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New relations for Morse–Smale systems with trivially embedded one-dimensional separatrices

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Abstract. New relations are established between the periodic structure of Morse–Smale systems (flows or diffeomorphisms) and the genus of Heegaard splittings of the supporting manifold. A sharp lower estimate is found for the number of non-closed heteroclinic curves of certain Morse–Smale diffeomorphisms on lens spaces.
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§ 1. Introduction

One of the central problems of the theory of dynamical systems is the relation between the dynamics and the topological structure of the supporting manifolds. It is particularly important for structurally stable dynamical systems, which exist on arbitrary compact manifolds. The simplest among them from many standpoints (finitely many non-wandering points, entropy zero, and so on) are Morse–Smale systems, for which, with the help of Morse's inequalities, one has found a connection between the Betti numbers and the structure of periodic points [1]. In [2] one can find analogues of Morse's inequalities for Morse–Smale flows without equilibria. Note that the non-wandering set of a Morse–Smale flow without equilibria consists of periodic trajectories. In [3] for such flows on manifolds of dimension $n \geqslant 4$ the author constructs a special decomposition of the supporting manifold into round handles and proves that if a manifold admits a decomposition into round handles, then there exists on it a Morse–Smale flow without equilibria. The topological structure of a 3-manifold admitting a Morse–Smale flow without equilibria is studied in [4], where it is shown that the supporting manifold is either a Seifert insufficiently large space (in Waldhausen's terminology [5]) or (if the manifold is sufficiently large) a special union of Seifert spaces and 'thick' tori (that is, the products $T^2 \times [0,1]$). We also point out the paper [6], which contains a classification of gradient-like flows without periodic trajectories in terms of Heegaard diagrams. However, that paper does not analyse the relation between the dynamical characteristics of the flow and the topological structure of the supporting manifold.

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In the present paper we establish new relations between the set of periodic orbits of a Morse–Smale system (a flow or a diffeomorphism) and the genus of the supporting manifold, which is assumed to be orientable (except in the 2-dimensional case) and closed. For a 2-manifold we mean its usual genus, and for a 3-manifold its Heegaard genus and the genus of a Heegaard splitting. In addition, we obtain an (attainable) lower bound for the number of non-closed heteroclinic curves for certain Morse–Smale diffeomorphisms of lens spaces.

Note that the Euler characteristic of a closed 3-manifold is always equal to 0 and therefore the well-known Euler–Hopf formula in the 3-dimensional case produces the same relation for all Morse–Smale systems (the sum of the indices of the equilibria is zero). Hence relations between the genus of a Heegaard splitting or the Heegaard genus of a manifold and the dynamical characteristics of a Morse–Smale system are more interesting.

We collected the basic concepts of the theory of smooth dynamical systems used throughout the paper in § 2.

Let f be a gradient-like diffeomorphism on a smooth closed oriented manifold M^3. Since the wandering set of such a diffeomorphism contains no heteroclinic points, one can show similarly to Lemma 1 in [7] that if p is a saddle periodic point of f, then the closure of each 1-dimensional unstable (stable) separatrix of p is homeomorphic to a closed interval and consists of the separatrix and two points, p and some sink periodic point ω (respectively, a source point α).

Let ω be a sink periodic point of f. Let $\mathcal{L}^u(\omega) = L_1^u \cup \dots \cup L_k^u$ be the set of all unstable 1-dimensional separatrices of saddle periodic points containing ω in their closure.

For $k > 0$ we select in $\mathbb{R}^3$ a system $l_1, \dots, l_k$ of distinct straight rays issuing from O.

Definition 1.1. The set of separatrices $\mathcal{L}^u(\omega)$ is said to be *trivially embedded in the manifold* M^3 if there exists a homeomorphism $\varphi\colon W^s(\omega) \to \mathbb{R}^3$ such that $\varphi(\omega) = O$ and $\varphi(L_i^u \cup \{O\}) = l_i$, $i = 1, \dots, k$.

In a similar way one defines a trivially embedded set of stable 1-dimensional separatrices $\mathcal{L}^s(\alpha) = L_1^s \cup \dots \cup L_m^s$ associated with a source α.

Note that in the case of a flow the sets $\mathcal{L}^u(\omega)$ and $\mathcal{L}^s(\alpha)$ are always trivially embedded. In the case of a diffeomorphism this is in general not so (see [8] and [9]).

Definition 1.2. We say that the set of 1-dimensional separatrices of saddle periodic points of a gradient-like diffeomorphism f is *trivially embedded* if the sets $\mathcal{L}^u(\omega)$ and $\mathcal{L}^s(\alpha)$ are trivially embedded for all sink and source periodic points ω and α.

Let f be a gradient-like diffeomorphism with trivially embedded set of 1-dimensional separatrices on a smooth closed oriented 3-manifold M^3. Let $\nu(f)$ be the number of saddle points, and $\mu(f)$ the number of sink and source periodic points of f.

It is shown in § 4 (Theorem 4.1) that M^3 can be represented as a Heegaard splitting of genus $h_D = (\nu(f) - \mu(f) + 2)/2$. It is shown in Theorem 4.2 that the number of saddle periodic points of the diffeomorphism f is not less than twice the Heegaard genus of M^3 and on each closed manifold M^3 of Heegaard genus

$h(M^3)$ there exists a gradient-like diffeomorphism f such that the number of saddle periodic points of f is $2h(M^3)$.

It follows from Proposition 2 in [8] that if a Morse–Smale diffeomorphism has an energy function, then the set of 1-dimensional separatrices of all saddle periodic points is trivially embedded. Hence the relation and the inequality obtained in Theorems 4.1 and 4.2 extend to gradient-like diffeomorphisms with an energy function. In this way we obtain the following result.

Corollary. *Let $f\colon M^3 \to M^3$ be a gradient-like Morse–Smale diffeomorphism with energy function on a closed 3-manifold M^3. Then M^3 can be represented as a Heegaard splitting of genus*

$$h_D = \frac{\nu(f) - \mu(f) + 2}{2}$$

and

$$h(M^3) \leqslant \frac{\nu(f) - \mu(f) + 2}{2}\,.$$

For completeness, at the end of § 4 in Theorem 4.3 we present similar relations for Morse–Smale diffeomorphisms of 2-dimensional closed surfaces (both oriented and non-oriented). These relations are consequences of Lefschetz's formula and [10], where it is proved that a Morse–Smale diffeomorphism induces an isomorphism of the 1-dimensional homology group with eigenvalues that are roots of unity.

Note that in Theorem 4.3 we make no assumptions about the set of heteroclinic points or about the character of the embedding in M_g^2 of the invariant manifolds of periodic points. The formula in this theorem is similar to the Euler–Poincaré formula connecting the sum of the indices of the singular points of a vector field on a 2-dimensional surface with the Euler characteristic. However, the proof of this formula for Morse–Smale diffeomorphisms is basically different from the proof of a similar formula for flows since no power of the diffeomorphism can be embedded in a flow in the case when there exist heteroclinic trajectories.

In § 5 we consider Morse–Smale flows on closed oriented 3-manifolds and in Theorem 5.1 we prove relations and estimates relating the numbers of equilibria and saddle periodic points with the genus of a Heegaard splitting of the manifold under consideration.

In § 6 we consider Morse–Smale diffeomorphisms on lens spaces $L_{p,q}$ with non-wandering sets containing precisely 4 periodic points and the trivially embedded set of separatrices of saddle periodic points. In Theorem 6.1 we show that the non-wandering set of such a diffeomorphism consists precisely of one sink, one source, and two saddle fixed points with distinct Morse indices. We also demonstrate that such diffeomorphisms are gradient-like and their wandering sets contain at least p non-closed heteroclinic curves whose end-points are saddle points.

The third section is technical; it contains a topological result (Theorem 3.3) which is in a certain sense a generalization of the annulus theorem. S. V. Matveev has kindly informed us that the result of Theorem 3.3 is fairly plain for experts in the topology of 3-manifolds and can be obtained from Haken's theory of normal surfaces. However, neither a statement of this result nor a direct proof of it is available. Theorem 3.3 is an immediate consequence of Theorem 3.2, the detailed proof of which we feel it necessary to present.

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§ 2. Main definitions

We recall several concepts and facts concerning Morse–Smale flows and diffeomorphisms. A decent source here are the monographs [11], [12], and the survey papers [1], [13]–[16]. Throughout what follows the manifold M^n of dimension $n \geqslant 3$ is assumed to be orientable unless otherwise stated. 2-manifolds (surfaces) can be either orientable or non-orientable.

Morse–Smale diffeomorphisms and flows. Let f be a C^1-smooth diffeomorphism of a closed n-manifold M $(n \geqslant 2)$ endowed with a Riemannian metric d. An f-invariant subset Λ of M is said to be *hyperbolic* if the restriction $T_\Lambda M$ of the tangent bundle TM of M to Λ can be represented as a Whitney sum $E^s_\Lambda \oplus E^u_\Lambda$ of df-invariant subbundles E^s_Λ and E^u_Λ, $\dim E^s_\Lambda + \dim E^u_\Lambda = n$, and there exist positive constants C_s, $C_u > 0$, and λ, $0 < \lambda < 1$, such that

$$\|df^n(v)\| \leqslant C_s\lambda^n\|v\|, \qquad v \in E^s_\Lambda, \quad n > 0,$$
$$\|df^{-n}(v)\| \leqslant C_u\lambda^n\|v\|, \qquad v \in E^u_\Lambda, \quad n > 0.$$

The *stable manifold of a point* $x \in \Lambda$ is the point set $W^s(x) = \{y \in M^n : d(f^k x, f^k y) \to 0,\ k \to +\infty\}$. The *unstable manifold* $W^u(x)$ of a point $x \in \Lambda$ is defined as its stable manifold with respect to the diffeomorphism f^{-1}. A point $x \in M$ is said to be *non-wandering* if for an arbitrary neighbourhood $U(x)$ of it and an arbitrary positive integer N there exists $n_0 \in \mathbb{Z}$, $|n_0| \geqslant N$, such that $f^{n_0}(x) \in U(x)$. We shall denote the set of non-wandering points of f by $NW(f)$.

A diffeomorphism of a manifold generates a discrete-time dynamical system (a cascade), which is the system of iterates of the diffeomorphism and which we shall normally identify with it.

A diffeomorphism f is called a *Morse–Smale diffeomorphism* if its non-wandering set $NW(f)$ is hyperbolic, consists of finitely many points, and the invariant manifolds $W^s(x)$ and $W^u(y)$ intersect transversally for all points $x, y \in NW(f)$. Since the non-wandering set $NW(f)$ is finite, it consists of periodic points. In similar fashion one defines a Morse Smale flow (a dynamical system with continuous time).

A periodic point $p \in NW(f)$ of a diffeomorphism f is a *saddle* point if its stable and unstable manifolds have non-zero topological dimensions. A periodic point $p \in NW(f)$ is a *node* if either $\dim W^s(p) = \dim M$ (and then p is a *sink*) or $\dim W^u(p) = \dim M$ (then p is a *source*).

The *Morse index of a periodic point* is the dimension of its unstable manifold.

An *unstable* (a *stable*) *separatrix* of a saddle periodic point p is a connected component of the set $W^u(p) \setminus p$ (respectively, of $W^s(p) \setminus p$).

One can define in the standard manner a saddle trajectory, a stable, and an unstable periodic trajectory and also a saddle point, a stable node and an unstable node of a flow. In what follows we formulate the main definitions only for diffeomorphisms (they are similar for flows).

A Morse–Smale diffeomorphism f is said to be *gradient-like* if for arbitrary periodic points $p,\ q \in NW(f)$, $p \neq q$, it follows from $W^u(p) \cap W^s(q) \neq \varnothing$ that $\dim W^s(p) < \dim W^s(q)$.

A point $x \in M$ in the transversal intersection of invariant manifolds $W^s(p)$ and $W^u(q)$, where $p, q \in NW(f)$, is said to be *heteroclinic* if $\dim W^s(p) = \dim W^s(q)$. It is an immediate consequence of the definition that a gradient-like Morse–Smale diffeomorphism has no heteroclinic points. Since a Morse–Smale diffeomorphism is structurally stable, the converse result also holds.

Lyapunov function and energy function. Let $f\colon M \to M$ be a diffeomorphism. A function $\varphi\colon M \to \mathbb{R}$ is called a *Lyapunov function* if $\varphi(f(x)) \leqslant \varphi(x)$ for all $x \in M$ and the equality holds only for non-wandering points. If $f\colon M \to M$ is a Morse–Smale diffeomorphism, then there exists a smooth Lyapunov function all of whose critical points are non-degenerate in the sense of the Morse theory.

A Lyapunov function of a Morse–Smale diffeomorphism $f\colon M \to M$ is called an *energy function* if all its critical points are non-degenerate and form the non-wandering set of the diffeomorphism f. Each periodic point of f is a critical point of each Lyapunov function. Hence the core of the definition of an energy function is the condition that each of its critical points is non-wandering.

