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# NATURAL EXTENSION OF THE CLASS OF REGIONS IN EMBEDDING THEOREMS

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The basic method for proving imbedding theorems for classes of differentiable functions is to represent the functions using integrals or series (see, for example, [1]–[5]). The class of regions under consideration is defined depending on the "body of the respresentation" (i.e. the set of points where the values of the function participate in the representation).

Thus, in Sobolev's imbedding theorems for the spaces $W_p^{(l)}(\Omega)$ with norm $\Sigma_{|\alpha|\le l}\|D^\alpha f\|_{L_p(\Omega)}$, $\Omega\subset E_n$, the function $f(x)$ is represented in terms of the values $D^\alpha f(y)$, $|\alpha|\le l$ at the points $y$ of some $n$-dimensional cone with vertex at the point $x$. Then regions $\Omega$ are considered which are star relative to a ball (each of their points can be reached from within as the vertex of some cone). Imbedding theorems are formulated for these regions $\Omega$ just as for the whole space $E_n$. Functions of the class $W_p^{(l)}(\Omega)$ can be extended beyond the limits of this type of region $\Omega$ onto $E_n$ preserving differential properties (Calderón's result). On the other hand, examples show that for regions of a more general form (some of whose points can only be reached as the vertex of a curvilinear cone with degenerate (zero) angle) the statement of imbedding theorems in general must be altered.

The present article examines the function spaces $W_p^l(\Omega)$ and $B_{p,\theta}^l(\Omega)$ with a vector value of $l$. We give representations of functions which are convenient in studying these spaces. The "body of the representation" is a "horn" $\mathfrak{R}=\mathfrak{R}(l)$ whose form is determined by the parameter $l$. It is then shown that such representations lead in a natural way to the possibility of extending the class of regions $\Omega\subset E_n$ for which imbedding theorems have the same form as for $E_n$.

These results are definitive and complete the study of the corresponding questions considered in articles mentioned above.

## §1. Integral representation of functions in terms of derivatives

Let $\mu(t)\in C^\infty(E_1)$, where $\mu(t)=1$ for $t\ge\gamma>\delta>0$, $\mu(t)=0$ for $t\le\delta$. We also consider the function $\omega(t)=(d^{s+1}/dt^{s+}\ (t^s\mu(t)/s!)$, for which, obviously,

$$\int\omega(t)\,dt=\mu(+\infty)=1. \tag{1}$$

We set $\xi(t)=t^{s+1}\mu'(t)/s!$ Then

$$\xi^{(s+1)}(t)=\frac{d^{s+1}}{dt^{s+1}}\left[\frac{t^{s+1}}{s!}\mu'(t)\right]=\frac{d^{s+1}}{dt^{s+1}}\left[t\left(\frac{t^s}{s!}\mu(t)\right)-s\left(\frac{t^s}{s!}\mu(t)\right)\right]$$
$$=t\omega'(t)+(s+1)\omega(t)-s\omega(t)=t\omega'(t)+\omega(t).$$

We assume that $f(x)$ is a locally summable function with generalized Sobolev derivatives $D_i^{l_i}f(x)=\partial^{l_i}f(x)/\partial x_i^{l_i}$ $(i=1,\cdots,n)$ in the region of $n$-dimensional Euclidean space for which this is necessary according to the sense of the following transformations.

Let $l_i$ be natural numbers,

$$\sigma = (\sigma_1, \ldots, \sigma_n),\ |\sigma| = \sum_{j=1}^{n} \sigma_j,\ \sigma_j = \frac{1}{l_j} > 0,\ h > 0,\ \frac{y}{h^\sigma} = \left(\frac{y_1}{h^{\sigma_1}}, \ldots, \frac{y_n}{h^{\sigma_n}}\right),$$

$$f_{h\sigma}(x) = h^{-|\sigma|} \int \prod_{j=1}^{n} \omega_j\left(\frac{y_j}{h^{\sigma_j}}\right) f(x+y)\, dy = h^{-|\sigma|} \int \Phi_0\left(\frac{y}{h^\sigma}\right) f(x+y)\, dy, \tag{2}$$

where $\omega_j(t)$ is the function $\omega(t)$ introduced above with parameters depending on $j$.

Hence, setting $\Pi^{(i)} = \Pi_{1 \le j \le n, j \ne i}$ we obtain

$$\frac{\partial}{\partial h} f_{h\sigma}(x) = \sum_{i=1}^{n} \sigma_i \int h^{-1-|\sigma|} \prod^{(i)} \omega_j\left(\frac{y_j}{h^{\sigma_j}}\right) \xi_i^{(s_i+1)}\left(\frac{y_i}{h^{\sigma_i}}\right) f(x+y)\, dy,$$

since

$$\frac{\partial}{\partial h}\left[\frac{1}{h}\omega\left(\frac{y_i}{h}\right)\right] = -\frac{1}{h}\left[\frac{1}{h}\omega\left(\frac{y_i}{h}\right) + \frac{1}{h}\cdot\frac{y_i}{h}\omega'\left(\frac{y_i}{h}\right)\right] = -\frac{1}{h^2}\xi^{(s+1)}\left(\frac{y_i}{h}\right).$$

