits a fundamental system of neighbourhoods of zero consisting of subgroups of the group $(M, +)$ is a complete sublattice of the lattice of all (R, τ)-module topologies.

An example of a topological ring (R, τ) and of an infinite R-module M such that the lattice of all (R, τ)-module topologies contains the only element, namely, the antidiscrete topology, has been built in [8].

Ibidem it is proved that if R is a discrete ring and M is an infinite R-module then the lattice Ω of all R-module topologies on M contains a non-discrete separated topology, and hence Ω contains at least three elements: the discrete topology, the antidiscrete topology, and the above mentioned topology. Moreover, the following theorem can be proved:

3.2 Theorem. If a ring is discrete, finite or countable and the module over the ring is infinite then the lattice Ω of its module topologies contains a countable set of strongly incomparable coatoms (see Definition 2.3).

3.3 Remark. Let Ω be a lattice mentioned in Theorem 3.2. The lattice Ω contains:

A sublattice which is isomorphic to the real line equipped with the natural order by Remark 2.4,

An infinite decreasing incondensable chain which is anti-isomorphic to the lattice of naturals equipped with the natural order by Remark 2.5.

3.4 Problem. Let M be a module over an infinite uncountable discrete ring. Is it true that the lattice of all its module topologies contains:

1) An infinite subset consisting of separate topologies;

2) A sublattice isomorphic to the real line equipped with the natural order;

3) A finite incondensable chain of an arbitrary length and, in particular, n-coatoms for every natural n.

We pass on to the question of the existence of the topologies preceding the given one and of their construction.

[5] i.e. of those (R, τ)-module topologies μ on M such that the topological module (M, μ) admits a fundamental system of neighbourhoods of zero consisting of submodules.

The first construction

Theorem 3.5 gives a construction of topologies preceding the given one and establishes some their properties.

3.5 Theorem. (see [16].) Let (R, τ) be a topological ring, Ω be a complete sublattice of the lattice of all (linear) R-module topologies on R-module M.

If $\tau_0 \in \Omega$ and there exists a coatom $\tau_* \in \Omega$ such that $\tau_0 \not\leq \tau_*$ then the topology $\tau_1 = \tau_0 \wedge \tau_*$ precedes τ_0.

If, moreover, Ω contains the discrete topology, then the following assertions are valid:

1) τ_1 is separated iff so are τ_0 and τ_*;

2) (G, τ_0) admits a fundamental system of neighbourhoods of zero closed in (G, τ_1) iff τ_* is separated.

3.6 Remark. The following theorem points out a case when every topology preceding τ_0 in Ω can be obtained by the construction of Theorem 3.5.

3.7 Theorem. (see [16].) Let:

R be a discrete skew-field;

M be a vector space over R;

Ω be the lattice of all linear R-module topologies[6] on M.

If $\tau_1, \tau_0 \in \Omega$, $\tau_1 \prec \tau_0$ in Ω, τ_0 is metrizable[7] and one of the following assertions hold:

1) M has the countable dimension over R;

2) (M, τ_0) is a complete topological group;

then there exists a coatom $\tau_* \in \Omega$ such that $\tau_1 = \tau_0 \wedge \tau_*$.

3.8 Remark. To obtain a separate topology preceding the given one applying the construction of Theorem 3.5 it is necessary for Ω to contain a coatom which is a separate topology and which is not comparable with τ_0.

The above mentioned necessary condition needs not to be always valid. Theorems 3.9 and 3.10 give examples of a topological ring (R, τ), an R-module M and a sublattice Ω of the lattice of all (R, τ)-module topologies on M such that every coatom in Ω is not a separate topology.

[6] see the footnote 5.

[7] A topology τ on the set G is said to be metrizable if it is defined by some metric; if τ is a group topology on the group G then it is metrizable iff τ satisfies the first countability axiom and is separated.