Genus and Heegaard splitting. Let B^n be the standard unit closed n-dimensional ball in the Euclidean space $\mathbb{R}^n$. A set homeomorphic to the product $B^i \times B^{n-i}$ is called a *handle of index* i, $0 \leqslant i \leqslant n$. The subset $\partial B^i \times B^{n-i}$ of a handle of index i is called its *foot*. We say that a manifold M can be obtained from a manifold N by *gluing several handles* if $\operatorname{clos}(M - N)$ is a disconnected union of handles with feet in ∂N (see the details in [17]).

A 3-manifold D^3_g is called a 3-*dimensional handlebody with* g *handles* if D^3_g can be obtained from B^3 by gluing $g \geqslant 0$ handles of index 1. By a *Heegaard splitting* of genus $g \geqslant 0$ of a closed 3-manifold M^3 one means a representation of M^3 as a sewing of two handlebodies with g handles by means of a homeomorphism identifying their boundaries. The *Heegaard genus* of M^3 is the smallest g such that M^3 has a corresponding Heegaard splitting.

§ 3. Embedding a surface in the product of the surface and an interval

Let $M_{g,k}$ be an orientable compact connected surface of genus $g \geqslant 0$ with boundary $\partial M_{g,k}$ that is either empty for $k = 0$ or consists of $k \geqslant 1$ disjoint simple closed curves $C_1, \dots, C_k$. We set $\mathcal{C}_k = \bigcup_{i=1}^k C_i$

Let $P_{g,k}$ be the 3-manifold that is the direct product of $M_{g,k}$ and the interval $[0,1]$, $P_{g,k} = M_{g,k} \times [0,1]$. We set

$$M^0_{g,k} = M_{g,k} \times \{0\}, \qquad M^1_{g,k} = M_{g,k} \times \{1\}.$$

Then the boundary $\partial P_{g,k}$ of the manifold $P_{g,k}$ consists of two compact surfaces $M^0_{g,k}$ and $M^1_{g,k}$ and a set that is either empty or is the union of k disjoint closed annuli $C_1 \times [0,1], \dots, C_k \times [0,1]$. In that case we set $\mathcal{K}_k = \bigcup_{i=1}^k C_i \times [0,1]$.

In this section we study the topological properties of the connected components of the set $P_{g,k} \setminus Q$, where Q is a 2-dimensional compact orientable manifold (a surface) smoothly embedded in the set $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$, with boundary ∂Q that is either empty (if Q is closed) or consists of finitely many simple closed curves. Here we shall always assume that the manifold Q is properly embedded in $P_{g,k}$. Recall that a compact 2-manifold N is properly embedded in a compact 3-manifold M if the interior of N lies in the interior of M, the boundary of N lies in the boundary of M, and N is transversal to the boundary of M.[1]

The main result of this section is Theorem 3.3 stating that if Q is a closed oriented surface of genus g not bounding a subdomain of $P_{g,k}$, then the closure of each component of $P_{g,k} \setminus Q$ is homeomorphic to $P_{g,k}$.

For the proof of this result we prove several lemmas and Theorems 3.1 and 3.2, which are of independent interest. Note that Theorem 3.3 is an immediate consequence of Theorem 3.2.

We say that a surface $Q \subset P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ separates the surfaces $M^0_{g,k}$ and $M^1_{g,k}$ in $P_{g,k}$ if $M^0_{g,k}$ and $M^1_{g,k}$ lie in different connected components of the set $P_{g,k} \setminus Q$. Otherwise we shall say that the surface $Q \subset P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ does not separate the surfaces $M^0_{g,k}$ and $M^1_{g,k}$.

The next lemma and its two corollaries contain conditions ensuring that Q in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ separates or does not separate the surfaces $M^0_{g,k}$ and $M^1_{g,k}$.

Lemma 3.1. *A connected closed oriented surface $Q \subset P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ does not separate the surfaces $M^0_{g,k}$ and $M^1_{g,k}$ if and only if Q bounds a subdomain A of $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ with boundary $\partial A = Q$.*

Proof. We prove the *necessity*. Assume that the surface $Q \subset P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ does not separate the surfaces $M^0_{g,k}$ and $M^1_{g,k}$.

We can assume without loss of generality that the manifold $M_{g,k}$ is embedded in the space $\mathbb{R}^3$ in the standard fashion. Then $P_{g,k}$ can be regarded as a 'fattening' of this manifold in $\mathbb{R}^3$. It is sufficient to consider the case $k = 0$ because for $k \neq 0$ the result follows from the case $k = 0$.

The set $\mathbb{R}^3 \setminus (M^0_{g,0} \cup M^1_{g,0})$ is a union of three disjoint domains: an unbounded subdomain A_1 of $\mathbb{R}^3$ with boundary $M^0_{g,0}$; the domain $A_2 = P_{g,0} \setminus (M^0_{g,0} \cup M^1_{g,0})$; a domain A_3 compactly lying in $\mathbb{R}^3$ and bounded by the surface $M^1_{g,0}$. Since the manifold Q is also embedded in $\mathbb{R}^3$, it also partitions $\mathbb{R}^3$ into an unbounded domain B_1 and a bounded domain B_2 (see, for instance, [18], Section 26, Theorem 6).

After the consideration of all possible cases of the mutual position of the domains A_i, B_i and the manifolds $M^j_{g,0}$, Q we arrive at the required result: B_2 lies entirely in A_2.

Sufficency is an immediate consequence of the definition of non-separability.

Corollary 3.1. *Let Q be a connected closed orientable surface embedded in the domain $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ so that there exists a simple arc γ with end-points x_1 and x_2 lying in $M^0_{g,k}$ and $M^1_{g,k}$, respectively, such that γ intersects Q transversally at a unique point. Then Q separates $M^0_{g,k}$ and $M^1_{g,k}$.*

Proof. Assume the contrary. Then by Lemma 3.1 the surface Q bounds a domain A, which contradicts the existence of an arc γ with the required properties.

[1]The transversality of ∂N to ∂M means that at each point $x \in (\partial M \cap \partial N)$ the picture of the intersection of M and N is similar to the intersection of the xy-plane with the yz half-plane in $\mathbb{R}^3$.

Corollary 3.2. *Let Q be a connected compact orientable surface with boundary ∂Q consisting of $r > 0$ closed curves that is embedded in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$, $k > 0$, so that there exists at least one curve C_i in the set $\mathcal{C}_k$, $i \in \{1, \dots, k\}$, that is homotopic to precisely one closed curve S in ∂Q. Then Q separates $M^0_{g,k}$ and $M^1_{g,k}$.*

Proof. We can assume that $P_{g,k}$ is a part of the manifold $P_{g,0} = M_{g,0} \times [0,1]$ that is a direct product of a closed orientable 2-manifold $M_{g,0} \supset M_{g,k}$ of genus g and the interval $[0,1]$.

We seal the boundary of Q with disjoint smooth 2-dimensional discs lying in $P_{g,0} \setminus P_{g,k}$ so that the result is a smooth closed orientable manifold Q'. Then there exists a path on $C_i \times [0,1]$ connecting points $x_1 \in M^0_{g,k}, x_2 \in M^1_{g,k}$, and intersecting S, and therefore also Q', at a unique point. By Corollary 3.1 we obtain that Q' separates $M^0_{g,0}$ from $M^1_{g,0}$ in $P_{g,0}$, therefore Q separates the manifolds $M^0_{g,k}$ and $M^1_{g,k}$ in $P_{g,k}$.

Theorem 3.1. *Let Q be a connected orientable 2-manifold of genus $r \geqslant 0$ embedded in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$, $r < g$. Then Q does not separate the manifolds $M^0_{g,k}$ and $M^1_{g,k}$ in $P_{g,k}$.*

Proof. First, we carry out the proof for $k = 0$ using induction on the genus g, starting from $g = 1$. Since Q is properly embedded in $P_{g,0} \setminus (M^0_{g,0} \cup M^1_{g,0})$, it too has no boundary and is a closed manifold. For $g = 1$ the manifold $P_{g,0}$ is a direct product of a 2-torus and the interval $[0,1]$, that is, it is an irreducible 3-manifold (this means that each 2-sphere cylindrically embedded in $P_{1,0} \setminus (M^0_{1,0} \cup M^1_{1,0})$ bounds an open 3-ball). The only oriented closed 2-manifold of genus $r < 1$ is a 2-sphere, therefore Q is a sphere bounding a 3-ball in $P_{1,0}$. Hence, by Lemma 3.1 the theorem holds for $g = 1$.

Assume now that the theorem holds for all genera smaller than g, $g > 1$. We claim that it also holds for genus g. Let Q be an oriented closed 2-manifold of genus $r < g$ embedded in $P_{g,0} \setminus (M^0_{g,0} \cup M^1_{g,0})$. Consider a closed smooth curve $c \subset M_{g,0}$ not partitioning $M_{g,0}$ and the two-dimensional annulus $K_0 = c \times [0,1]$. After a small perturbation of Q we can assume that the intersection $Q \cap K_0$ is either empty or consists of p smooth disjoint closed curves c_i, $i = 1, \dots, p$. If the intersection $Q \cap K_0$ is empty or all the curves c_i are homotopic to zero in the annulus K_0, then there exists a simple arc with interior in K_0 and end-points in distinct components of the boundary of the annulus K_0. Thus means that the surface Q does not separate $M^0_{g,0}$ from $M^1_{g,0}$, which proves Theorem 3.1 for $k = 0$.

Assume now that $Q \cap K_0 \neq \varnothing$ and among the curves c_i there exists at least one curve (for instance, c_1) not homotopic to zero in K_0.

We cut the manifold $P_{g,0}$ along K_0. Since c does not partition $M_{g,0}$, this operation produces a connected 2-manifold $M_{g-1,2}$ and a connected 3-manifold $P_{g-1,2}$, which is the direct product of the manifold $M_{g-1,2}$ by the interval $[0,1]$, with boundary containing two annuli K_1 and K_2 corresponding to the annulus K_0.

Since $Q \cap K_0 \neq \varnothing$, cutting Q along K_0 we obtain a compact (in general, not necessarily connected) 2-manifold, which we denote by Q_1. We can assume that the manifold $P_{g-1,2}$ is a part of a manifold $P_{g-1,0}$, where $M_{g-1,2} \subset M_{g-1,0}$, and the set $P_{g-1,0} \setminus P_{g-1,2}$ is a disjoint union of two sets whose closures are 3-balls b_1 and b_2 intersecting $P_{g-1,2}$ along K_1 and K_2, respectively.

Let c_j^i, $j = 1, \dots, p$, be curves on K_i, $i = 1, 2$, corresponding to the c_j after cutting the manifold $P_{g,0}$ along the annulus K_0.

We now attach to all closed curves c_j^i, $j = 1, \dots, p$, disjoint two-dimensional discs d_j^i lying in the interior of the ball b_i (Fig. 1). This produces a closed orientable surface $Q' = Q_1 \cup \bigcup_{j=1}^{p} d_j^i$ (we emphasize that Q' can consist of several disjoint closed connected orientable 2-manifolds $Q'_1, \dots, Q'_q$ properly embedded in $P_{g-1,0}$, in spite of the fact that Q is connected). Since the curve c_1 is not homotopic to zero in K_0, it is not homotopic to zero in $P_{g,0}$. Hence c_1 is not homotopic to zero in Q, therefore the genus of each component Q'_i is smaller than the genus of Q.

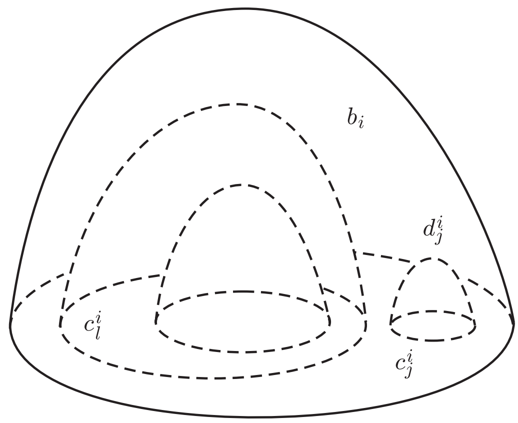

Figure 1. The discs d_j^i

By the induction hypothesis the surface Q'_i does not separate $M^0_{g-1,0}$ and $M^1_{g-1,0}$. Hence by Lemma 3.1 the surface Q'_i (for each $i \in \{1, \dots, q\}$) bounds a subdomain of $P_{g-1,0}$ and therefore $M^0_{g-1,0}$ and $M^1_{g-1,0}$ lie in the same connected component of the set $P_{g-1,0} \setminus \bigcup_{i=1}^{q} Q'_i$. Then there exists a path ω connecting points $x \in M^0_{g-1,0}$, $y \in M^1_{g-1,0}$ and disjoint from the manifold $Q' = \bigcup_{i=1}^{q} Q'_i$. Hence if $\omega \cap \operatorname{int}(b_1 \cup b_2) = \varnothing$, then the surface Q does not separate $M^0_{g,0}$ and $M^1_{g,0}$, which completes the proof of Theorem 3.1.