Integrating the last integral by parts several times over $y_1, \cdots, y_n$ and taking into account that the kernel is finite, we obtain (for $s_j = l_j$)

$$\frac{\partial}{\partial h} f_{h\sigma}(x) = -\sum_{i=1}^{n} \int h^{-|\sigma|} \Phi_i\left(\frac{y}{h^\sigma}\right) D_i^{l_i} f(x+y)\, dy,$$

where

$$\Phi_i(x) = (-1)^{l_i} \prod^{(i)} \omega_j(x_j)\, \xi_i'(x_i), \tag{3}$$

so that the kernels $\Phi_i(x) \in C_0^\infty(E_n)$ and are concentrated in any cube given in advance from the first coordinate angle.

A consequence of equation (3) which is essential for later estimates is the following equation, true for any $\alpha$:

$$\int D^\alpha \Phi_i(x)\, dx = 0 \quad (i = 1, \ldots, n). \tag{4}$$

For a function $f \in L^{\text{loc}}$ we have $f_{h\sigma} \to f$ $(h \to 0)$ in the sense of convergence in $L^{\text{loc}}$ and for almost all $x$. The former follows by the continuity with respect to translation of $f(x)$ in $L^{\text{loc}}$, and the latter is true at the Lebesgue points of the function $f(x)$ (see for example [6], Russian p. 237).

Integrating $\partial f_{h\sigma}/\partial h$, we obtain the representation $(r > 0)$

$$f(x) = f_{r\sigma}(x) + \int_0^r \sum_{i=1}^{n} \int h^{-|\sigma|} \Phi_i\left(\frac{y}{h^\sigma}\right) D_i^{l_i} f(x+y)\, dy\, dh. \tag{5}$$

Equation (5) holds for almost all $x$ in the set for which the right side is defined.

Because of the arbitrariness mentioned above of the support of $\Phi_i(x)$, the actual integration in the right side of (5) is only over the points $y \in \mathfrak{R}(l) = \mathfrak{R}(l, r, a, \epsilon)$:

$$\mathcal{R}(l, r, a, \varepsilon) = \{y; y_i > 0,\ 0 < a_i h < y_i^{l_i} < (a_i + \varepsilon) h\ (i = 1, \dots, n),\ 0 < h < r\}.$$

Designating the operation of vector summation of two sets by the sign +, we find that the body of the representation in (5) is the horn $x + \mathfrak{R}(l)$ of "curvature" $l$, "radius" $r$, and arbitrarily small "opening" $\epsilon > 0$. The vector parameter $a = (a_1, \cdots, a_n)$ can have any positive coordinates.

We let $\mathfrak{R}(l) = \mathfrak{R}(l, r, a, \epsilon)$ designate a horn obtained from the one under consideration by a symmetry transformation with respect to one or several in succession of the coordinate hyperplanes. Thus there are $2^n$ distinct horns in all with fixed parameters $l$, $r$, $a$, $\epsilon$.

We note that for identical values of $l_1 = \cdots = l_n$ the horn $\mathfrak{R}(l)$ becomes a cone.

## §2. Integral representation of functions in terms of differences

We now find a representation of the function $f(x)$ in terms of difference operators.

Let $\eta_i(t) \in C_0^\infty(E_1)$, $a_i > 0$,

$$\operatorname{supp} \eta_i \subset \left[\frac{a_i^{\sigma_i}}{2}, \frac{(a_i + \varepsilon)^{\sigma_i}}{2}\right], \quad \int \eta_i(t)\, dt = 1, \quad \zeta_i(t) = t\eta_i(t), \quad 0 < \delta < 1,$$

$$\chi_i(t) = \frac{1}{A_i} \sum_{k=0}^{m_i} \frac{(-1)^{m_i - k}}{(1 + k\delta)^2} C_{m_i}^k \eta_i\left(\frac{t}{1 + k\delta}\right), \tag{6}$$

where

$$A_i = \int_0^1 (1 - t^\delta)^{m_i} dt = \sum_{k=0}^{m_i} \frac{(-1)^k}{1 + k\delta} C_{m_i}^k > 0.$$

We also introduce the averages

$$f_{h^\sigma}(x) = h^{-|\sigma|} \int \prod_{i=1}^n \chi_i\left(\frac{y_i}{h^{\sigma_i}}\right) f(x + y)\, dy,$$

$$\hat{f}_{h^\sigma}(x) = (f_{h^\sigma})_{h^\sigma}(x) = h^{-2|\sigma|} \iint \prod_{i=1}^n \chi_i\left(\frac{y_i}{h^{\sigma_i}}\right) \chi_i\left(\frac{z_i}{h^{\sigma_i}}\right) f(x + y + z)\, dy\, dz. \tag{7}$$

Here we assume that $l_i = 1/\sigma_i$ $(i = 1, \cdots, n)$ are any positive numbers.