3.9 Theorem. (see [11].) Let (R, τ_ξ) be a complete topological non-discrete skew-field such that topology τ_ξ is defined by a real absolute value ξ.[8] If M is an R-module and Ω is the lattice of all (R, τ_ξ)-module topologies on M then the topology $\tau \in \Omega$ is a coatom iff there exists a subspace N of the vector space M of dimension 1 such that τ is the infimum of topologies $\mathbf{1}_\Omega$ and $\tau_{\{N\}}$ [9] in the lattice of all group[10] topologies of the group $(M, +)$.

In particular there is no coatom in Ω which is a separate topology.

3.10 Theorem. (see [13].) Let (R, τ_ξ) be a complete non-discrete topological skew-field such that topology τ_ξ is defined by a real non-archimedean absolute value ξ,[11] M be a vector space over R, Ω be the lattice of all (R, τ_ξ)-module topologies admitting a fundamental system of neighbourhoods of zero consisting of subgroups.

Then Ω contains a coatom which is a separate topology iff the dimension of the vector space M is a measurable cardinal.[12]

3.11 Remark. If the topological vector space (M, τ_0) is not complete and its dimension is infinite and uncountable then (taking into account Theorem 2.8) then the construction of the topology preceding the given one can be changed, and we get

The second construction

3.12 Theorem. If (R, τ) is a topological ring, M is an R-module, Ω is the lattice of all (linear) (R, τ)-module topologies on M, $\tau_0 \in \Omega$ is a separated topology and (M, τ_0) is not complete, $(\widehat{R}, \widehat{\tau})$ is the completion of the topological ring (R, τ) and $(\widehat{M}, \widehat{\tau}_0)$ is the completion of the topological module (M, τ),[13] $\widehat{\Omega}$ is a lattice of all (linear) $(\widehat{R}, \widehat{\tau})$-module topologies on $\widehat{M}$, τ_* is a coatom in $\widehat{\Omega}$ which is not comparable with $\widehat{\tau}_0$, then the topology $\tau_1 = (\widehat{\tau}_0 \wedge \tau_*) \mid_M$ precedes the topology τ_0, and:

[8] A mapping $\xi : R \to \mathbb{R}$ is said to be an absolute value if $\xi(a + b) \leqslant \xi(a) + \xi(b)$, $\xi(-a) = \xi(a)$ and $\xi(ab) = \xi(a)\xi(b)$ for every $a, b \in R$.

[9] Write $\tau_{\{N\}}$ for such a group topology on $(M, +)$ that the $\{N\}$ is a fundamental system of neigbourhoods of zero in $(M, \tau_{\{N\}})$.

[10] not the (R, τ)-module!

[11] i.e. $\xi(a + b) \leqslant \max\{\xi(a), \xi(b)\}$.

[12] A cardinal $\mathfrak{m}$ is said to be measurable iff there exists an ultrafilter on the set of cardinality $\mathfrak{m}$ with empty intersection containing the intersection of every its countable subset. The problem of the existence of measurable cardinals has not been resolved yet. Every "known" cardinal is known to be not measurable. Hence, for every "known" vector space the lattice under consideration Ω contains no coatom which is a separated topology.

[13] $(\widehat{M}, \widehat{\tau}_0)$ is known to be a topological $(\widehat{R}, \widehat{\tau})$-module.

If the topological ring (R, τ) is discrete, then the topology τ_1 is separated iff so are τ_0 and $\tau_* |_M$;

If the topological ring (R, τ) is discrete, then the topological module (M, τ_0) admits a fundamental system of neighbourhoods of zero closed in (M, τ_1) iff the topology τ_* is separated.

It follows easily from theorems 2.8, 2.9 and 3.7 that every linear topology preceding a metrizable topology τ_0 in the lattice of all linear module topologies on a vector space over a discrete skew-field is obtained by that construction.

So the question of the construction of a topology preceding the given metrizable topology in the lattice of all linear module topologies of a vector space over a discrete skew-field may be considered to be resolved.