Assume that $\omega \cap \operatorname{int}(b_1 \cup b_2) \neq \varnothing$. Since Q' lies at a positive distance from $M^0_{g-1,0} \cup M^1_{g-1,0}$, we can assume without loss of generality that $x \in M^0_{g-1,2}$ and $y \in M^1_{g-1,2}$. We can also assume without loss of generality that ω intersects the annuli K_i transversally at the entrance and the exit from the balls b_i, $i = 1, 2$. Let $(x_1, x_2) \subset \omega \cap (b_1 \cup b_2)$ be the first arc in ω on the way from x to y lying in $\operatorname{int}(b_1 \cup b_2)$, with end-points in the boundary of $b_1 \cup b_2$. We shall assume for definiteness that $(x_1, x_2) \subset \omega \cap \operatorname{int} b_1$, $x_1, x_2 \in K_1$. Since the discs d_j^1, $j = 1, \dots, p$, are disjoint, (x_1, x_2) lies in one connected component of $b_1 - \bigcup_{j=1}^{p} d_j^1$, say, K_{12}. Hence the points x_1 and x_2 also lie in the same component $C_{12} \subset \operatorname{clos} K_{12} \cap K_1$ of the set $K_1 - \bigcup_{j=1}^{p} c_j^1$. Thus, we can replace (x_1, x_2) by an arc $(x_1, x_2)' \subset C_{12}$ with the same end-points x_1, x_2.

In the neighbourhood of x_1 and x_2 on the arc ω we select points y_1 and y_2 lying outside $b_1 \cup b_2$ and join then by a smooth arc (y_1, y_2) lying in the neighbourhood of $(x_1, x_2)'$ outside $\operatorname{clos}(b_1 \cup b_2)$ and disjoint from Q'. Carrying out the same construction with all components of the set $\omega \cap \operatorname{int}(b_1 \cup b_2)$ we obtain an arc

$$h \subset P_{g-1,0} \setminus \operatorname{int}(b_1 \cup b_2)$$

connecting points x and y. Hence in $P_{g,0}$ the arc h connects the points $x \in M^0_{g,0}$ and $y \in M^1_{g,0}$ without intersecting Q. Thus, Theorem 3.1 is proved in the case when $M_{g,0}$ has no boundary.

If $M_{g,k}$ is a manifold with boundary then we can assume that it is a part of the manifold $M_{g,0}$ and therefore $P_{g,k}$ is a part of the manifold $P_{g,0}$ that is a direct product of $M_{g,0}$ and the interval $[0, 1]$.

If Q is a manifold with boundary of genus r lower than g, then its boundary consists of finitely many simple disjoint curves. Since the manifold Q is properly embedded, we can attach to each component of its boundary a smooth 2-disc lying in $P_{g,0} \setminus P_{g,k}$. This produces a closed oriented 2-manifold Q' of genus r smoothly embedded in $P_{g,0}$. By the above there exists a path connecting points in distinct components of the boundary of $P_{g,0}$ and disjoint from Q'. Using now the above arguments on the modification of the path ω we construct another path connecting a pair of points, one in $M^0_{g,k}$ and the other in $M^1_{g,k}$, and disjoint from Q. The proof of Theorem 3.1 is now complete.

For the proof of Theorem 3.2 we require several technical results on embeddings of 2-dimensional annuli in $P_{g,k}$ for $g \geqslant 1$. Here, as before, all embeddings of the annuli under consideration are assumed to be proper.

On the surface $M_{g,k}$ we take two closed curves c and e, each not partitioning $M_{g,k}$ and intersecting at a unique point x. Since $g \geqslant 1$, such curves exist. Let K_e (respectively, K_c) be an annulus of the form $e \times [0, 1] \subset P_{g,k}$ (respectively, of the form $K_c = c \times [0, 1] \subset P_{g,k}$) with boundary curves $e_0 = e \times \{0\}$ and $e_1 = e \times \{1\}$ (respectively, $c_0 = c \times \{0\}$ and $c_1 = c \times \{1\}$). We set $x_0 = x \times \{0\}$, $x_1 = x \times \{1\}$, and $A_x = K_c \cap K_e$.

Lemma 3.2. *Let K be a closed annulus embedded in $P_{g,k}$ so that it is bounded by the closed curves c_0, c_1 and intersects transversally the annuli K_c and K_e. Then there exists a diffeomorphism $f\colon P_{g,k} \to P_{g,k}$ homotopic to the identity on $M^0_{g,k} \cup M^1_{g,k}$ such that*

(1) *the annulus $f(K)$ intersects transversally the annuli K_c and K_e;*
(2) *the set $\gamma = f(K) \cap K_c$ consists of several closed curves non-homotopic in K_c, each intersecting A_x at a unique point;*
(3) *the intersection $\operatorname{int} f(K) \cap \operatorname{int} K_e$ is precisely one simple arc $a(K_e, f(K))$ with end-points x_0 and x_1.*

Proof. By the assumptions of the lemma K is transversal to K_c, therefore the set $\operatorname{int} K \cap \operatorname{int} K_c$ is a finite (possibly empty) system of simple closed curves. If some curves in this system are non-homotopic to zero in K_c, then there exists an orientation-preserving diffeomorphism $h_0\colon K_c \to K_c$ isotopic to the identity diffeomorphism and actually equal to the identity on $c_0 \cup c_1$ that takes each closed curve in the set $\operatorname{int} K \cap \operatorname{int} K_c$ non-homotopic to zero in K_c to a closed curve intersecting A_x at a unique point. Hence the diffeomorphism h_0 can be extended to an orientation-preserving diffeomorphism $h_1\colon P_{g,k} \to P_{g,k}$ such that

(1) h_1 coincides with h_0 on K_c;
(2) the diffeomorphism h_1 is the identity outside a small neighbourhood of the annulus K_c and on the set $M^0_{g,k} \cup M^1_{g,k}$;
(3) the annulus $h_1(K)$ intersects transversally the annuli K_c and K_e.

In the rest of the proof of the lemma, for convenience we denote the annulus $h_1(K)$ by K again.

If the set $\gamma = K \cap K_c$ contains closed curves bounding two-dimensional discs in K_c, then we choose a closed curve $s \subset \gamma$ such that the open 2-disc $D_c \subset K_c$ bounded by s contains no points in γ.

The closed curve s also bounds an open 2-disc D on K. (Otherwise s is not homotopic to zero in K and therefore not homotopic to zero in $P_{g,k}$, which is impossible.)

Hence $D_c \cup s \cup D$ is a piecewise smooth 2-sphere $S^2_1 \subset P_{g,k}$. By the irreducibility of $P_{g,k}$ the sphere S^2_1 bounds in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ an open 3-ball b_1 with closure $\operatorname{clos} b_1$ that is a closed 3-ball. Obviously, $\operatorname{clos} b_1 \cap (M^0_{g,k} \cup M^1_{g,k}) = \varnothing$, therefore there exists a diffeomorphism $f_1\colon P_{g,k} \to P_{g,k}$ that is the identity in the neighbourhood of the set $\operatorname{clos} b_1$ and is such that the intersection $\operatorname{int} f_1(K) \cap \operatorname{int} K_c$ contains at least one closed curve fewer (the diffeomorphisms $f_1\colon P_{g,k} \to P_{g,k}$ can be conceived as pressing the ball b_1 through the disc D_c). Continuing this process we finally obtain a diffeomorphism $f_2\colon P_{g,k} \to P_{g,k}$ such that

(1) the annulus $f_2(K)$ intersects K_c and K_e transversally;
(2) the set $f_2(K) \cap K_c$ consists of closed curves in K_c non-homotopic to zero, each intersecting A_x at a unique point.

If the intersection $\operatorname{int} f_2(K) \cap \operatorname{int} K_e$ consists of precisely one arc with endpoints x_0 and x_1, then we can set $f = f_2$ and the proof is complete.

Assume that this is not the case and denote $f_2(K)$ by K once again.

The annulus K intersects K_e transversally and the boundaries of these annuli intersect at two points. Hence the intersection $K \cap K_e$ consists of several closed curves and an open arc $a(K_e, K)$.

Since the closure of $a(K_e, K)$ connects distinct components of the boundaries of K_e and K, cutting both annuli along $\operatorname{clos} a(K_e, K)$ converts them into discs. Hence all closed curves in $\operatorname{int} K \cap \operatorname{int} K_e$ are homotopic to zero in both K_e and K. Similarly to the case when $K_c \cap K$ consists of closed curves homotopic in K_c we can construct a diffeomorphism $f_3\colon P_{g,k} \to P_{g,k}$ such that the set $f_3(K) \cap K_c$ coincides with $K \cap K_c$ and the set $f_3(K)$ consists of a single arc $a(K_e, K)$. Setting $f = f_3$ we obtain the statement of the lemma.

Lemma 3.3. *Let K be an annulus embedded in $P_{g,k}$ with boundary curves c_0 and c_1. Then there exists a diffeomorphism of the manifold $P_{g,k}$ onto itself which is the identity on $M^0_{g,k} \cup M^1_{g,k}$ and takes K into K_c.*

Proof. We can assume without loss of generality that K intersects transversally the annuli K_c and K_e. Then by Lemma 3.2 we can also assume that

(1) the intersection $\operatorname{int} K \cap \operatorname{int} K_e$ consists of one arc $a(K_e, K)$;

(2) the intersection $\operatorname{int} K \cap \operatorname{int} K_c$ consists only of closed simple curves not homotopic to zero in K_c and intersecting each A_x at a unique point.

We consider now the case $g = 1, k = 0$, that is, the manifold $P_{1,0}$, which is the direct product of a 2-torus and a closed interval.

Consider a homeomorphism $h_0 \colon M^0_{1,0} \cup M^1_{1,0} \cup K \to M^0_{1,0} \cup M^1_{1,0} \cup K_c$ such that

(1) h_0 is identical on the set $M^0_{1,0} \cup M^1_{1,0}$;
(2) the restriction of h_0 to the annulus K is a diffeomorphism $h_0 \colon K \to K_c$;
(3) $h_0(a(K_e, K)) = A_x$.

Let Q_c (respectively, Q) be the manifold with boundary obtained by cutting $P_{1,0}$ along the annulus K_c (along K). By construction the boundary of Q_c (of Q) is a 2-torus Γ_c (respectively, Γ). In addition, there exists a 2-dimensional disc D_c (respectively, D) obtained from the annulus K_e by cutting it along K_c (along K) whose boundary ∂D_c (respectively, ∂D) is a meridian, that is, a simple closed curve not homotopic to zero on the torus Γ_c (on Γ), but homotopic to zero in Q_c (in Q). Since $P_{1,0}$ is an irreducible manifold, it now follows that Q_c (respectively, Q) is a solid torus, that is, a 3-manifold with boundary homeomorphic to the direct product of the two-dimensional disc and a circle.

The homeomorphism h_0 induces a homeomorphism $\widetilde{h}_0 \colon \Gamma \to \Gamma_c$ transforming the meridian ∂D into ∂D_c. Hence $\widetilde{h}_0$ can be extended to a homeomorphism $\widetilde{h}_1 \colon Q \to Q_c$ that is a diffeomorphism on $\operatorname{int} Q$. By construction $\widetilde{h}_1$ induces a diffeomorphism $h_1 \colon P_{0,1} \to P_{0,1}$ coinciding with h_0 on $M^0_{1,0} \cup M^1_{1,0} \cup K$. This proves the lemma in the case $g = 1$, $k = 0$.

Consider now the remaining case when g, k is a pair distinct from $g = 1$, $k = 0$.

We start from the picture when the curves $\operatorname{clos} a(K_e, K)$ and $\operatorname{clos} A_x$ intersect only at the end-points $x_0 = \{x\} \times 0$ and $x_1 = \{x\} \times 1$.

Then the annuli K and K_c intersect only along their boundary curves $c_0 = c \times 0$ and $c_1 = c \times 1$. Hence the union $K \cup K_c$ is a two-dimensional torus, which we denote by T.

Two subcases are now possible:

(a) the simple closed curve $S = \operatorname{clos} a(K_e, K) \cup \operatorname{clos} A_x$ is homotopic to zero in K_e;
(b) the curve S is not homotopic to zero in K_e.

In case (a) the curve S bounds on K_e a disc d and is non-homotopic to zero on the torus T (since $\operatorname{clos} a(K_e, K)$ (respectively, $\operatorname{clos} A_x$) does not partition the annulus K_e (respectively, K)). Hence the torus T bounds a solid torus in $P_{g,k}$ and therefore there exists a diffeomorphism homotopic to the identity that is the identity on $M^0_{g,k} \cup M^1_{g,k}$ and transforms K into K_c. The proof of the lemma in this case is complete.

In case (b) there exists a simple open arc $\gamma \subset \operatorname{int} K_e$ with end-points γ_0 and γ_1 such that:

(1) $\gamma_0 \in e_0$ and $\gamma_0 \neq x_0$, $\gamma_1 \in e_1$ and $\gamma_1 \neq x_1$;
(2) the arc γ and the torus T have precisely one common point.

It follows by the transversality of the intersection of K and K_c that there exists in a tubular half-neighbourhood of the torus T a smooth 2-torus $\widetilde{T}$ properly embedded in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ such that the arc γ intersects $\widetilde{T}$ at a unique point.

By Corollary 3.1 the torus $\widetilde{T}$ separates in $P_{g,k}$ the surfaces $M^0_{g,k}$ and $M^1_{g,k}$. Then, however, Theorem 3.1 yields $g = 1, k = 0$, which is impossible.