It is obvious that

$$\frac{\partial}{\partial h} \hat{f}_{h^\sigma}(x) = 2 \frac{\partial}{\partial t} (f_{h^\sigma})_{t^\sigma} \Big|_{t=h}.$$

We also note that

$$\frac{\partial}{\partial h} \int \frac{1}{h} \chi_i\left(\frac{t}{h}\right) g(x_i + t)\, dt$$

$$= \frac{\partial}{\partial h} \int A_i^{-1} \sum_{k=0}^{m_i} \frac{(-1)^{m_i - k}}{1 + k\delta} C_{m_i}^k \eta_i(\tau)\, g(x_i + \tau h + k\delta h\tau)\, d\tau =$$

$$= A_i^{-1} \int \tau\eta_i(\tau) \sum_{k=0}^{m_i} (-1)^{m_i-k} C_{m_i}^k g'(x_i + \tau h + k\delta h\tau)\, d\tau$$

$$= A_i^{-1} \int \frac{1}{h} \zeta_i\left(\frac{t}{h}\right) \Delta_1^{m_i}(\delta t)\, g'(x_i + t)\, dt,$$

where $\Delta_i^m(t) f(x)$ designates the $m$th order difference with increment $t$ in the direction of the $x_i$-axis.

We now find $\partial \hat{f}_{h^\sigma}(x)/\partial h$ by using the last two remarks and the fact that differentiation of the average can be carried over to the kernel:

$$\frac{\partial}{\partial h} \hat{f}_{h^\sigma}(x) = -2 \sum_{i=1}^{n} \frac{\sigma_i}{A_i} h^{-2|\sigma|} \iint \prod^{(i)} \chi_j\left(\frac{y_j}{h^{\sigma_j}}\right) \chi_j\left(\frac{z_j}{h^{\sigma_j}}\right)$$

$$\times h^{\sigma_i - 1} \frac{\partial}{\partial y_i} \chi_i\left(\frac{y_i}{h^{\sigma_i}}\right) \zeta_i\left(\frac{z_i}{h^{\sigma_i}}\right) \Delta_i^{m_i}(\delta z_i)\, f(x + y + z)\, dy\, dz$$

$$= -\sum_{i=1}^{n} h^{-1-|\sigma|-\sigma_i} \int_{E_n} \int_0^\infty \Psi_i\left(\frac{y}{h^\sigma}\right) \zeta_i\left(\frac{t}{h^{\sigma_i}}\right) \Delta_i^{m_i}(\delta t)\, f(x + y + te_i)\, dt\, dy,$$

where $e_i$ is the unit coordinate vector of the $x_i$-axis.

By the choice of the constants $A_i$ we have

$$h^{-2|\sigma|} \iint \prod_{i=1}^{n} \chi_i\left(\frac{y_i}{h^{\sigma_i}}\right) \chi_i\left(\frac{z_i}{h^{\sigma_i}}\right) dy\, dz = 1,$$

so that for almost all $x$ we have (in the $L^{\mathrm{loc}}$ sense) that

$$\hat{f}_{h^\sigma}(x) \to f(x) \quad (h \to 0).$$

Using the fundamental lemma of integral calculus, we now obtain the representation

$$f(x) = \hat{f}_{r^\sigma}(x) + \int_0^r \sum_{i=1}^{n} \int_{E_n} \int_0^\infty h^{-1-|\sigma|-\sigma_i} \Psi_i\left(\frac{y}{h^\sigma}\right) \zeta_i\left(\frac{t}{h^{\sigma_i}}\right) \Delta_i^{m_i}(\delta t) f(x+y+te_i)\, dt\, dy\, dh. \tag{8}$$

We note that the following estimates are valid for the support of the kernel of the $i$th term in (8):

$$a_j^{\sigma_j} < \frac{y_j}{h^{\sigma_j}} < (1 + m_j\delta)(a_j + \varepsilon)^{\sigma_j} \quad (1 \leqslant j \leqslant n;\ j \neq i),$$

$$\frac{a_i^{\sigma_i}}{2} < \frac{y_i}{h^{\sigma_i}} < (1 + m_i\delta) \frac{(a_i + \varepsilon)^{\sigma_i}}{2}, \quad \frac{a_i^{\sigma_i}}{2} < \frac{t}{h^{\sigma_i}} < \frac{(a_i + \varepsilon)^{\sigma_i}}{2}.$$

It is clear that for suitable choice of the parameters $a_i > 0$, $\delta > 0$, $\epsilon > 0$ the body of the representation in (8) can be any horn $x + \mathfrak{R}(l)$.

## §3. Imbedding $W$-spaces and extending functions

We let $\chi(G) = \chi(G;\, x)$ designate the characteristic function of the set $G$, and we let $G_1 + G_2$ designate the vector sum of the sets. Let $U \subset E_n$ be an open set, and let the function $f(x)$ be defined for $x \in U + \mathfrak{R}(l)$.