3.13 Remark. Let M be a module over a discrete ring (R, τ_R) and Ω be a lattice of all (linear) (R, τ_R)-module topologies. It is known that every precompact separate topology $\tau \in \Omega$ is coarser then every separate coatom $\tau_* \in \Omega$. Hence the first construction of the separated topology preceding the given one is not applicable for a precompact separated topology. One can apply the second construction to obtain a separated topology preceding the given precompact separate topology. To do that one should take a coatom $\tau_* \in \widehat{\Omega}$ such that it is not a separated topology and $\tau_* |_M$ is the discrete topology.

3.14 Remark. Let the condition of Theorem 3.12 be valid, the topological ring (R, τ) be discrete, and topology τ_1 be metrizable. If the topology τ_* is not separated and the topology $\tau_* |_M$ is discrete then the topology τ_1 in Theorem 3.12 is metrizable.

4. The lattice of all group topologies

4.1 Remark. Let G be a group and Ω be the set of all group topologies on G. Then the following assertions hold:

1) The set Ω is a complete lattice and needs not to be modular (see [18]);

2) If the group G is abelian then the lattice Ω is modular ([30]), and needs to be neither a distributive lattice nor a lattice with complements ([28]).

3) The set Ω_1 of all group topologies $\tau \in \Omega$ such that the set G is thin in (G, τ)[14] is a complete modular sublattice of Ω.

[14]The group G is said to be thin in a group topology τ provided for every neighbourhood of the identity U in (G, τ) there exists a neighbourhood of the identity V in (G, τ) such that for every $g \in G$ it holds that $g^{-1}Vg \subseteq U$.

4) The set Ω_2 of all group topologies τ such that the topological group (G, τ) admits a fundamental system of neighbourhoods of the identity consisting of subgroups is a complete sublattice in Ω.

5) The set Ω_3 of all group topologies τ such that the topological group (G, τ) admits a fundamental system of neighbourhoods of the identity consisting of normal divisors is a complete modular sublattice in every of lattices Ω, Ω_1, Ω_2.

6) Since every topology of a finite group is defined by its least normal divisor then the lattice all group of topologies of every finite group is isomorphic to the lattice of all its normal divisors and coincides with the lattice of all group topologies τ such that the topological group (G, τ) admits a fundamental system of neighbourhoods of zero consisting of normal divisors and hence is modular.

4.2 Remark. An example of an infinite group with the two element lattice of all group topologies is constructed in [29] assuming CH (i.e. $2^{\aleph_0} = \aleph_1$);

An example of a countable group admitting no separated group topologies and such that the set of all coatoms in its lattice of all group topologies is finite is constructed in [27].

4.3 Problem. Let G be an infinite (countable) group admitting no separated group topologies, and Ω be a lattice of all group topologies on G. Is it true that the set of all coatoms in Ω is finite.

4.4 Problem. Is it true that there exists a countable group with two element lattice of all group topologies.

4.5 Theorem. Let G be a countable nilpotent group and Ω be the lattice of all group topologies on G. Then:

1) Ω contains a countable set I of coatoms such that elements of I are strongly incomparable (see [33]), hence (see Remark 2.4) Ω contains a sublattice isomorphic to the real line equipped with the natural order;

2) Ω contains an incondensable chain of separate topologies which is anti-isomorphic to the lattice of naturals equipped with the natural order, and, in particular, it contains n-coatoms for every $n \in \mathbb{N}$.

Using the construction of separated topology on a countable group from [25] one can obtain the following

4.6 Theorem. If a countable group G admits a separated group topology then the lattice Ω of all group topologies on G contains a countable set I of coatoms such that elements of I are strongly incomparable; hence (see Remark 2.4) the lattice Ω contains a sublattice isomorphic to the real line equipped with the natural order.

4.7 Remark. Note that for every topology τ defined on a subgroup A of an abelian group G there exists the only topology μ on G such that A is open in (G, μ) and $\mu\mid_A = \tau$. Therefore the lattice Ω of all group topologies of an infinite abelian group contains the lattice of all topologies of every its (countable) subgroup as a sublattice. Hence (see remarks 2.4 and 2.5) Ω contains both a sublattice isomorphic to the real line equipped with the natural order, and an unrefinable chain anti-isomorphic to the lattice of naturals.