It remains to consider the case when $\operatorname{int} a(K_e, K)$ intersects A_x. Let $S_A \subset A_x$ be a closed arc intersecting the closure of $a(K_e, K)$ precisely at the end-points and let $S_K \subset a(K_e, K)$ be an arc with the same end-points as S_A. Then the union $S_1 = S_K \cup S_A$ is a simple closed curve in the annulus K_e. Passing through the end-points of the arcs S_K and S_A there are two closed curves $C_1, C_2 \subset K \cap K_c$ not homotopic to zero in either K or K_c. Hence they bound annuli $A_K \subset K$ and $A_c \subset K_c$. The intersection $K \cap K_c$ contains no curves homotopic to zero in the annuli K and K_c, and the arcs $S_K \subset a(K_e, K)$ and $S_A \subset A_x$ do not intersect at their interior points, therefore the annuli $A_K \subset K$ and $A_c \subset K_c$ intersect precisely along their boundaries. Hence the union $A_K \cup A_c = T_A$ is a 2-torus. Since the arcs S_K and S_A do not partition the annuli A_K and A_c, respectively, the curve S_1 is not homotopic to zero in T_A.

Two cases are now formally admissible:

(a_1) the curve S_1 is homotopic to zero in K_e;

(b_1) the curve S_1 is not homotopic to zero in K_e.

As before (in the discussion of case (b)) one demonstrates that (b_1) cannot occur.

In case (a_1) the curve S_1 bounds a disc $d_1 \subset K_e$ and therefore the torus T_A bounds a solid torus in $P_{g,k}$. Hence there exists a deformation of the annulus A_K into A_c across this solid torus such that the arc S_K is deformed into S_A across the disc d_1. As a result, the annulus K is deformed into an annulus K' whose intersection P with K_c contains the annulus A_c; at the remaining points P coincides with the intersection of K and K_c. Since S_A and S_K have common end-points, K' can be deformed into an annulus K_1 disjoint from K_c along the annulus A_c. In addition, one can avoid the occurrence of new points of intersection. Then the resulting annulus K_1 has fewer curves in its intersection with K_c than K, at least by 1. Accordingly, the curve $a(K_e, K_1)$ intersects A_x in fewer points than $a(K_e, K)$.

Continuing this process we obtain an annulus $h(K)$ produced by some diffeomorphism $h\colon P_{g,k} \to P_{g,k}$ that is the identity on the set $M^0_{g,k} \cap M^1_{g,k}$ and intersects K_c only along the closed curves c_0 and c_1. This case has been already considered, therefore the proof of the lemma is complete.

This lemma has an important consequence.

Corollary 3.3. *Let K be an annulus embedded in $P_{g,k}$ with boundary curves c_0 and c_1. Then after cutting the manifold $P_{g,k}$ along the annulus K one obtains a manifold $P_{g-1,k+2}$ that is the direct product of the 2-manifold $M_{g-1,k+2}$ by the closed interval $[0,1]$.*

Lemma 3.4. *Let Q be a connected compact orientable 2-manifold of genus $g \geqslant 1$ embedded in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$, with boundary ∂Q consisting of k closed curves, separating the surfaces $M^0_{g,k}$ and $M^1_{g,k}$. Let K be an annulus with boundary curves c_0 and c_1 embedded in $P_{g,k}$ so that $K \cap Q$ consists of finitely many simple closed curves not homotopic to zero in Q. Then all these curves are not homotopic to zero in K either.*

Proof. We start with one important observation. Since Q is properly embedded, separates $M^0_{g,k}$ from $M^1_{g,k}$, and the number of the components of the boundary of Q

is equal to that of the boundary of $M_{g,k}$, each annulus $C_i \times [0,1] \subset \mathcal{K}_k$ contains precisely one closed curve $\widetilde{C}_i$ in ∂Q homotopic to C_i (recall that we denote by C_i, $i = 1, \dots, k$, all closed curves in $\partial M_{g,k}$).

We proceed to the proof of the lemma. Assume the contrary. Then $K \cap Q$ contains a curve s bounding on K a 2-dimensional disc d_K disjoint from $K \cap Q$. Selecting a sufficiently small neighbourhood of the curve s we can find an open annulus K_s in this neighbourhood that contains s, is disjoint from the curves in $Q \cap K$ distinct from s, and has the following property: after cutting this annulus away from Q the resulting holes can be sealed by disjoint 2-dimensional discs d_1 and d_2. Let Q' be the surface obtained after the sealing and let g' be the highest genus of the connected components of the surface Q'. We claim that $g' \leqslant g - 1$. For assume the contrary: let $g' = g$. Then Q' decomposes into two connected components, one of which (we denote it by Q'_1) is a plane 2-manifold with boundary in $\bigcup_{i=1}^{k} \widetilde{C}_i$. Then, however, Q'_1 has genus 0 and by Corollary 3.2 separates the surfaces $M^0_{g,k}$ and $M^1_{g,k}$ in $P_{g,k}$. Since $g > 0$, we arrive at a contradiction with Theorem 3.1, which shows that $g' \leqslant g - 1$.

By Theorem 3.1 none of the connected components of Q' separates $M^0_{g,k}$ and $M^1_{g,k}$ in $P_{g,k}$, that is, there exists a path ω from $M^0_{g,k}$ to $M^1_{g,k}$ disjoint from Q'. By the assumptions of the lemma Q separates the boundary components $M^0_{g,k}$ and $M^1_{g,k}$. Hence $\omega \cap K_s \neq \varnothing$. The set $d_1 \cup d_2 \cup K_s$ is a 2-sphere bounding a closed ball $B \subset P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$. Hence the part of ω lying in B can be replaced by a path lying in a small neighbourhood of K_s and disjoint from B and Q. Then, however, ω is disjoint from Q, which contradicts the hypothesis of the lemma.

The next result is a fairly simple fact in two-dimensional topology; however, we present its proof for completeness.

Lemma 3.5. *Let Q be an orientable 2-manifold of genus $g \geqslant 1$ with boundary consisting of $k \geqslant 0$ simple closed curves, and let $s_1, \dots, s_m$, $m \geqslant 3$, be a system of simple closed curves on Q satisfying the following conditions*:

(1) $s_i \cap s_j = \varnothing$ *for* $i \neq j$;
(2) $s_i \cap \partial Q = \varnothing$ *for each* $i \in \{1, \dots, m\}$;
(3) s_i *is not homotopic to zero on Q for each* $i \in \{1, \dots, m\}$;
(4) *there exists a connected component of the set $Q \setminus \bigcup_{1=1}^{m} s_i$ of genus at least $g - 1$ containing for $k \neq 0$ the boundary ∂Q of the manifold Q.*

Then there exist at least two curves in the set $s_1, \dots, s_m$ bounding a connected component of the set $Q \setminus \bigcup_{1=1}^{m} s_i$ homeomorphic to an open 2-dimensional annulus.

Proof. If the system $s_1, \dots, s_m$ contains a pair of curves bounding a 2-dimensional annulus K, then the set $K \setminus \bigcup_{1=1}^{m} s_i$ is a union of open annuli because the s_i are not homotopic to zero in Q. Hence the lemma holds in this case.

Assume now the contrary: no pair of closed curves in the system $s_1, \dots, s_m$ is the boundary of a 2-dimensional annulus. First, we consider the case of a system $s_1, \dots, s_m$ containing at least one closed curve (for instance, s_1) not partitioning Q. By the assumptions of the lemma there exists a connected component of $Q \setminus \bigcup_{1=1}^{m} s_i$ of genus higher than $g - 2$ that contains the boundary of Q. Hence $Q \setminus (s_1 \cup s_2)$ decomposes into two components, Q_1 and Q_2, such that Q_1 has genus higher than

$g-2$ and contains the boundary of Q, while Q_2 is a plane domain bounded by the curves s_1 and s_2. In view of our assumption (that the contrary holds), Q_2 is a triply connected domain.

By the hypothesis of the lemma s_3 is not homotopic to zero in Q and by our assumption to the contrary it does not bound a 2-dimensional annulus in conjunction with s_1 or s_2. Hence s_3 lies in Q_1. Then, however, it follows by the assumption of the lemma that one of the components of $Q_1 \setminus s_3$ has genus $g-1$ and contains the boundary of Q. Hence s_2 and s_3 bound a 2-dimensional annulus in Q_1, a contradiction.

It remains to consider the case when each closed curve s_i, $i \in \{1,\dots,m\}$, partitions Q into two components. Then $Q \setminus s_1$ decomposes into components Q_1 and Q_2 such that Q_1 has genus $g-1$ and contains the boundary of the manifold Q, and Q_2 has genus 1 and its boundary has only one component, the curve s_1.

Let Q_* be the one of the components Q_1 and Q_2 containing s_2. Then s_2 partitions Q_* into two components, one of them plane and not containing components of the boundary of the original manifold Q. In that case, however, s_1 and s_2 bound a two-dimensional annulus in Q_*. We arrive at a contradiction again, which completes the proof.

Lemma 3.6. *Let Q be a connected compact orientable 2-manifold of genus $g \geqslant 1$ embedded in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$ and separating the surfaces $M^0_{g,k}$ and $M^1_{g,k}$, with boundary ∂Q consisting of k closed curves. Then there exists an annulus K with boundary curves c_0 and c_1 that is embedded in $P_{g,k}$ so that K is transversal to Q and the intersection $K \cap Q$ consists precisely of a unique simple closed curve not partitioning Q and not homotopic to zero in K.*

Proof. At the first step we modify the annulus K_c in a small neighbourhood of the intersection $Q \cap K_c$ so that the resulting annulus K_0 has the same boundary as K_c and intersects Q transversally. Then the intersection $Q \cap K_0$ consists of finitely many closed curves.

We claim that by deforming K_0 we can obtain another annulus K_1 whose intersection with Q contains no closed curves homotopic to zero in Q.

In fact, assume that $Q \cap K_0$ contains closed curves homotopic to zero in Q. Then there exists a curve $s \subset Q \cap K_0$ bounding a disc d_Q on Q with interior disjoint from $Q \cap K_0$. The curve s bounds a disc d_K on K_0 (for otherwise s is not homotopic to zero in K_0, and therefore in $P_{g,k}$, which is impossible because s bounds a disc d_Q).

Since $g \geqslant 1$, the manifold $P_{g,k}$ is irreducible. The union $d_Q \cup d_K \cup s$ is a cylindrically embedded sphere, therefore it bounds a ball in $P_{g,k}$ (with interior disjoint from the annulus K_0). Hence there exists a deformation of K_0 identical on its boundary that transforms it into another annulus, whose intersection with Q contains one curve fewer. Continuing in this manner we obtain an annulus K_1 such that $K_1 \cap Q \neq \varnothing$ and $K_1 \cap Q$ consists of finitely many simple closed curves $s_1, \dots, s_m$ not homotopic to zero in Q. By Lemma 3.4 all these curves are not homotopic to zero in K_1 either.

We cut the manifold $P_{g,k}$ along the annulus K_1. Then by Corollary 3.3 we obtain a connected manifold $P_{g-1,k+2}$ that is a direct product of the manifold $M_{g-1,k+2}$ by the closed interval $[0,1]$. Corresponding to K_1 there are annuli $K^1_1, K^2_1 \subset \partial P_{g-1,k+2}$, and corresponding to the sets $C_i \times [0,1]$ in $\mathcal{K}_k$ there are subsets of $\partial P_{g-1,k+2}$, which we denote by H_i, $i=1,\dots,k$.

Assume first that $K_1 \cap Q$ consists of precisely one curve (that is, $m=1$). We claim that the closed curve $s_1 = Q \cap K_1$ does not partition Q. Assume the contrary. Then s_1 partitions Q into two components and after cutting $P_{g,k}$ along the annulus K_1 embedded in $P_{g-1,k+2}$ one obtains two disjoint 2-manifolds Q_1 and Q_2 corresponding to the components of $Q \setminus s_1$. Corresponding to the curve s_1 there will be the curves $t_i = Q_i \cap K_1^i$, $i=1,2$.

By construction t_1 is the unique closed curve in the boundary of Q_1 that is homotopic to a boundary component of the annulus K_1^1. Hence by Corollary 3.2 the surface Q_1 separates in $P_{g-1,k+2}$ the manifolds $M^0_{g-1,k+2}$ and $M^1_{g-1,k+2}$. Since $Q_1 \cap K_1^2 = \varnothing$, there exists a path in K_1^2 connecting points in $M^0_{g-1,k+2}$ and $M^1_{g-1,k+2}$ and disjoint from Q_1. This is a contradiction establishing the lemma for $m=1$.

Consider now the case $m>1$. Since Q separates the manifolds $M^0_{1,k}$ and $M^1_{1,k}$ in $P_{g,k}$ by assumption, it follows that $m \geqslant 3$ (if m is 2, then there exists a path in the annulus K_1 connecting points in the manifolds $M^0_{g,k}$ and $M^1_{g,k}$ and intersecting Q only at two points, which is impossible).