Along with the function $f(x)$ we consider the function

$$\tilde{f}(x) = (\chi(U + \mathcal{R}) f)_{r\sigma}(x)$$
$$+ \int_0^r \sum_{i=1}^n \int h^{-|\sigma|} \Phi_i\left(\frac{y}{h^\sigma}\right) \chi(U + \mathcal{R}; x + y) D_i^{l_i} f(x + y)\, dy\, dh. \tag{9}$$

Comparing (5) and (9), we see that $\tilde{f}(x) = f(x)$, $x \in U$. At the same time $\tilde{f}(x)$ is defined for all $x \in E_n$ using formula (9), and by the same token $\tilde{f}(x)$ is an extension of the function $f(x)$ beyond $U \subset E_n$ onto $E_n$. Of course, $\tilde{f}(x)$ does not necessarily coincide with $f(x)$ outside $U$. It is then natural to consider the functions $f(x)$, $\tilde{f}(x)$ defined on different copies (sheets) of Euclidean $n$-space which are glued together by the set $U$.

Lemma 1. *For* $1 \le p \le \infty$, $\Sigma_{i=1}^n \alpha_i/l_i < 1$ *we have*

$$\| D^\alpha \tilde{f} \|_{L_p(E_n)} \leqslant \delta \sum_{j=1}^n \| D_j^{l_j} f \|_{L_p(U+\mathcal{R})} + C(\delta) \| f \|_{L_p(U+\mathcal{R})}, \tag{10}$$

*and for* $1 < p < \infty$, $\Sigma_{i=1}^n \alpha_i/l_i = 1$

$$\| D^\alpha \tilde{f} \|_{L_p(E_n)} \leqslant C \sum_{j=1}^n \| D_j^{l_j} f \|_{L_p(U+\mathcal{R})} + C \| f \|_{L_p(U+\mathcal{R})}, \tag{11}$$

*where* $\delta > 0$ *is arbitrarily small and* $C$, $\delta$ *do not depend on* $f$.

The proof is obtained by estimating the derivatives in the right side of formula (9). To estimate (10) it suffices to apply Hölder's inequality, taking $r$ sufficiently small in (9).

Inequality (11) is more complicated. It is based on estimates of singular integral operators which generalize the well-known Calderón-Zygmund estimates.

We make the following assertion.

Theorem 1. *Let the support of the function* $\Phi(x) \in C_0^\infty(E_n)$ *be contained inside the first coordinate angle and let* $\int \Phi(x) dx = 0$. *Let* $1 < p < \infty$ *and for* $u(x) \in L_p(E_n)$ *let*

$$v_{\varepsilon r}(x) = \int_\varepsilon^r \int h^{-1-|\sigma|} \Phi\left(\frac{y}{h^\sigma}\right) u(x + y)\, dy\, dh.$$

*Then*

1) $\| v_{\varepsilon r} \|_{L_p(E_n)} \leqslant C_p \| u \|_{L_p(E_n)}$,
2) $v_{\varepsilon r}(x) \to v_{0r}(x)$ *in* $L_p(E_n)$ *as* $\varepsilon \to 0$.

Theorem 1 is proved in [7] and [8] (see also [9]). We also note that the case $l_1 = \cdots = l_n$ of Lemma 1 is contained in results of Calderón [10] and Smith [11].

For a vector $l = (l_1, \cdots, l_n)$ with positive integral coordinates, we consider the function space $W_p^l(\Omega)$, $\Omega \subset E_n$ with norm

$$\| f \|_{W_p^l(\Omega)} = \| f \|_{L_p(\Omega)} + \sum_{i=1}^n \| D_i^{l_i} f \|_{L_p(\Omega)}.$$

We note that the operation of extending the function $f(x)$, $x \in U + \mathcal{R}$ beyond $U$ onto $E_n$ given by formula (9) is linear by construction, and by Lemma 1 it is also a bounded operation from $W_p^l(U+\mathfrak{R})$ into $W_p^l(E_n)$.

We shall say that $\Omega \in A(W_p^l)$ if $\Omega$ is a region for which there exist a finite covering by open sets $U_k (k = 1, \cdots, K)$ and a choice of $\mathfrak{R}_k(l)$ such that

1°. $\Omega = \bigcup U_k$.

2°. For each $k = 1, \cdots, K$ there is a bounded linear extension

$$W_p^l(\Omega) \ni f(x) \to f_k(x) \in W_p^l(U_k + \mathcal{R}_k), \quad f_k(x) = f(x)\ (x \in U_k).$$

We note that one of the ways of obtaining the bounded linear extension of the function $f(x)$ required in 2° is indicated in Lemma 1. Using it we easily give some sufficient conditions for $\Omega \in A(W_p^l)$ which are connected with the extension of $f(x)$ onto the auxiliary many-sheeted Euclidean space.

