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ABSTRACT DEGENERATE NON-SCALAR VOLTERRA EQUATIONS

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In this paper, we contribute to the existing theory of abstract degenerate Volterra integro-differential equations by investigating abstract degenerate non-scalar Volterra equations. We consider the generation of (A, k, B)-regularized C-pseudoresolvent families in Banach spaces, as well as their analytical properties and hyperbolic perturbation results.

Keywords: *abstract degenerate differential equation, non-scalar Volterra equation, degenerate (A, k)-regularized C-pseudoresolvent family, degenerate fractional resolvent family, well-posedness.*

1. Introduction and preliminaries

The theory of abstract degenerate Volterra integro-differential equations becomes very popular over the last few years. The main purpose of this paper, which is written in an expository manner, is to show how the techniques established in [1] and [2] can be helpful in the analysis of a substantially large class of abstract degenerate Volterra integral equations of non-scalar type.

Let X and Y be two complex Banach spaces satisfying that Y is continuously embedded in X, let the operator $C \in L(X)$ be injective, and let $\tau \in (0, \infty]$. The norm in X, resp. Y, will be denoted by $\| \cdot \|_X$, resp. $\| \cdot \|_Y$; $[R(C)]$ denotes the Banach space $R(C)$ equipped with the norm $\|x\|_{R(C)} = \|C^{-1}x\|_X$, $x \in R(C)$, and, for a given closed linear operator A in X, $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$, $x \in D(A)$. By B we denote a closed linear operator with domain and range contained in X. Suppose F is a subspace of X. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax,\ x \in D(A_{|F})$.

Let $A(t)$ be a locally integrable function from $[0, \tau)$ into $L(Y, X)$. Unless stated otherwise, we assume that $A(t)$ is not of scalar type, i. e., that there does not exist $a \in L^1_{\text{loc}}[0, \tau)$, $a \neq 0$, and a closed linear operator A in X such that $Y = [D(A)]$ and $A(t) = a(t)A$ for a. e. $t \in [0, \tau)$.

In the sequel, we will basically follow the notation employed in the monograph of J. Prüss [1] and our previous paper [2] on abstract non-degenerate equations of non-scalar type (cf. also [3; 4] for some other relevant references in this direction); the meaning of symbol A will be clear from the context. We refer the reader to [1] and [5; 6] for further information concerning abstract non-degenerate Volterra equations of scalar type. For the basic source of information on abstract degenerate Volterra integro-differential equations and abstract degenerate fractional differential equations, we may refer to the monographs [7–11], and references cited therein. Concerning the vector-valued Laplace transform, we recommend for the reader the monograph [12].

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We need to introduce the following condition

(P1): $k(t)$ is Laplace transformable, i. e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t} k(t)\, dt := \int_0^\infty e^{-\lambda t} k(t)\, dt$ exists for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > \beta$. Put $\mathrm{abs}(k) := \inf\{\mathrm{Re}(\lambda) : \tilde{k}(\lambda) \text{ exists}\}$.

Let us recall that a function $k \in L^1_{\mathrm{loc}}[0,\tau)$ is called a kernel on $[0,\tau)$ iff for every function $\phi \in C([0,\tau))$ the preassumption $\int_0^t k(t-s)\phi(s)\, ds = 0,\ t \in [0,\tau)$, implies $\phi(t) = 0,\ t \in [0,\tau)$. Set $\Theta(t) := \int_0^t k(s)\, ds,\ t \in [0,\tau)$. The principal branch is always used to take the powers and the abbreviation $*$ stands for the finite convolution product. Set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ $(\alpha > 0,\ t > 0)$, where $\Gamma(\cdot)$ denotes the Gamma function, and $g_0(t) :=$the Dirac delta distribution.

2. Degenerate (A,k)-regularized C-pseudoresolvent families

Before we start this section by introducing our basic concepts in Definition 1, we would like to recall that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two given complex Banach spaces as well as that Y is continuously embedded in X. Further on, we assume that the operator $C \in L(X)$ is injective, $\tau \in (0,\infty]$ and $B : D(B) \to X$ is a closed linear operator on X.

Definition 1. Let $k \in C[0,\tau)$ and $k \neq 0$. Consider the following linear degenerate Volterra equation:

$$Bu(t) = f(t) + \int_0^t A(t-s)u(s)\, ds,\ \ t \in [0,\tau), \tag{1}$$

where $\tau \in (0,\infty]$, $f \in C([0,\tau) : X)$ and $A \in L^1_{\mathrm{loc}}([0,\tau) : L(Y,X))$. Then a function $u \in C([0,\tau) : [D(B)])$ is said to be:

(i) *a strong solution* of (1) iff $u \in L^\infty_{\mathrm{loc}}([0,\tau) : Y)$ and (1) holds on $[0,\tau)$,

(ii) a mild solution of (1) iff there exist a sequence (f_n) in $C([0,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : [D(B)])$ such that $u_n(t)$ is a strong solution of (1) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ as well as $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0,\tau)$.