We claim that at least one component of the set $Q \setminus \bigcup_{i=1}^m s_i$ is homeomorphic to an open 2-dimensional annulus and is bounded by two curves from the system $s_1,\dots,s_m$.

By Lemma 3.5 it is sufficient to show that the set $Q \setminus \bigcup_{i=1}^m s_i$ has at least one connected component of genus at least $g-1$, containing in addition the boundary ∂Q of the manifold Q if $k \neq 0$.

By hypothesis the manifold Q separates $M^0_{g,k}$ and $M^1_{g,k}$, therefore by Corollary 3.2 after cutting $P_{g,k}$ along the curve K_1 one obtains a manifold Q_1 of genus g_1 corresponding to a connected component of the set $Q \setminus \bigcup_{i=1}^m s_i$ that is smoothly embedded in $P_{g-1,k+2}$ and separates $M^0_{g-1,k+2}$ and $M^1_{g-1,k+2}$. Hence by Theorem 3.1 we obtain the inequality $g-1 \leqslant g_1 \leqslant g$.

We shall now show that if $k \neq 0$, then the boundary of the surface Q lies entirely in the component Q_1.

Assume the contrary, that is, assume that the boundary of Q lies in distinct connected components of $Q \setminus \bigcup_{i=1}^m s_i$. Then there exists an index $i \in \{1,2,\dots,k\}$ such that $Q_1 \cap H_i = \varnothing$. Then, however, there exists a path on H_i connecting points in the manifolds $M^0_{0,k+2}$ and $M^1_{0,k+2}$, but disjoint from Q_1, which is impossible.

Thus, the set $Q \setminus \bigcup_{i=1}^m s_i$ has at least one component (say, A) homeomorphic to an open annulus, with boundary curves in the family $s_1, \dots, s_m$. We shall assume for definiteness that A is bounded by s_1 and s_2. These curves bound an annulus in K_1 (say, A_*) because s_1 and s_2 are not homotopic to zero in K_1 either. Replacing A_* by A we obtain an annulus K_1' intersecting Q in the annulus A and having a common boundary with K_1. We can now deform K_1' in the neighbourhood of A so that the resulting annulus K_1'' intersects Q transversally and contains one or two curves fewer in its intersection with Q.

Continuing in this manner we obtain the required annulus K with boundary curves c_0 and c_1 that intersects Q in precisely one simple closed curve non-homotopic to zero in K. It follows from the above that this curve does not partition Q. The proof of the lemma is now complete.

Theorem 3.2. *Let Q be a smooth connected compact orientable 2-manifold of genus $g \geqslant 0$ embedded in $P_{g,k} \setminus (M^0_{g,k} \cup M^1_{g,k})$, $k \geqslant 0$, with boundary ∂Q consisting of k closed curves. Assume that Q separates 2-manifolds $M^0_{g,k}$ and $M^1_{g,k}$ in $P_{g,k}$. Then the set $P_{g,k} \setminus Q$ consists of two components and the closure of each component is homeomorphic to $P_{g,k}$.*

Proof. For $i = 0, 1$ we denote by $A^i_{g,k}$ the closure of the connected component of the set $P_{g,k} \setminus Q$ whose boundary contains $M^i_{g,k}$.

We prove the theorem by induction on the genus $g \geqslant 0$. We start with $g = 0$ and the case when $k = 0$. Then for $g = 0$ and each $k \geqslant 1$ we prove the theorem by induction on k.

For $k = 0$ the manifolds $M^0_{0,0}$ and $M^1_{0,0}$ are 2-spheres. Moreover, we can assume without loss of generality that $P_{0,0}$ is a closed standard 3-dimensional annulus (that is, a closed subdomain in Euclidean 3-space bounded by concentric 2-spheres). Since the 2-sphere Q separates $M^0_{0,0}$ and $M^1_{0,0}$, it partitions the standard 3-dimensional annulus into two domains such that the boundary of each domain contains precisely one of the manifolds $M^0_{0,0}$ and $M^1_{0,0}$. Since Q is a sphere cylindrically embedded in $P_{0,0}$, the closures $A^i_{0,0}$, $i = 0, 1$, of these domains are homeomorphic to a closed standard 3-dimensional annulus, that is, to $P_{0,0}$ (see, for example, [19]).

We shall now show that for each $k \geqslant 1$ and $g = 0$ the set $P_{0,k} \setminus Q$ consists of two components (with closures $A^0_{0,k}$ and $A^1_{0,k}$) and there exists a homeomorphism $\psi_{i,0,k}\colon A^i_{0,k} \to P_{0,k}$ such that $\psi_{i,0,k}\big|_{M^i_{0,k}} = \mathrm{id}$, $\psi_{i,0,k}(Q) = M^j_{0,k}$, where $i = 0, 1$, $j \neq i$.

For $k = 1$ the manifold $P_{0,1}$ is a closed 3-ball and Q is a smooth 2-dimensional disc intersecting transversally the boundary of $P_{0,1}$ in a smooth simple closed curve l. Note that the boundary of $P_{0,1}$ is homeomorphic to the 2-sphere that is the union of the discs $M^0_{0,1}$, $M^1_{0,1}$ and a cylinder. Thus, l partitions the boundary of $P_{0,1}$ into two discs such that the union of each disc and Q is a piecewise smooth sphere embedded in $P_{0,1}$. Hence Q partitions $P_{0,1}$ into two closed balls $A^0_{0,1}$ and, $A^1_{0,1}$. For each $i = 0, 1$ we can now construct in the standard way a homeomorphism $\psi_{i,0,1}\colon A^i_{0,1} \to P_{0,1}$ such that $\psi_{i,0,1}\big|_{M^i_{0,1}} = \mathrm{id}$, $\psi_{i,0,1}(Q) = M^j_{0,1}$, $j \neq i$.

Assume now that $k > 1$ and that for all l such that $0 < l < k$, and $i = 0, 1$ the set $P_{0,l} \setminus Q$ consists of two components (with closures $A^0_{0,l}$ and $A^1_{0,l}$) and there exists a homeomorphism $\psi_{i,0,l}\colon A^i_{0,l} \to P_{0,l}$ such that $\psi_{i,0,l}\big|_{M^i_{0,l}} = \mathrm{id}$ and $\psi_{i,0,l}(Q) = M^j_{0,l}$, where $i = 0, 1$, $j \neq i$. We claim that all these results hold also for $l = k$.

Consider the manifold $P_{0,k}$, $k \geqslant 2$. Let x and y be arbitrary points in distinct connected components of the boundary of $M^0_{0,k}$, and let $h_0 \subset M^0_{0,k}$ be a simple smooth arc with end-points x and y disjoint from other components of the boundary of $M^0_{0,k}$.

We consider the closed contour γ_0 formed by h_0 and the three arcs $h_x = x \times [0,1]$, $h_1 = h_0 \times 1$ ($h_1 \subset M^1_{0,k}$), and $h_y = y \times [0,1]$. We set $D_0 = h_0 \times [0,1]$. By hypothesis the boundary of D_0 is γ_0. By hypothesis the set $Q \cap \mathcal{K}_k$ consists of precisely k closed curves, and the surface Q separates $M^0_{0,k}$ and $M^1_{0,k}$, therefore each curve in $Q \cap \mathcal{K}_k$ lies in precisely one annulus in the set $\mathcal{K}_k$ and is not homotopic to zero there. Let K_x and K_y be the annuli in $\mathcal{K}_k$ containing the points x and y, and let c_x

and c_y be the components of the boundary of Q lying in K_x and K_y, respectively. By construction the intersections $h_x \cap c_x$ and $h_y \cap c_y$ are non-empty. If these intersections are transversal and consist of a single point each, then we set $Q_0 = Q$, but if one of these conditions fails, then we set $Q_0 = F_0(Q)$, where $F_0\colon P_{0,k} \to P_{0,k}$ is a diffeomorphism isotopic to the identity such that F_0 fixes the points in the set $M^0_{0,k} \cup M^1_{0,k}$, the intersections $h_x \cap F_0(c_x)$ and $h_y \cap F_0(c_y)$ are transversal and consist of a single point each. We now prove the existence of a diffeomorphism F_0. Since the curves c_x and c_y are not homotopic to zero in the annuli K_x and K_y, respectively, there exist orientation-preserving diffeomorphisms $f_x\colon K_x \to K_x$ and $f_y\colon K_y \to K_y$ fixing the points in the boundary of K_x or K_y, respectively, such that the intersections $f_x(c_x) \cap h_x$ and $f_y(c_y) \cap h_y$ are transversal and consist of a single point each. Since the diffeomorphisms f_x and f_y are isotopic to the identity diffeomorphisms on K_x and K_y, respectively, there exists a diffeomorphism F_0 isotopic to the identity such that F_0 fixes the points in the set $M^0_{0,k} \cup M^1_{0,k}$ and coincides with f_x and f_y on the sets K_x and K_y, respectively. Then F_0 is the required diffeomorphism by construction.

Next, there exists a diffeomorphism $F_1\colon P_{0,k} \to P_{0,k}$ equal to the identity outside a small neighbourhood of the set $D_0 \cap Q_0$ such that the 2-manifold $Q_1 = F_1(Q_0)$ intersects the 2-dimensional disc D_0 transversally along one arc and several closed curves. Deforming Q_1 we now remove these closed curves from the intersection in the standard fashion. Formally, this means that there exists a diffeomorphism $F_2\colon P_{0,k} \to P_{0,k}$ such that $F_2(Q_1)$ intersects D_0 transversally along one arc d_0. We set $Q_2 = F_2(Q_1)$.

Let $M^i_{0,k-1}$ be the 2-manifold with boundary obtained by cutting the manifold $M^i_{0,k}$ along the curve h_i, $i = 0, 1$, let $P_{0,k-1}$ be the 3-manifold obtained by cutting $P_{0,k}$ along the 2-dimensional disc D_0, and Q_3 the 2-manifold obtained by cutting the manifold Q_2 along d_0. Note that, by construction, the manifold $P_{0,k-1}$ is the direct product of a manifold $M_{0,k-1}$ and the interval $[0, 1]$ so that $M^i_{0,k-1} = M_{0,k-1} \times \{i\}$.

Next, let D_1 and D_2 be the 2-dimensional discs resulting from the cut along the disc D_0 and let $G\colon D_1 \to D_2$ be the diffeomorphism gluing D_1 and D_2 together into the disc D_0. By the induction hypothesis the set $P_{0,k-1} \setminus Q_3$ consists of two components with closures $A^i_{0,k-1}$ homeomorphic to $P_{0,k-1}$; let $\psi_{i,0,k-1}$, $i = 0, 1$, be the corresponding homeomorphisms. Let K_0 be the 2-dimensional annulus on the boundary $\mathcal{K}_{k-1}$ of the manifold $P_{0,k-1}$ containing the 2-dimensional discs D_j, $j = 1, 2$. For each $i \in \{0, 1\}$ the discs $\psi_{i,0,k-1}(D_1 \cap A^i_{0,k-1})$ and $\psi_{i,0,k-1}(D_2 \cap A^i_{0,k-1})$ are disjoint, therefore by the properties of the homeomorphism $\psi_{i,0,k-1}$ there exists an orientation-preserving homeomorphism $\phi_i\colon K_0 \to K_0$ such that

$$\phi_i(\psi_{i,0,k-1}(D_1 \cap A^i_{0,k-1})) = D_1, \qquad \phi_i(\psi_{i,0,k-1}(D_2 \cap A^i_{0,k-1})) = D_2$$

and ϕ_i is the identity on $K_0 \cap M^i_{0,k-1}$.

Since ϕ_i preserves orientation, it is isotopic to the identity map. Hence ϕ_i can be extended to an orientation-preserving homeomorphism $\Phi_i\colon P_{0,k-1} \to P_{0,k-1}$ such that

$$\Phi_i(K_0) = K_0, \qquad \Phi_i(\psi_{i,0,k-1}(D_j \cap A^i_{0,k-1})) = D_j, \quad j = 1, 2, \qquad \Phi_i\big|_{M^i_{0,k-1}} = \mathrm{id}\,.$$

Then the map

$$\theta_i\big|_{A^i_{0,k-1}\cap D_1} = \psi^{-1}_{i,0,k-1}\Phi^{-1}_i G^{-1}\Phi_i\psi_{i,0,k-1}G\big|_{A^i_{0,k-1}\cap D_1}$$

is a homeomorphism preserving the orientation of the set $A^i_{0,k-1}\cap D_1$ and θ_i can be extended from $A^i_{0,k-1}\cap D_1$ to an orientation-preserving homeomorphism $\Theta_i\colon A^i_{0,k-1}\to A^i_{0,k-1}$ coinciding with θ_i on $A^i_{0,k-1}\cap D_1$ that is the identity outside a small neighbourhood of $A^i_{0,k-1}\cap D_1$ (D_2 must be disjoint from this neighbourhood). Hence the homeomorphism $\Phi_i\psi_{i,0,k-1}\Theta_i$ takes $A^i_{0,k-1}$ to $P_{0,k-1}$ and satisfies the condition

$$\Phi_i\psi_{i,0,k-1}\Theta_i G\big|_{A^i_{0,k-1}\cap D_1} = G\Phi_i\psi_{i,0,k-1}\Theta_i\big|_{A^i_{0,k-1}\cap D_1}.$$

It follows by the construction of the manifold $P_{0,k-1}$ that the set $P_{0,k-1}\setminus Q_3$ consists of two components and the homeomorphism $\Phi_i\psi_{i,0,k-1}\Theta_i$ induces a homeomorphism $\psi_{i,0,k}$ onto $P_{0,k}$ of the closure $A^i_{0,k}$ of each of these components. This completes the proof of the theorem for $g=0$.