In the case when $U_k + \mathfrak{R}_k \subset \Omega$ $(k = 1, \cdots, K)$ the requirement of extending the functions drops out, and condition 2° is trivially fulfilled.

For an open set $U$, let $|||f|||_U$ designate a norm of the function $f(x)$ or its trace on manifolds in $U$ which is monotonic with respect to $U$ (i.e. $|||f|||_{U_1} \leq |||f|||_{U_2}$ if the set $U_1$ is contained in $U_2$).

Theorem 2. *Let $\Omega \in A(W_p^l)$ and let the monotonic norm $|||f|||_\Omega$ satisfy the condition*[1)]

$$|||f|||_\Omega \leqslant c \sum_{k=1}^{K} |||f|||_{U_k}.$$

*Then the imbedding theorem*

$$|||f|||_{E_n} \leqslant C \|f\|_{W_p^l(E_n)},$$

*implies the imbedding theorem*

$$|||f|||_\Omega \leqslant C' \|f\|_{W_p^l(\Omega)}.$$

The proof is contained in the following chain of inequalities:

$$|||f|||_\Omega \leqslant c \sum_{k=1}^{K} |||f|||_{U_k} = c \sum_{k=1}^{K} |||f_k|||_{U_k} \leqslant c \sum_{k=1}^{K} |||\tilde{f}_k|||_{E_n} \leqslant cC \sum_{k=1}^{K} \|\tilde{f}_k\|_{W_p^l(E_n)}$$
$$\leqslant C_1 \sum_{k=1}^{K} \|f_k\|_{W_p^l(U_k+\mathcal{R}_k)} \leqslant c' \|f\|_{W_p^l(\Omega)}.$$

The proof of Theorem 2 was based on the possibility of extending the function $f(x)$ onto a many-sheeted manifold. It is clear from the example of the region $\Omega$ representing an annulus with a cut $(n = 2)$ that for $\Omega \in A(W_p^l)$ a bounded linear extension of functions beyond $\Omega$ onto $E_n$ is not always possible. We therefore consider a narrower class of regions.

We shall say that $\Omega \in A_\epsilon(W_p^l)$ if $\Omega \in A(W_p^l)$ and for a covering $\{U_k\}$ and some $\epsilon > 0$ we have the additional condition

1) Burenkov [14] has studied the question of the validity of such a condition.

3°. $\Omega \subset \bigcup_{k=1}^{K} U_k^{(\epsilon)}$,

where $U_k^{(\epsilon)} = \{x;\ x \in U_k,\ \rho(x, \Omega \setminus U_k) < \epsilon\}$.

**Theorem 3.** *Let $\Omega \in A_\epsilon(W_p^l)$. Then functions in $W_p^l(\Omega)$, $1 < p < \infty$ permit bounded linear extension beyond $\Omega \subset E_n$ onto $E_n$.*

Proof. Let the infinitely differentiable functions $\eta_k(x)$ $(k = 1, \cdots, K)$ form a partition of unity corresponding to the covering $\{U_k\}$ of the region $\Omega$, i.e.

$$0 \leqslant \eta_k(x) \leqslant 1 \quad \text{for} \quad x \in E_n,$$

$$\sum_{\kappa=1}^{K} \eta_k(x) = 1 \quad \text{for} \quad x \in \Omega,$$

$$\eta_k(x) = 0 \quad \text{outside an } \epsilon\text{-neighborhood } U_k^{(\epsilon)}.$$

We further require that $\sup_{x \in E_n} |D^\alpha \eta_k(x)| \le C$ for $|\alpha| \le \max l_i$ $(k = 1, \cdots, K)$.

A system of such functions $\eta_k(x)$ is constructed in a well-known way (see, for example [12], Russian p. 162) by averaging the characteristic functions for $U_k^{(\epsilon)}$. The requirement that the derivatives be bounded is insured by property 3°.

For the functions $f(x)$, $x \in \Omega$ let the function $f_k(x)$ satisfy property 2°. We let $\tilde{f}_k(x)$ designate their extension (beyond $U_k$) onto $E_n$ corresponding to formula (9). It is now easily seen by the estimates of Lemma 1 that the function

$$\tilde{f}(x) = \sum_{\kappa=1}^{K} \eta_k(x) \tilde{f}_k(x)$$

is the desired bounded linear extension of the function $f(x)$ beyond $\Omega$ onto $E_n$.

We note that the case $l_1 = \cdots = l_n$ is contained in [10] and [11].

## §4. Imbedding *B*-spaces and extending functions

We now turn to these considerations for the function spaces $B_{p,\theta}^l(\Omega)$.

Let the region $\Omega \subset E_n$, $l = (l_1, \cdots, l_n)$, $l_i > 0$, $m_i = s_i + k_i > l_i > k_i \ge 0$ ($s_i$, $k_i$ *are integers*), $1 \le p \le \infty$, $1 \le \theta \le \infty$.