Since the lattice Ω of all group topologies of an abelian group G coincides with the lattice of all $(\mathbb{Z}, \tau_{\mathbb{Z}})$-module topologies on G where $\tau_{\mathbb{Z}}$ is the discrete topology then the construction of the topology τ_1 preceding the given topology τ_0 is the same as for the section 3.

Some constructions of topologies preceding the given one similar to the constructions of the section 3 are set forth in this section for some sublattices of the lattice of all the group topologies of an arbitrary group.

4.8 Remark. A non separated group topology τ_* on the group G is a coatom in the lattice of all group topologies on G iff $\tau_* = \tau_{\{H\}}$[15] for some normal divisor $H \subseteq G$ such that H contains no non-identical normal divisor of the group G.

4.9 Theorem. If G is an arbitrary group,

Ω is either a lattice of all group topologies τ such that G is a thin set in (G, τ), or a lattice of all group topologies admitting a fundamental system of neighbourhoods of zero consisting of normal divisors,

$\tau_0 \in \Omega$ is a separated topology,

$(\widehat{G}, \widehat{\tau}_0)$ is the completion of topological group (G, τ_0),[16]

$\widehat{\Omega}$ is a lattice of all group topologies τ on $\widehat{G}$ such that $\widehat{G}$ is a thin set in $(\widehat{G}, \tau)$, or a lattice of all group topologies on $\widehat{G}$ admitting a fundamental system of neighbourhoods of zero consisting of normal divisors,

$\tau_* \in \widehat{\Omega}$ is such a coatom that $\widehat{\tau}_0 \not\leq \tau_*$,

$\widehat{\tau}_1 = \widehat{\tau}_0 \wedge \tau_*$ in the lattice $\widehat{\Omega}$ and $\tau_1 = \widehat{\tau}_1\mid_G$,

then the following assertions are valid:

1) $\tau_1 \prec \tau_0$ in Ω;

2) The topology τ_1 is separated iff so are $\tau_*\mid_G$ and τ_0;

3) (G, τ_0) admits a fundamental system of neighbourhoods of zero which are closed in (G, τ_1) iff the topology τ_* is separated;

4) Let the coatom topology τ_* be not separated. and $H = [\{0\}]_{(G, \tau_*)}$. Then the topology τ_1 is separated iff $H \cap G = \{e\}$ where $\tau_* = \tau_{\{H\}}$ (see Remark 4.8).

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[15] see the footnote 9.

[16] This is a partial case when the left and the right uniformities of the topological group (G, τ_0) coincide and hence $(\widehat{G}, \widehat{\tau}_0)$ is also a topological group.

5) Let the topology τ_0 be metrizable. Then τ_1 is so iff the topology τ_* is not separated and the topology $\tau_* |_G$ is discrete.

Taking into account that every abelian group is thin in every its group topology, and the assertion 5) of Theorem 4.9 one obtains a

4.10 Corollary. Let G be an abelian group of a prime period, Ω be the lattice of all (linear) group topologies on G and $\tau_0 \in \Omega$ be a metrizable separated topology. Then there exists a metrizable separated topology $\tau_1 \in \Omega$ such that $\tau_1 \prec \tau_0$ iff the topological group (G, τ_0) is not complete.

Another construction of the topology preceding the given one in the lattice of all group topologies of an abelian group of finite period is set forth here. The idea of the construction can be found in [15]. The article [14] contains a partial case of the construction.

The construction uses the division of some elements of the group into a prime p.

The third construction

Let G be an abelian group of a finite period, Ω be the lattice of all group topologies on G admitting a fundamental system of neighbourhoods of zero consisting of subgroups, and $\tau_0 \in \Omega$ be a metrizable topology.