Assume now that for each $g\geqslant 1$ and $0\leqslant g'<g$ the set $P_{g',k}\setminus Q$ consists of two components (the closures of which we denote by $A^0_{g',k}$ and $A^1_{g',k}$) and there exists a homeomorphism $\psi_{i,g',k}\colon A^i_{g',k}\to P_{g',k}$ such that $\psi_{i,g',k}\big|_{M^i_{g',k}}=\mathrm{id}$ and $\psi_{i,g',k}(Q)=M^j_{g',k}$, where $i=0,1$, $j\neq i$. We claim that the theorem holds for genus g. By Lemma 3.6 there exists an annulus K with boundary curves $c_0=c\times 0\subset M^0_{g,k}$ and $c_1=c\times 1\subset M^1_{g,k}$ ($c\subset M_{g,k}$ is an arbitrary curve not partitioning $M_{g,k}$) intersecting Q transversally along a unique closed curve s not partitioning Q. By cutting $P_{g,k}$ along the annulus K we obtain by Corollary 3.3 a manifold $P_{g-1,k+2}$ with boundary containing components K_1 and K_2 corresponding to the annulus K. Let $G\colon K_1\to K_2$ be the sewing homeomorphism.

Since Q is not partitioned by the closed curve s, the genus of the 2-manifold Q' obtained by cutting Q also reduces by 1. By the induction hypothesis the set $P_{g-1,k+2}\setminus Q'$ consists of precisely two components with closures $A^1_{g-1,k+2}$ and $A^2_{g-1,k+2}$ homeomorphic to $P_{g-1,k+2}$ by means of homeomorphisms $\psi_{1,g-1,k+2}$ and $\psi_{2,g-1,k+2}$, respectively.

We set

$$\phi_i\big|_{A^i_{g-1,k+2}\cap K_1} = \psi^{-1}_{i,g-1,k+2}G^{-1}\psi_{i,g-1,k+2}G\big|_{A^i_{g-1,k+2}\cap K_1}.$$

It is now easy to verify that the homeomorphism ϕ_i preserves the orientation of $A^i_{g-1,k+2}\cap K_1$. Hence ϕ_i can be extended from $A^i_{g-1,k+2}\cap K_1$ to an orientation-preserving homeomorphism $\Phi_i\colon A^i_{g-1,k+2}\to A^i_{g-1,k+2}$ that coincides with ϕ_i on $A^i_{g-1,k+2}\cap K_1$ and is the identity outside a small neighbourhood of $A^i_{g-1,k+2}\cap K_1$ (the annulus K_2 must be disjoint from this neighbourhood). Then the homeomorphism $\psi_{i,g-1,k+2}\Phi_i\colon A^i_{g-1,k+2}\to P_{g-1,k+2}$ takes $A^i_{g-1,k+2}$ to $P_{g-1,k+2}$ and satisfies the condition

$$\psi_{i,g-1,k+2}\Phi_i G\big|_{A^i_{g-1,k+2}\cap K_1} = G\psi_{i,g-1,k+2}\Phi_i\big|_{A^i_{g-1,k+2}\cap K_1}.$$

Now, $\psi_{i,g-1,k+2}\Phi_i$ induces a homeomorphism of the closure of each component of the set $P_{g,k}\setminus Q$ onto $P_{g,k}$. This completes the proof of the theorem.

The next result is a consequence of Theorem 3.2.

Theorem 3.3. *Let Q be a closed orientable 2-manifold smoothly embedded in the interior of a manifold P diffeomorphic to the direct product of Q and the interval $[0,1]$ so that Q does not bound a subdomain of P. Then the closure of each component of $P \setminus Q$ is homeomorphic to P.*

§ 4. Heegaard splitting for diffeomorphisms

Theorem 4.1. *Let $f\colon M^3 \to M^3$ be a gradient-like diffeomorphism defined on a closed orientable 3-manifold M^3 such that the set of 1-dimensional separatrices of saddle periodic points of f is trivially embedded. Then M^3 can be represented by a Heegaard splitting of genus*

$$h_D = \frac{\nu(f) - \mu(f) + 2}{2}.$$

Proof. For the proof of the theorem we can assume without loss of generality that all periodic points of the diffeomorphism f are fixed. Let Ω^+ (respectively, Ω^-) be the set of all sinks (sources) and let W_1^u and W_1^s be the sets of unstable (respectively, stable) 1-dimensional manifolds of all saddle points of f; we set $\mathcal{A} = \Omega^+ \cup W_1^u$ and $\mathcal{R} = \Omega^- \cup W_1^s$. By construction the set $\mathcal{A}$ ($\mathcal{R}$) is a connected 1-dimensional continuum. Since the separatrix set of saddle periodic points of f is trivially embedded, for each sink ω (respectively, for each source α) there exists a closed neighbourhood U_ω (respectively, U_α) such that the boundary ∂U_ω (respectively, ∂U_α) is a smoothly embedded 2-sphere intersecting transversally each separatrix in the set $\mathcal{L}^u(\omega)$ (in $\mathcal{L}^s(\alpha)$) at a unique point (recall that $\mathcal{L}^u(\omega)$ (respectively, $\mathcal{L}^s(\alpha)$) is the set of all unstable (stable) separatrices of saddle periodic points with closures containing the point ω (respectively, α)). We select neighbourhoods U_ω (respectively, U_α) such that the neighbourhoods of distinct points are disjoint. Next, for each component γ^u (respectively, γ^s) of the unstable (stable) manifold in the set $\Gamma^u = W_1^u \setminus \bigcup_{\omega\in\Omega^+} U_\omega$ $\bigl($in $\Gamma^s = W_1^s \setminus \bigcup_{\alpha\in\Omega^-} U_\alpha\bigr)$ we select a closed tubular neighbourhood U_{γ^u} (respectively, U_{γ^s}) such that the intersection of U_{γ^u} (of U_{γ^s}) with each sphere ∂U_ω (∂U_α) is either empty or consists of at most 2 disjoint closed 2-dimensional discs (by a closed tubular neighbourhood of U_{γ^u} (of U_{γ^s}) we mean here a set diffeomorphic to the direct product of a closed 2-dimensional disc and the interval $[0,1]$; these closed tubular neighbourhoods must be disjoint).

We set

$$P^+ = \bigcup_{\omega\in\Omega^+} U_\omega \cup \bigcup_{\gamma^u\in\Gamma^u} U_{\gamma u}, \qquad P^- = \bigcup_{\alpha\in\Omega^-} U_\alpha \cup \bigcup_{\gamma^s\in\Gamma^s} U_{\gamma^s},$$
$$M^+ = \partial P^+, \qquad M^- = \partial P^-.$$

By construction M^+ (respectively, M^-) is a connected orientable piecewise smooth closed 2-dimensional surface of some genus g^+ (of genus g^-). Without loss of generality we can modify a neighbourhood P^+ (respectively, P^-) so that its boundary becomes a smooth 2-dimensional surface. Hence we assume in what follows that P^+ (respectively, P^-) has this property.

We claim that $g^+ = g^-$. Assume the contrary and also assume for definiteness that $g^+ > g^-$. The condition of the trivial embedding of the separatrix set in the manifold M^3 and the construction of the set P^+ allow one to construct a diffeomorphism φ of $P^+\setminus\mathcal{A}$ onto a direct product $(0,1]\times M^2_{g^+}$ such that $\varphi(M^+)=\{1\}\times M^2_{g^+}$, where $M^2_{g^+}$ is a smooth closed orientable surface of genus g^+. By construction the continuum $\mathcal{A}$ is an attractor, therefore there exists an integer $N>0$ such that $f^N(M^-)\subset \operatorname{int} P^+$. We set $M_N^-=f^N(M^-)$ and select $t_N\in(0,1)$ such that the interior of the set P bounded by the surfaces $M_N^+=\varphi^{-1}(\{t_N\}\times M^2_{g^+})$ and M^+ contains the surface M_N^-. By construction the set P is diffeomorphic to the product $[t_N,1]\times M^2_{g^+}$.

Since the genus of M_N^- is less than that of M^+ by our assumption to the contrary, it follows by Theorem 3.1 and Lemma 3.1 that the surface M_N^- bounds a subdomain Q of $\operatorname{int} P$. We set $M_1=Q\cup M_N^-\cup f^N(P^-)$. By construction M_1 is a clopen subset of the manifold M^3 and therefore coincides with M^3, which is impossible because M_1 is disjoint from the continuum $\mathcal{A}$.

Thus, the surfaces M^- and M^+ have the same genus. Since M_N^- does not bound a domain, it follows by Lemma 3.1 that M_N^- separates the surfaces M_N^+ and M^+ in P. By Theorem 3.3 the closure of the connected component of $P\setminus M_N^-$ whose boundary contains M_N^+ (we denote this closure by Q^+) is diffeomorphic to the direct product of the surface $M^2_{g^+}$ and $[0,1]$.

By construction the surface M_N^+ is the boundary of a closed set P_N^+ containing the continuum $\mathcal{A}$ and lying in P^+. In addition, P_N^+ is diffeomorphic to the set P_{g^+} obtained by attaching g^+ handles of index 1 to a closed 3-ball. Hence the union $M^{3+}=P_N^+\cup Q^+$ is also diffeomorphic to P_{g^+}. We set $M^{3-}=\operatorname{clos}(f^N(P^-))$. Since $g^+=g^-$, it follows that $M^3=M^{3+}\cup M^{3-}$, where M^{3+} and M^{3-} intersect precisely in their common boundary, which is diffeomorphic to a 2-dimensional surface of genus $g=g^+=g^-$, and each of them is diffeomorphic to a closed handlebody with g handles of index 1. This gives us a Heegaard splitting of genus g for M^3.

It remains to prove the formula

$$g=h_D=\frac{\nu(f)-\mu(f)+2}{2}.$$

We can assume without loss of generality that the set P^+ has the property $f(P^+)\subset \operatorname{int} P^+$. Since P^+ is diffeomorphic to the set P_g obtained by attaching g handles of index 1 to a closed 3-ball, there exists precisely g saddle fixed points $p_1,\dots,p_g$ in the continuum $\mathcal{A}$ and open neighbourhoods $U_1,\dots,U_g$ of the sets $W^s(p_i)\cap P^+$ such that $\bigcup_{i=1}^g U_i$ contains no fixed points distinct from $p_1,\dots,p_g$ and the set $P_*=P^+\setminus\bigcup_{i=1}^g U_i$ has the following properties:

(1) P_* is diffeomorphic to the closed 3-ball;

(2) $f(P_*)\subset \operatorname{int} P_*$.

It follows by Lefschetz's formula that the sum of the indices of the fixed points of f lying in the set P_* is 1. Let ν^+ and μ^+ be the numbers of saddle points and sinks lying in the attractor $\mathcal{A}$, respectively. Lefschetz's formula now yields the equality $\mu^+-(\nu^+-g)=1$. Hence $g=\nu^+-\mu^++1$.

Next, let ν^- and μ^- be the numbers of saddle fixed points and sources belonging to the repellor $\mathcal{R}$, respectively. Applying similar arguments to P^- and the diffeomorphism f^{-1} we obtain the formula $g=\nu^--\mu^-+1$.

Adding these formulae we obtain $2g = \nu - \mu + 2$. Hence

$$g = h_D = \frac{\nu(f) - \mu(f) + 2}{2}.$$

Theorem 4.2. *Let $f\colon M^3 \to M^3$ be a gradient-like diffeomorphism of a closed orientable 3-manifold M^3. If the set of 1-dimensional separatrices of saddle periodic points of f is trivially embedded, then the number of saddle periodic points of f is at least twice the Heegaard genus of M^3. On each closed manifold M^3 of Heegaard genus $h(M^3)$ there exists a gradient-like diffeomorphism f such that the number of saddle periodic points of f is $2h(M^3)$.*

Proof. By the definition of the Heegaard genus $h(M^3) \leqslant h_D$, therefore the estimate $2h(M^3) \leqslant \nu(f)$ follows by Theorem 4.1 and the inequality $\mu(f) \geqslant 2$ (because each Morse–Smale diffeomorphism on a compact manifold has at least 2 nodal periodic points: a sink and a source [1]).