Let $\Delta_i^s(t, \Omega) f(x) = \Delta_i^s(t) f(x)$ if the difference is constructed over points belonging to $\Omega$ together with the segment joining them, and let $\Delta_i^s(t, \Delta) f(x) = 0$ otherwise.

As usual, we take

$$\|f\|_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

We let $B_{p,\theta}^l(\Omega)$ designate the space of functions $f(x) \in L_p(\Omega)$ having generalized (Sobolev) derivatives $D_i^{k_i} f(x)$ $(i = 1, \cdots, n)$ with finite norm

$$\|f\|_{B_{p,\theta}^l(\Omega)} = \|f\|_{L_p(\Omega)} + \|f\|_{\mathscr{L}_{p,\theta}^l(\Omega)}, \tag{12}$$

where for $1 \le \theta < \infty$

$$\|f\|_{\mathscr{L}_{p,\theta}^l(\Omega)} = \sum_{i=1}^{n} \left\{ \int_0^\infty \| \Delta_i^{s_i}(t, \Omega) D_i^{\kappa_i} f \|_{L_p(\Omega)}^\theta \frac{dt}{t^{1+\theta(l_i - k_i)}} \right\}^{\frac{1}{\theta}}, \tag{13}$$

$$\|f\|_{\mathscr{L}^l_{p,\infty}(\Omega)} = \sum_{i=1}^{n} \operatorname*{ess\,sup}_{t>0} t^{-(l_i - k_i)} \|\Delta_i^{s_i}(t, \Omega) D_i^{k_i} f\|_{L_p(\Omega)} . \tag{13'}$$

This definition of the space $B^l_{p,\theta}(\Omega)$ for the region $\Omega$ is somewhat broader than that adopted in earlier articles, where differences at all points where they made sense participated in the construction of (13), (13′). This definition is more natural, since it corresponds to the local character of the classes $W^l_p(\Omega)$.

For later estimates we use Minkowski's inequality $(1 \le p \le \infty)$

$$\|\int\limits_Y \varphi(x, y)\, dy\|_{L_p(X)} \leqslant \int\limits_Y \|\varphi\|_{L_p(X)} dy \tag{14}$$

and Hardy's inequality $(1 \le p \le \infty)$

$$\|x^{-\gamma} F(x)\|_{L_p(x>0)} \leqslant C \|x^{-\gamma-1} f(x)\|_{L_p(x>0)} \tag{15}$$

where

$$F(x) = \int\limits_0^x f(t)\, dt \quad \left(\gamma > \frac{1}{p}\right), \qquad F(x) = \int\limits_x^\infty f(t)\, dt \quad \left(\gamma < \frac{1}{p}\right).$$

It is well known that Hardy's inequality reduces to Minkowski's on replacing $t$ by $xr$.

Let $e_i$ be the unit coordinate vectors. Then

$$\Delta_i^s(t) D_i^k f(x) = \int\limits_0^t d\xi_1 \ldots \int\limits_0^t D_i^{k+s} f(x + \xi_1 e_i + \ldots + \xi_s e_i)\, d\xi_s$$

so that, using (14), we obtain

$$\|\Delta_i^s(t, \Omega) D_i^k f\|_{L_p(\Omega)} \leqslant t^s \|D_i^{k+s} f\|_{L_p(\Omega)}. \tag{16}$$

The definition of the norm in the space $B^l_{p,\theta}(\Omega)$ depends on the parameters $s_i$, $k_i$. It is shown below that for a region $\Omega \in A'(B^l_{p,\theta})$ the norms with different values of $s_i$, $k_i$ are equivalent if only $s_i + k_i > l_i > k_i \ge 0$ ($s_i$, $k_i$ are integers).

We note that the norm in $B^l_{p,\theta}(\Omega)$ with parameters $s_i$, $k_i$ is estimated using (16) in terms of the norm with parameters $s'_i$, $k'_i$ if $s_i + k_i = s'_i + k'_i$, $s'_i \le s_i$, $k'_i \ge k_i$. The same can be said concerning the case $s'_i < s_i$, $k'_i = k_i$. Here we must only take into account that the difference of order $s_i$ is expressed in the form of a linear combination of translations of the difference of order $s'_i$. These norm estimates are true for any region $\Omega$.

Below we obtain their converse for a region $\Omega$ of the class $A'(B^l_{p,\theta})$; this will complete the assertion that the norms are equivalent. Returning to this question only at the very end, we then add that in the left sides of the estimates below the $k_i$ could be the maximum possible and the $s_i$ the minimum possible, while in the right sides $k_i = 0$ and the $s_i$ are arbitrarily large.

It is also shown that the norm in $B^l_{p,\theta}(\Omega)$ remains equivalent if in its construction we integrate over the step from zero to any number $T > 0$.