4.11 Definition. Let A, B be abelian groups, $\alpha : A \to B$ some mapping (not necessary a homomorphism), τ be a group topology on A, admitting a fundamental system of neighbourhoods of zero consisting of subgroups. We write τ/α for the finest of all group topologies μ on B, admitting a fundamental system of neighbourhoods of zero consisting of subgroups and such that the mapping $\alpha : (A, \tau) \to (B, \mu)$ is continuous in zero.

We set forth some considerations establishing properties of the topology τ_1 preceding the topology τ_0.

The group G can be decomposed into a direct sum of all its primary components G_i, $1 \leqslant i \leqslant n$.

If $\tau_1 \prec \tau_0$ then

$\tau_1 |_{G_i} \preccurlyeq \tau_0 |_{G_i}$ for every $i \leqslant n$;

there exists the only index i_0 such that $\tau_1 |_{G_{i_0}} \prec \tau_0 |_{Gi_0}$;

$\tau_1 = \bigwedge_{i=1}^{n} (\tau_0 |_{G_i})$.[17]

[17] By Remark 4.7 one may assume $\tau_0 |_{G_i} \in \Omega$.

So the construction of the topology on the group G of a finite period preceding the given one is reduced to its partial case for the primary group of period p^n.

To simplify the presentation of the construction we assume the group G to be isomorphic to $\bigoplus_{i\in I} \left(\mathbb{Z}_{p^2}\right)_i$, where p is a prime.

We write $\omega : G \to G$ for a homomorphism such that $\omega(g) = pg$ and H for $\ker\omega$.

If $\tau_1 \in \Omega$ then $\tau_1 \prec \tau_0$ iff either $\tau_1 |_H \prec \tau_0 |_H$ and $\tau_1/\omega = \tau_0/\omega$, or $\tau_1 |_H = \tau_0 |_H$ and $\tau_1/\omega \prec \tau_0/\omega$.

To construct a topology τ_1 such that $\tau_1 |_H \prec \tau_0 |_H$ and $\tau_1/\omega = \tau_0/\omega$, we do the following:

Since H is a group of the prime period p then H is a $\mathbb{Z}_p$-module.

Using the first or the second construction from the section 3, we construct such a topology τ_1' on H that $\tau_1' \prec \tau_0 |_H$ in the lattice of all group topologies on H. One may assume (see Remark 4.7) that $\tau_1' \in \Omega$ and by setting $\tau_1 = \tau_1' \wedge \tau_0$ obtains the desired topology.

The construction of a topology τ_1 such that $\tau_1 |_H = \tau_0 |_H$ and $\tau_1/\omega \prec \tau_0/\omega$ is the following: one takes:

A coatom $\tau_* \in \widehat{\Omega}$ such that $\tau_* \geqslant \tau_0$ and τ_* is incomparable with τ_0/ω;

An open subgroup C in (G, τ_*);

Defines a mapping $\alpha : C \to G$ such that $\omega\alpha$ is the identity mapping of C.

Then the topology $\tau_1 = (\tau_*/\alpha) \wedge \tau_0$ is the desired one.

5. The lattice of ring topologies

5.1 Theorem. Let R be a ring. Then the following assertions hold:

1) The set of all ring topologies is a complete lattice by the inclusion and is not a sublattice of the lattice of all group topologies on the group $(R, +)$ and, moreover, needs not to be modular.

2) The set of all ring topologies τ on the ring R such that the topological ring (R, τ) is bounded from the left (bounded from the right, bounded)[18] is a sublattice of the lattice of all group topologies on the group $(R, +)$ and hence is modular.

3) The set of all ring topologies τ on the ring R such that (R, τ) admits a fundamental system of neighbourhoods of zero consisting of the

[18]A subset M of a topological ring R is said to be a bounded from the left (right) one, provided for every neighbourhood U of zero in R there exists a neighbourhood U of zero in R such that $V \cdot M \subseteq U$ $(M \cdot V \subseteq U)$; the subset is said to be bounded provided it is bounded both from the left and from the right.

left (right, two-sided) ideals is a complete modular sublattice of the lattice of all group topologies on the group $(R, +)$.