Assume that M^3 can be represented as a Heegaard splitting of genus $g \geqslant 0$, that is, M^3 can be obtained by gluing together two closed sets M_g^{3+} and M_g^{3-} along their boundaries, where both M_g^{3+} and M_g^{3-} are diffeomorphic to the set P_g obtained by attaching g handles of index 1 to a 3-ball. On M^3 one can easily construct a Morse–Smale flow f^t without periodic trajectories such that

(1) the trajectories of the flow are transversal to the boundary of M_g^{3+} (of M_g^{3-});
(2) the non-wandering set of f^t consists precisely of one stable node and g saddle points lying in $\operatorname{int} M_g^{3+}$ and one unstable node and g saddle points in $\operatorname{int} M_g^{3-}$;
(3) the 1-dimensional separatrices of saddle points in M_g^{3+} (M_g^{3-}) make up a wedge of circles with the stable (respectively, unstable) node as the wedge point.[2]

A unit-time translation along the trajectories of the flow f^t produces a gradient-like diffeomorphism f of the manifold M^3. It follows by the construction that the number of saddle periodic points of f is $2g$, which is equal to $2h(M^3)$.

We consider now Morse-Smale diffeomorphisms of closed surfaces, both orientable and non-orientable.

Theorem 4.3. *Let $f\colon M_g^2 \to M_g^2$ be a Morse–Smale diffeomorphism of a closed surface M_g^2 of genus g (the genus g is non-negative if M_g^2 is orientable, and $g \geqslant 1$*

[2] The construction and the classification of such flows are carried out in [6], where they are called polar flows. We outline the construction here for the reader's convenience. Assume that M^3 is represented as a Heegaard splitting of genus $g \geqslant 0$. On a manifold M_g^3 with boundary diffeomorphic to a handlebody with g handles of index 1 one easily constructs a Morse–Smale vector field transversal to the boundary with unique source (or unique sink) and g saddle points such that their 1-dimensional separatrices make up a wedge of circles with wedge point at the node. On the basis of this example one easily constructs a Morse–Smale flow without periodic trajectories on M^3 with $2g$ saddle points.

if M_g^2 is non-orientable). Let $\nu(f)$ be the number of saddle points, and $\mu(f)$ the number of sink and source periodic points of f. Then

$$g = \frac{\nu(f) - \mu(f) + 2}{2} \quad \text{if } M_g^2 \text{ is orientable,}$$
$$g = \nu(f) - \mu(f) + 2 \quad \text{if } M_g^2 \text{ is non-orientable.}$$

Proof. This is an immediate consequence of Lefschetz's formula and [10]. We present the corresponding argument for completeness. We start with the case when M_g^2 is an orientable surface. For the proof of the theorem we can assume without loss of generality that all periodic points of f are fixed and the restriction of f to the unstable manifold of each saddle periodic point preserves its orientation. Thus, the saddle fixed points of f have index -1, while the sinks and the sources have index $+1$. By [10] a Morse–Smale diffeomorphism of a closed manifold induces an isomorphism of the first homology group described by a matrix A with eigenvalues that are roots of unity. Hence we can also assume without loss of generality that all eigenvalues of this matrix are equal to 1 and its trace is $2g$. Then it follows by Lefschetz's formula that $2 - 2g = \mu(f) - \nu(f)$, which gives us the required expression.

Assume now that M_g^2 is a non-orientable surface of genus g. Then it possesses a two-sheeted cover by an orientable surface $\widetilde{M}_{g-1}^2$ of genus $g-1$. As is known, there exists in our case a diffeomorphism $\overline{f}$ on $\widetilde{M}_{g-1}^2$ covering f. Obviously, the number of saddle points and the number of sinks and sources of the diffeomorphism $\overline{f}$ are twice the corresponding numbers for f. Applying to $\overline{f}$ the above-proved formula we obtain $2(g-1) = 2\nu(f) - 2\mu(f) + 2$. This yields the required expression for non-orientable surfaces.

§ 5. Heegaard splitting and Morse–Smale flows

We start with Morse–Smale flows without periodic trajectories on a closed 3-manifold M^3. For a flow g^t let $\nu(g^t)$ be the number of its saddle points and $\mu(g^t)$ the number of its nodes.

Lemma 5.1. *Let f^t be a Morse–Smale flow without periodic trajectories on a closed 3-manifold M^3. Assume that f^t has $\nu(f^t)$ saddle points and $\mu(f^t)$ nodes. Then the manifold M^3 can be represented as a Heegaard splitting of genus*

$$h_D = \frac{\nu(f^t) - \mu(f^t) + 2}{2}.$$

Proof. Consider the diffeomorphism f obtained from the flow f^t by the unit-time translation along its trajectories. Then f is obviously a gradient-like diffeomorphism, the 1-dimensional separatrices of saddle periodic points are trajectories of the flow f^t, and therefore the set of these separatrices is trivially embedded (this is an immediate consequence of the fact that f^t is topologically conjugate to a linear flow in the neighbourhood of each node). Applying now Theorem 4.1 to f we obtain the result of the lemma.

Corollary 5.1. *Let f^t be a Morse–Smale flow without periodic trajectories on a closed 3-manifold M^3. Assume that f^t has $\nu(f^t)$ saddle points and $\mu(f^t)$ nodes. Then*

$$h(M^3) \leqslant \frac{\nu(f^t) - \mu(f^t) + 2}{2},$$

where $h(M^3)$ is the Heegaard genus of M^3.

Proof. This is a consequence of Lemma 5.1 and the definition of the Heegaard genus as the least g such that there exists a Heegaard splitting of genus g of M^3.

We consider now Morse–Smale flows that can have periodic trajectories alongside equilibria. In particular, flows may have only periodic trajectories and not possess equilibria. In the last case the result below yields an alternative description (to [3]) of the underlying manifold in terms of Heegaard splittings. In addition, we obtain an estimate of the number of saddle points and saddle periodic trajectories in terms of the Heegaard genus.

Theorem 5.1. *Let f^t be a Morse–Smale genus on a closed 3-manifold M^3 with $\nu(f^t) \geqslant 0$ saddle points, $\mu(f^t) \geqslant 0$ nodes, $s(f^t) \geqslant 0$ saddle and $r(f^t) \geqslant 0$ stable and unstable periodic trajectories. Then M^3 can be represented as a Heegaard splitting of genus*

$$h_D = \frac{\nu(f^t) - \mu(f^t) + 2}{2} + s(f^t)$$

and the Heegaard genus $h(M^3)$ of the manifold M^3 has the estimate

$$h(M^3) \leqslant \frac{\nu(f^t) + r(f^t)}{2} + s(f^t).$$

Proof. We reduce the general case of the theorem to the case of a Morse–Smale flow without periodic trajectories by eliminating the latter and increasing the number of saddle points and nodes. For definiteness we consider the case of an attracting periodic trajectory l_0. This trajectory has an open attracting neighbourhood $U(l_0)$ that possesses a homeomorphism φ onto the standard solid torus $Q = d \times S^1$, where $d = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, such that $\varphi(l_0) = S^1_* = \{0, 0\} \times S^1$. It is easy to construct in Q a vector field with two singularities lying in S^1_*, a saddle point of index 1 and an attracting node, such that the closure of each 1-dimensional separatrix of the saddle point contains the node and the union of the closures of these separatrices forms the circle S^1_*. The circle S^1_* also has an attracting neighbourhood homeomorphic to a solid torus. We replace the original vector field in $U(l_0)$ by a vector field with saddle point and node of the above-described form so that the result is a Morse–Smale field with the 'old' non-wandering set outside $U(l_0)$ (see Fig. 2). In a similar fashion, on the repelling periodic trajectory we 'place' one saddle point of index 2 and a repelling node and we place two saddle points on the saddle periodic trajectory (see Fig. 3). As a result, we obtain a Morse–Smale flow g^t without periodic trajectories with non-wandering set containing $\nu(f^t) + 2s(f^t) + r(f^t)$ saddle points and $\mu(f^t) + r(f^t)$ nodes. Applying now Lemma 5.1 to the flow g^t we obtain

$$h_D = \frac{\nu(f^t) + 2s(f^t) + r(f^t) - (\mu(f^t) + r(f^t)) + 2}{2} = \frac{\nu(f^t) - \mu(f^t) + 2}{2} + s(f^t).$$

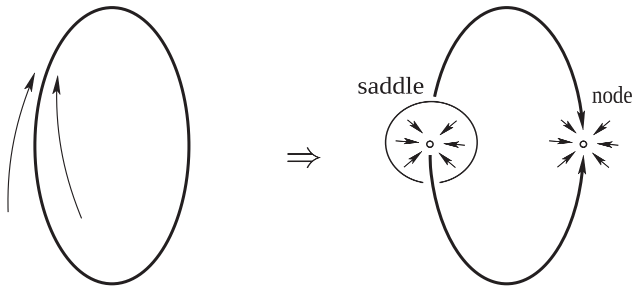

Figure 2. 'Breakdown' of a periodic trajectory and birth of a saddle point and a node

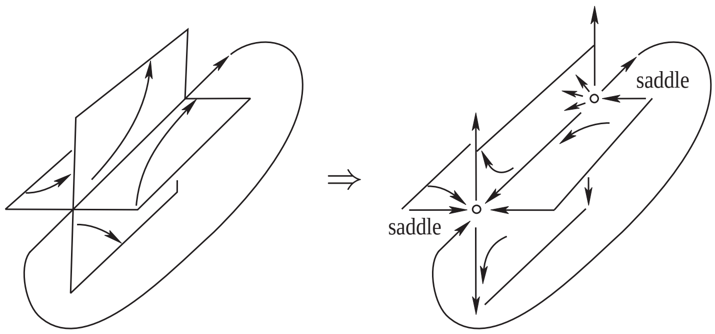

Figure 3. 'Breakdown' of a periodic trajectory and birth of two saddle points

On the other hand, since g^t is a Morse–Smale flow, the quantity $\mu(f^t)+r(f^t)$ is at least 2. By the definition of the Heegaard genus $h(M^3)\leqslant h_D$. Hence

$$h(M^3)\leqslant h_D\leqslant \frac{\nu(f^t)+2s(f^t)+r(f^t)}{2}=\frac{\nu(f^t)+r(f^t)}{2}+s(f^t).$$

§ 6. Estimate of the number of heteroclinic curves on lens spaces

Recall that if $W^u(p)\cap W^s(q)\neq\varnothing$ for periodic points p and q of a Morse–Smale diffeomorphism and $\dim W^s(p)<\dim W^s(q)$, then the connected component of the intersection $W^u(p)\cap W^s(q)$ is called the *heteroclinic submanifold*. If the manifold has dimension 3, then each heteroclinic submanifold is either a simple closed curve (homeomorphic to a circle) or a non-closed curve without self-intersections (homeomorphic to an open interval). We say that such curves are *heteroclinic*. Note that the absence of heteroclinic points does not mean the absence of heteroclinic curves. In particular, a gradient-like diffeomorphism can (and sometimes, must: see Theorem 6.1) have heteroclinic curves.

Recall that if p and q are coprime integers, $p\geqslant 3$, $1\leqslant q<p$, then the *lens space* $L_{p,q}$ is the 3-manifold obtained by taking the quotient of the standard sphere $S^3=\{(z,w)\in\mathbb{C}^2: |z|^2+|w|^2=1\}$ by the action

$$(z,w)\mapsto\left(z\exp\frac{2\pi i}{p},w\exp\frac{2\pi iq}{p}\right).$$

The lens space $L_{p,q}$ can also be obtained by gluing together two solid tori along their boundaries (2-dimensional tori) by means of the linear automorphism of the torus of the following form:

$$\begin{cases} \overline{x} = rx + py; \\ \overline{y} = sx + qy, \end{cases}$$

where the integers r and s satisfy the relation $ps - qr = \pm 1$.

We add to the list of lens spaces the manifold $S^2 \times S^1 = L_{0,1}$, which is not normally included, but can formally be obtained by gluing two solid tori in the above-described manner.

The main result of this section is as follows.

Theorem 6.1. *Let $f\colon L_{p,q} \to L_{p,q}$ be a Morse–Smale diffeomorphism with wandering set consisting precisely of 4 periodic points. Then*

(1) *f is gradient-like*;
(2) *f has one sink, one source, and two saddle fixed points with distinct Morse indices*;
(3) *if the set of 1-dimensional separatrices of fixed points of f is trivially embedded, then there exist at least p heteroclinic non-closed curves with end-points at saddle fixed points.*

Proof. Let k be a positive integer such that all periodic points of the diffeomorphism $g = f^k$ are fixed and the restrictions of g to the stable and the unstable manifolds of each fixed point preserve the orientations of these manifolds. Since g is a Morse–Smale diffeomorphism, its non-wandering set contains at least one source α and one sink ω. On the other hand its non-wandering set cannot consist of two sinks and two sources. Hence the non-wandering set of g consists either of a sink, a source, and two saddle fixed points or of three fixed points that are either sinks or sources and one saddle fixed point. The last option is impossible because in that case the underlying manifold must be a 3-sphere by [9]. Hence g has a sink, a source, and two saddle fixed points σ_1 and σ_2. By Lefschetz's formula the sum of the indices of the fixed points of the diffeomorphism g is 0, therefore the indices of the points σ_1 and σ_2 are different. Hence the dimensions of the unstable manifolds of the points σ_1 and σ_2 are also different.[3]

Thus, the non-wandering set of the original diffeomorphism f has a source α, a sink ω, and two saddle fixed points σ_1 and σ_2. Assume for definiteness that $\dim W^u(\sigma_1) = 2$ and $\dim W^u(\sigma_2) = 1$.