We consider the function $\tilde{f}(x)$ obtained as a result of modifying the right side in (8):

$$\tilde{f}(x) = (\widehat{\chi(U+\mathscr{R})f})_{r\sigma}(x)$$

$$+\int_0^r \sum_{i=1}^n \int_{E_n} \int_0^\infty h^{-1-|\sigma|-\sigma_i}\Psi_i\left(\frac{y}{h^\sigma}\right)\zeta_i\left(\frac{t}{h^{\sigma_i}}\right)\Delta_i^{m_i}(\delta t, U+\mathscr{R})f(x+y+te_i)\,dt\,dy\,dh, \quad (17)$$

where the open set $U \subset E_n$, $\chi(U + \mathfrak{R}) = \chi(U + \mathfrak{R}; x)$ is the characteristic function of the set $U + \mathfrak{R}$.

Comparing formulas (8) and (17) for the function $f(x)$, $x \in U + \mathfrak{R}$, we conclude that $\tilde{f}(x)$ represents an extension of the function $f(x)$ beyond $U$ onto $E_n$.

**Lemma 2.** *The function $\tilde{f}(x)$ defined by equation (17) satisfies the estimate*

$$\|\tilde{f}\|_{B^l_{p,\theta}(E_n)} \leqslant C\|f\|_{B^l_{p,\theta}(U+\mathscr{R})} \quad (18)$$

*with the constant $C$ not depending on $f(x)$.*

The proof is an obvious consequence of

**Lemma 3.** *For $0 \le \tau < H \le \infty$, $\sigma_i = 1/l_i$ the function*

$$G_{\tau,H}(x) = \int_\tau^H \int_{E_n} \int_0^\infty h^{-1-|\sigma|-\sigma_i}\Psi_i\left(\frac{y}{h^\sigma}\right)\zeta_i\left(\frac{t}{h^{\sigma_i}}\right)t^{\frac{1}{\theta}+l_i}g(t, x+y+te_i)\,dt\,dy\,dh \quad (19)$$

*satisfies the estimate*

$$\|G_{\tau,H}\|_{\mathscr{L}^l_{p,\theta}(E_n)} \leqslant C\int_0^1 u^{l_i}\left\{\int_{u\tau^{\sigma_i}}^{uH^{\sigma_i}} \|g\|^\theta_{L_p(E_n)}\,dt\right\}^{\frac{1}{\theta}} du$$

$$\leqslant C\left\{\int_0^{H^{\sigma_i}} \|g\|^\theta_{L_p(E_n)}\,dt\right\}^{\frac{1}{\theta}} \leqslant C\|g\|_{L_{p,\theta}(E^+_{n+1})}, \quad (20)$$

*where*

$$E^+_{n+1} = E_n \times (0, \infty), \quad \|g\|_{L_{p,\theta}(E^+_{n+1})} = \|\|g\|_{L_p(E_n)}\|_{L_\theta(0,\infty)} < \infty,$$

*and the constant $C$ does not depend on $g$, $\tau$, $H$.*

**Proof.** Let $\chi_{\tau,H}(t)$ be the characteristic function of the interval $(\tau, H)$. Carrying the differentiation over to the kernel and using estimate (16) and Minkowski's inequality (14), we obtain

$$\|\Delta_j^{s_j}(t)D_j^{k_j}G_{\tau,H}\|_{L_p(E_n)} = \|\Delta_j^{s_j}(t)D_j^{k_j}\left(\int_0^{t^{l_j}} + \int_{t^{l_j}}^\infty\right)$$

$$\times \chi_{\tau,H}(h)\int_{E^+_{n+1}}\int_0^\infty h^{-1-|\sigma|-\sigma_i}\Psi_i\left(\frac{y}{h^\sigma}\right)\zeta_i\left(\frac{v}{h^{\sigma_i}}\right)v^{\frac{1}{\theta}+l_i}g(v, x+y+ve_i)\,dv\,dy\,dh\|_{L_p(E_n)} \leqslant$$

$$\leqslant C_1\left(\int_0^{t^{l_j}} + t^{s_j}\int_{t^{l_j}}^{\infty} h^{-s_j\sigma_j}\right)h^{-k_j\sigma_j}\chi_{\tau,H}(h)\,h^{-1-\sigma_i}\int_0^{ch^{\sigma_i}} v^{\frac{1}{\theta}+l_i}\|g\|_{L_p(E_n)}dv\,dh.$$

Applying Hardy's inequality (15) after replacing $t^{l_j}$ by $t$ and then again applying Minkowski's inequality (14) after substituting $t^{\sigma_i} = h,\ v = hu$, we obtain

$$\left\{\int_0^\infty \|\Delta_j^{s_j}(t)\,D_j^{k_j}G_{\tau,H}\|^\theta_{L_p(E_n)}\frac{dt}{t^{1+\theta(l_j-k_j)}}\right\}^{\frac{1}{\theta}}$$

$$\leqslant C_2\left\{\int_0^\infty t^{-k_j\sigma_j\theta}\chi_{\tau,H}(t)\,t^{(-1-\sigma_i)\theta}\left[\int_0^{t^{\sigma_i}} v^{\frac{1}{\theta}+l_i}\|g\|_{L_p(E_n)}dv\right]^\theta\frac{dt}{t^{1-k_j\sigma_j\theta}}\right\}^{\frac{1}{\theta}}$$