4) The lattice of all ring topologies on the finite ring R is isomorphic to the lattice of all two-sided ideals of the ring and hence is modular.

There exists a ring R admitting only the discrete and the antidiscrete topology (see [6]), i.e. the lattice of all its ring topologies is a two element lattice.

5.2 Remark. Let R be a countable ring (infinite domain). Using the construction of the ring topology contained in [2] for a countable ring and in [7] for an infinite domain one can obtain a countable set of strongly incomparable coatoms in the lattice Ω of all ring topologies on R. Hence there exists a sublattice in Ω isomorphic to the real line equipped with the natural order (see Remark 2.4). [19]

5.3 Theorem. (see [34].) The lattice of all ring topologies Ω of a countable nilpotent ring contains an unrefinable decreasing chain consisting of separated topologies which is anti-isomorphic to the a lattice of naturals; hence it contains an unrefinable decreasing chain of an arbitrary finite length. In particular, Ω contains an n-coatom for an arbitrary $n \in \mathbb{N}$.

5.4 Problem. Is it true that the lattice of all ring topologies of every countable (infinite associatively-commutative) ring contains "neighbouring" topologies and, in particular, incondensable chains of an arbitrary finite length.

Since the lattice of all ring topologies needs not to be modular then the presence of coatoms does not permit to obtain topologies preceding the given one using the constructions similar to the constructions mentioned in the section 3.

As it had been said above yet the lattices mentioned in Remark 5.1, items 2) and 3), are modular. Theorem 5.6 proves that the constructions of topologies preceding the given one which are similar to those of the section 3 are available for the lattices mentioned in Remark 5.1.

5.5 Remark. A non separated ring topology τ_* on the ring R is a coatom in the lattice of all ring topologies on R iff $\tau_* = \tau_{\{J\}}$ [20] for some ideal $J \subseteq R$ such that J contains no nonzero ideal of the ring R.

[19] Such a set of strongly incomparable coatoms for a countable nilpotent ring has been built in [34].

[20] see the footnote 9.

5.6 Theorem. Let R be a ring;

Ω be the lattice of all ring topologies on R bounded from the left (bounded from the right, bounded) and $\tau_0 \in \Omega$; $(\widehat{R}, \widehat{\tau}_0)$ be the completion of the topological ring (R, τ_0);

$\widehat{\Omega}$ be the lattice of all ring topologies on $\widehat{R}$ bounded from the left (right, bounded); it is clear that $\widehat{\tau}_0 \in \widehat{\Omega}$;

$\tau_* \in \widehat{\Omega}$ be an arbitrary coatom incomparable with $\widehat{\tau}_0$, and $\tau_1 = (\widehat{\tau}_0 \wedge \tau_*) |_R$.

Then the following assertions hold:

1) $\tau_1 \prec \tau_0$ in the lattice Ω;

2) $\tau_1 \prec \tau_0$;

3) The topology τ_1 is separated iff so are τ_0 and $\tau_* |_R$;

4) (R, τ_0) admits a fundamental system of neighbourhoods of zero consisting of sets closed in (R, τ_1) iff the topology τ_* is separate;

5) If τ_* is not separate, then the topology τ_1 is separated iff $J \cap R = \{0\}$, where $\tau_* = \tau_{\{J\}}$ (see Remark 5.5).

6) If the topology τ_0 is metrizable then τ_1 is so iff τ_* is not separated and the topology $\tau_* |_R$ is discrete.

5.7 Remark. Theorem 5.6 holds also for the case when Ω is a lattice of all ring topologies τ on R such that (R, τ) admits a fundamental system of neighbourhoods of zero consisting of the left (right, two sided) ideals.

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CONTACT INFORMATION

V. I. Arnautov Institutul de matematică şi Informatică, str. Academiei, 5, Chişinău, MD-2028, Moldova
E-Mail: arnautov@math.md

K. M. Filippov Ibidem
E-Mail: Ibidem

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