We claim that f is gradient-like. Since the non-wandering set is finite, it follows by transversality arguments that $W^s(\sigma_1)$ (respectively, $W^u(\sigma_2)$) is disjoint from $W^u(\sigma_1)$ and $W^u(\sigma_2)$ (respectively, from $W^s(\sigma_2)$ and $W^s(\sigma_1)$). On the other hand if $W^u(\sigma_1) \cap W^s(\sigma_2) \neq \varnothing$, then $1 = \dim W^s(\sigma_1) < \dim W^s(\sigma_2) = 2$, which means that f is gradient-like.

We now set $l_1 = W^s(\sigma_1) \cup \alpha$ and $l_2 = W^u(\sigma_2) \cup \omega$. Let $\mathbf{V} = D^2 \times S^1$ be the standard solid torus, and $\mathbf{T} = \partial D^2 \times S^1$ the standard torus (D^2 is the standard disc $\{(x,y) : x^2 + y^2 \leqslant 1\}$).

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[3]Since the restriction of the diffeomorphism g to the stable and the unstable manifolds of fixed points preserves their orientation, the index of an arbitrary fixed point p of g is $(-1)^{\dim W^u(p)}$.

Since the closure of each 1-dimensional separatrix of the point σ_1 (of σ_2) contains a point α (respectively, ω), the set l_1 (respectively, l_2) is homeomorphic to a circle and we have $\overline{W^u(\sigma_1)}\setminus W^u(\sigma_1) \subset l_2$ (respectively, $\overline{W^s(\sigma_2)}\setminus W^s(\sigma_2) \subset l_1$). Moreover, the set of 1-dimensional separatrices of the diffeomorphism f is trivially embedded by assumption, therefore there exist homeomorphisms $\varphi_i \colon \mathbf{V} \to L_{p,q}$, $i = 1, 2$, with the following properties:

(1) the restriction of φ_i to the set $\mathbf{V} \setminus \{(0,0)\} \times S^1$ is a diffeomorphism and $l_i = \varphi_i(\{(0,0)\} \times S^1)$;
(2) the surface $T_1 = \varphi_1(\partial D^2 \times S^1)$ intersects transversally the manifold $W^u(\sigma_1)$ along a unique closed curve c_1, and the surface $T_2 = \varphi_2(\partial D^2 \times S^1)$ intersects transversally the manifold $W^s(\sigma_2)$ along a unique closed curve μ_2;
(3) the unstable manifold $W^u(\sigma_1)$ intersects transversally the surface T_2;
(4) if the set $W^u(\sigma_1) \cap W^s(\sigma_2)$ contains non-closed heteroclinic curves, then $\mu_2 \cap W^u(\sigma_1)$ is non-empty and is a finite point set G_2 such that each point in G_2 lies on some non-closed heteroclinic trajectory and no two points belong to the same heteroclinic curve (see Fig. 4).

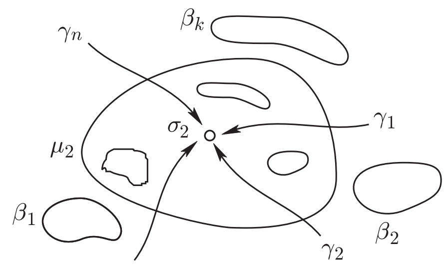

Figure 4. Construction of the curve μ_2

We set $V_i = \varphi_i(D^2 \times S^1)$.

Conditions (1) and (3) can obviously be fulfilled. We now explain how condition (4) can be fulfilled. Let f_2 be the restriction of the diffeomorphism f to the manifold $W^s(\sigma_2)$. Note that if the intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is non-empty and contains non-closed heteroclinic curves, then their number is finite in view of transversality; let it be n, $n > 0$. We denote these heteroclinic curves by $\gamma_1, \dots, \gamma_n$ and observe that each γ_i, $i \in \{1, \dots, n\}$, has precisely two end-points, σ_1 and σ_2. If the intersection $W^u(\sigma_1) \cap W^s(\sigma_2)$ is non-empty and contains closed heteroclinic curves, then their set decomposes into the orbits of several (say k, $k > 0$) fixed closed curves $\beta_1, \dots, \beta_k$. In addition, due to the diffeomorphism f_2 the ω-limit set of the orbit of each β_j, $j \in \{1, \dots, k\}$, is the point σ_2. It follows by these properties of heteroclinic curves in the set $W^u(\sigma_1) \cap W^s(\sigma_2)$ that there exists a smooth closed curve μ_2 on the manifold $W^s(\sigma_2)$ such that

(a) μ_2 bounds a domain homeomorphic to a disc with σ_2 in its interior;
(b) μ_2 is disjoint from all closed heteroclinic curves;
(c) if the set of non-closed heteroclinic curves is non-empty, then μ_2 intersects each curve γ_i, $i \in \{1, \dots, n\}$, in precisely one point (see Fig. 4).

We can now construct a surface T_2 containing the curve μ_2 and having properties (1)–(4).

Since $\overline{W^u(\sigma_1)}\setminus W^u(\sigma_1) \subset l_2$, the intersection $W^u(\sigma_1)\cap T_2$ is a non-empty compact set consisting of finitely many disjoint simple smooth closed curves. Let m be a positive integer such that $f^m(T_1) \subset \operatorname{int} V_2$. We set $c_m = f^m(c_1)$ and denote by $K_{1m} \subset W^u(\sigma_1)$ the closed domain homeomorphic to an annulus and bounded by the curves c_1 and c_m. Then the set $K_{1m}\cap T_2$ contains a curve (we denote it by μ_1) such that μ_1 is homotopic to c_1 in K_{1m} and in the interior of the closed domain K_{12} bounded by the curves c_1 and μ_1 (and homeomorphic to an annulus) there exist no other curves in $W^u(\sigma_1) \cap T_2$ homotopic to c_1 in K_{12}.

We claim that the set $V_1^* = \overline{L_{p,q} \setminus V_2}$ is homeomorphic to the standard solid torus. It is sufficient to show that the closed domain $V_{12} = L_{p,q} \setminus (\operatorname{int} V_1 \cup \operatorname{int} V_2)$ is homeomorphic to the product of the standard torus and a closed interval.

We set $T_m = f^m(T_1)$. This surface T_m bounds a set V_m that is a solid torus, that is, the image under the map $\psi = f^m \circ \varphi_1$ of the product $D^2 \times S^1$; moreover, $l_1 = \psi(\{(0,0)\} \times S^1)$. We set $\widehat{T}_1 = \psi^{-1}(T_1)$, $\widehat{T}_2 = \psi^{-1}(T_2)$, $\widehat{T}_m = \psi^{-1}(T_m)$, and select r, $0<r<1$, such that the solid torus $D_r^2 \times S^1$, where $D_r^2 = \{(x,y) : x^2+y^2 \leqslant r\}$, contains no points of the surface $\widehat{T}_1$. Let $\widehat{T}_r$ be the boundary of the set $D_r^2 \times S^1$, and $\widehat{V}_{rm}$ the closed domain bounded by the surfaces $\widehat{T}_r$ and $\widehat{T}_m$ (by construction $\widehat{T}_m = \partial D^2 \times S^1$). The domain $\widehat{V}_{rm}$ is homeomorphic to the direct product of a torus and a closed interval and the surface $\widehat{T}_1$ partitions $\widehat{V}_{rm}$ into two parts, each homeomorphic to the same direct product by Theorem 3.3. Then, however, the domain $\widehat{V}_{1m}$ bounded by the surfaces $\widehat{T}_1$ and $\widehat{T}_m$ is also homeomorphic to the direct product of a torus and a closed interval. In a similar way one demonstrates that the domain $\widehat{V}_{2m}$ bounded by $\widehat{T}_2$ and $\widehat{T}_m$ is homeomorphic to the direct product of a torus and a closed interval. Then, however, the set $\widehat{V}_{12} = \widehat{V}_{1m} \setminus \widehat{V}_{2m}$, and therefore also the set $V_{12} = \psi(\widehat{V}_{12})$, is homeomorphic to the direct product of a torus and a closed interval.

Since by construction the domain K_{12} lies in the set $V_{1m} = \psi(\widehat{V}_{1m})$, the curve μ_1 is homotopic to c_1 in V_{1m}. The fundamental group of T_2 is isomorphic to the fundamental group of V_{1m}, therefore the curve μ_1 is homotopic to c_1 in V_{1m}. Hence the curve μ_1 is not homotopic to zero in T_2. On the other hand, the fundamental groups of V_{12} and V_{1m} are isomorphic, therefore μ_1 is homotopic to c_1 also in the set V_{12}. Since $V_1 \subset V_1^*$, the curve c_1 is homotopic to zero in V_1^*. Since $V_{12} \subset V_1^*$, c_1 and μ_1 are homotopic in V_1^* and therefore μ_1 is homotopic to zero in V_1^*.

We have thus represented the manifold L_{pq} as a union of the sets V_1^* and V_2, which are homeomorphic images of the standard solid torus $\mathbf{V}$ and intersect in the 2-torus $T_2 = V_1^* \cap V_2$ (see Fig. 5). Hence we have represented the manifold L_{pq} as the lens space $L_{p',q'}$. Moreover, the curves μ_1 and μ_2 are not homotopic to zero in T_2 and μ_1 (respectively, μ_2) is homotopic to zero in V_1^* (in V_2). By [20], Assertion 2.11 (see also [17]) $p = p'$ and the integers q and q' are related by one of the two congruences $q = \pm q' \bmod p$ or $qq' = \pm 1 \bmod p$.

We represent the torus $\mathbf{T}$ as the quotient of the Euclidean plane by the discrete group Γ of transformations of the form $\overline{x} = x + m$, $\overline{y} = y + n$, where m and n are integers and the variables x, y are selected so that the meridian of the torus $\mathbf{T}$ corresponds to an element of Γ of the form $\overline{x} = x$, $\overline{y} = y + 1$. Let Ψ be the

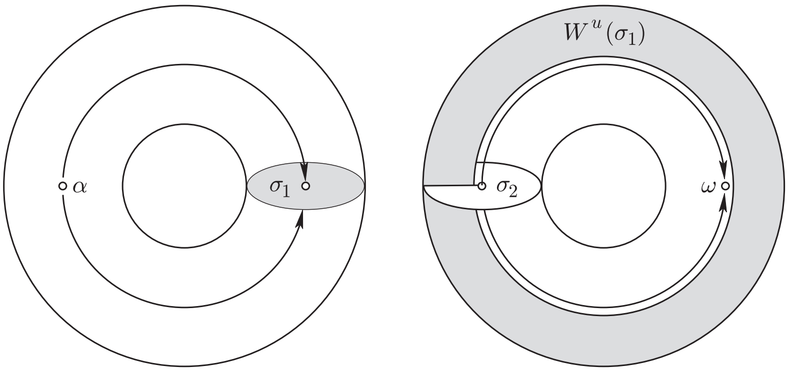

Figure 5. The solid tori V_1^* and V_2

automorphism of $\mathbf{T}$ defined by the map

$$\begin{cases} \overline{x} = rx + py; \\ \overline{y} = sx + q'y, \end{cases}$$

where the expressions on the right-hand sides are taken modulo 1. Then there exists homeomorphisms $\widetilde{\varphi}_1 \colon \mathbf{V} \to V_1^*$ and $\widetilde{\varphi}_2 \colon \mathbf{V} \to V_2$ such that $\widetilde{\varphi}_1\big|_{\mathbf{T}} = \widetilde{\varphi}_2 \circ \Psi\big|_{\mathbf{T}}$.

In view of the form of Ψ, the intersection $\Psi(\widetilde{\varphi}_1^{-1}(\mu_1)) \cap \widetilde{\varphi}_2^{-1}(\mu_2)$ contains at least p points. Since $\widetilde{\varphi}_2(\Psi(\widetilde{\varphi}_1^{-1}(\mu_1))) = \mu_1$ and $\widetilde{\varphi}_2(\varphi_2^{-1}(\mu_2)) = \mu_2$, it follows that $\mu_1 \cap \mu_2 = \widetilde{\varphi}_2(\Psi(\widetilde{\varphi}_1^{-1}(\mu_1)) \cap \widetilde{\varphi}_2^{-1}(\mu_2))$ and therefore this intersection also contains at most p points. By our choice of μ_2 all these points lie on non-closed heteroclinic curves and no two of them lie on the same heteroclinic curve. This completes the proof of the theorem.

Remark. One easily constructs a Morse–Smale diffeomorphism of the lens space $L_{p,q}$ that has precisely p non-closed heteroclinic curves; that is, the estimate in Theorem 6.1 is sharp. On the other hand one can construct a Morse–Smale diffeomorphism with more than p non-closed heteroclinic curves (more precisely, with $p+2k$ such curves, where k is an arbitrary positive integer).

One can also include the 3-sphere $S^3 = L_{1,0} = L_{1,q}$ in the list of lens spaces of this theorem; then, however, the theorem holds for it under the following additional assumption: the non-wandering set of the Morse–Smale diffeomorphism contains two saddle fixed points. In particular, it is shown in [21] that the estimate for the sphere can be proved without the assumption about the trivial embedding of the separatrices.

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