$$\leqslant C_3\left\{\int_0^\infty \chi_{\tau,H}(h^{l_i})\left[\int_0^1 u^{\frac{1}{\theta}+l_i}\|g(hu,x)\|_{L_p(E_n)}du\right]^\theta dh\right\}^{\frac{1}{\theta}}$$

$$\leqslant C_3\int_0^1 u^{\frac{1}{\theta}+l_i}\left\{\int_0^\infty \chi_{\tau,H}(h^{l_i})\|g(hu,x)\|^\theta_{L_p(E_n)}dh\right\}^{\frac{1}{\theta}}dv$$

$$= C_3\int_0^1 u^{l_i}\left\{\int_0^\infty \chi_{\tau,H}\left(\left(\frac{t}{u}\right)^{l_i}\right)\|g(t,x)\|^\theta_{L_p(E_n)}dt\right\}^{\frac{1}{\theta}},$$

as was to be proved.

For the spaces $B^l_{p,\theta}(\Omega)$ we have theorems analogous (and proved analogously) to Theorems 2 and 3 for the spaces $W^l_p(\Omega)$.

Theorem 4. *If, under the conditions of Theorem* 2, $\Omega \in A(B^l_{p,\theta})$ *satisfies the imbedding theorem*

$$|||f|||_{E_n} \leqslant C\,\|f\|_{B^l_{p,\theta}(E_n)},$$

*then it also satisfies the imbedding theorem*

$$|||f|||_\Omega \leqslant C'\,\|f\|_{B^l_{p,\theta}(\Omega)}.$$

We finally return to the question of the equivalence of norms in the spaces $B^l_{p,\theta}(\Omega)$. Let $\Omega \in A(B^l_{p,\theta})$. We first see the equivalence of norms differing only by the upper limits of the integration over the step. For this it suffices to estimate

$$\left\{\int_T^\infty \|\Delta_i^{s_i}(t,\Omega)\,D_i^{k_i}f\|^\theta_{L_p(\Omega)}\frac{dt}{t^{1+\theta(l_i-k_i)}}\right\}^{\frac{1}{\theta}} \leqslant C(T)\,\|D_i^{k_i}f\|_{L_p(\Omega)}$$

in terms of the norm $\|f\|_{B^l_{p,\theta}(\Omega)}$ (with upper limit $T < \infty$ of integration over the step). But such an estimate follows from Theorem 4.

Now let $\Omega \in A(B^l_{p,\theta})$ and, in addition, let the following condition be fulfilled for the norm $|||f|||_\Omega = \|f\|_{B^l_{p,\theta}(\Omega)}$ defined in terms of maximal $k_i$ and minimal $s_i$ in integrating over the step from 0 to some positive $T < \infty$:

$$|||f|||_\Omega \leqslant c \sum_{k=1}^{K} |||f|||_{U_k}.$$

We designate the class of such regions $\Omega$ by $A'(B^l_{p,\theta})$. It is easy to show that

$$A_\varepsilon(B^l_{p,\theta}) \subset A'(B^l_{p,\theta}) \subset A(B^l_{p,\theta}).$$

A consequence of Theorem 4 and our previous reasoning and estimates is

Theorem 5. *For $\Omega \in A'(B^l_{p,\theta})$ the norms $\|f\|_{B^l_{p,\theta}(\Omega)}$ with different upper limits of integration over the step and different values of $s_i$, $k_i$ are equivalent if $s_i + k_i > l_i > k_i \geq 0$ ($s_i$, $k_i$ are integers).*

Theorem 6. *Let $\Omega \in A_\epsilon(B^l_{p,\theta})$. Then functions in $B^l_{p,\theta}(\Omega)$ permit bounded linear extension beyond $\Omega \subset E_n$ onto $E_n$.*

Concerning the proof of Theorem 6, which is analogous to that of Theorem 3, it is worth noting only that the operation of multiplication by the function $\eta(x) \in C^\infty(E_n)$, $|D^\alpha \eta| \leq C_\alpha$ ($|\alpha| \leq 1 + \max l_i$) is bounded in the space $B^l_{p,\theta}(E_n)$, as immediately follows from the possibility of taking a sufficiently high order difference in constructing the norm in $B^l_{p,\theta}(E_n)$.

We conclude by noting that there are examples (of Nikol'skiĭ, Il'in, Burenkov) showing the essential nature of the restriction adopted for the regions $\Omega$ for which the same imbedding theorems hold as for the whole space. Burenkov established [16] that if we take a horn $\Omega$ for $\mathfrak{R}$, then for all the imbedding theorems to hold for the spaces $B^l_{p,\theta}(\mathfrak{R})$ or $W^l_p(\mathfrak{R})$ it is necessary that the "curvature" of the horn coincide with the parameter $l$ of the function space.

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