The abstract Cauchy problem (1) is said to be (kC)-well posed (C-well posed, if $k(t) \equiv 1$) iff for every $y \in Y$, there exists a unique strong solution of

$$Bu(t;y) = k(t)Cy + \int_0^t A(t-s)u(s;y)\, ds,\ \ t \in [0,\tau) \tag{2}$$

and if $u(t;y_n) \to 0$ in $[D(B)]$, uniformly on compact subsets of $[0,\tau)$, whenever (y_n) is a zero sequence in Y; (1) is said to be a-regularly (kC)-well posed (a-regularly C-well posed, if $k(t) \equiv 1$), where $a \in L^1_{\mathrm{loc}}([0,\tau))$, iff (1) is (kC)-well posed and if the equation

$$Bu(t) = (a * k)(t)Cx + \int_0^t A(t-s)u(s)\, ds,\ \ t \in [0,\tau)$$

admits a unique strong solution for every $x \in X$.

We would like to point out that every strong solution of (1) is also a mild solution of (1) as well as that the notions introduced in Definition 1 generalize the corresponding one from [2, Definition 1], given in the case that $B = I$. It is also clear that the concept of a strong (mild) solution of (1) and the concept of a (kC)-well posedness of (1) can be defined in some other ways; details are left to the interested reader.

The following definition will be crucial for our further work.

Definition 2. Let $\tau \in (0, \infty]$, $k \in C[0, \tau)$, $k \neq 0$ and $A \in L^1_{loc}([0, \tau) : L(Y, X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X, [D(B)])$ is called an (A, k, B)-regularized C-pseudoresolvent family iff the following holds:

(S1) The mappings $t \mapsto S(t)x$, $t \in [0, \tau)$ and $t \mapsto BS(t)x$, $t \in [0, \tau)$ are continuous in X for every fixed $x \in X$, $BS(0) = k(0)C$ and $S(t)C = CS(t)$, $t \in [0, \tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0, \tau)$. Then (S2) means $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0, \tau)$, and $(U(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$.

(S3) The resolvent equations

$$BS(t)y = k(t)Cy + \int_0^t A(t-s)\,dU(s)y,\ t \in [0, \tau),\ y \in Y, \tag{3}$$

$$BS(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds,\ t \in [0, \tau),\ y \in Y, \tag{4}$$

hold; (3), resp. (4), is called the first resolvent equation, resp. the second resolvent equation.

An (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be an (A, k, B)-regularized C-resolvent family if additionally:

(S4) For every $y \in Y$, $S(\cdot)y \in L^\infty_{loc}([0, \tau) : Y)$.

An operator family $(S(t))_{t\in[0,\tau)}$ in $L(X, [D(B)])$ is called a weak (A, k, B)-regularized C-pseudoresolvent family iff (S1) and (4) hold. Finally, a weak (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be a-regular ($a \in L^1_{loc}([0, \tau))$) iff $a * S(\cdot)x \in C([0, \tau) : Y)$, $x \in \overline{Y}^X$.

Let us agree on the following: A (weak) (A, k, B)-regularized C-(pseudo)resolvent family with $k(t) \equiv g_{\alpha+1}(t)$, where $\alpha \geq 0$, is also called a (weak) α-times integrated (A, B)-regularized C-(pseudo)resolvent family, whereas a (weak) 0-times integrated (A, B)-regularized C-(pseudo)resolvent family is also said to be a (weak) (A, B)-regularized C-(pseudo)resolvent family. A (weak) (A, k, B)-regularized C-(pseudo)resolvent family is also called a (weak) (A, k, B)-regularized (pseudo)resolvent family ((weak) (A, B)-regularized (pseudo)resolvent family) if $C = I$ (if $C = I$ and $k(t) \equiv 1$).

As in non-degenerate case, the integral appearing in the first resolvent equation (3) is understood in the sense of discussion following [1, Definition 6.2, p. 153]. Observe also that the condition (S3) can be rewritten in the following equivalent form:

(S3)′

$$BU(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds,\ t \in [0, \tau),\ y \in Y,$$

$$BU(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds,\ t \in [0, \tau),\ y \in Y.$$

By the norm continuity we mean the continuity in $L(X)$ and, in many places, we do not distinguish $S(\cdot)$ $(U(\cdot))$ and its restriction to Y.

The notion of an (A, k, B)-regularized C-uniqueness family plays a crucial role in proving the uniqueness of solutions of abstract degenerate Cauchy problem (1).

Definition 3. Let $\tau \in (0, \infty]$, $k \in C[0, \tau)$, $k \neq 0$ and $A \in L^1_{\text{loc}}([0, \tau) : L(Y, X))$. A strongly continuous operator family $(V(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be an (A, k, B)-regularized C-uniqueness family iff

$$V(t)By = k(t)Cy + \int_0^t V(t-s)A(s)y\,ds,\ t \in [0, \tau),\ y \in Y \cap D(B). \tag{5}$$

Before stating the following propositions, whose proofs can be deduced as in non-degenerate case (cf. [1] and [2]), we want to observe that the notion of an (A, k, I)-regularized C-uniqueness family is a special case of the notion of a weak (A, k, I)-regularized C-pseudoresolvent family and that the assertion of [2, Proposition 2 (i)] holds even in the case that the condition $S(t)C = CS(t)$, $t \in [0, \tau)$ is disregarded (cf. also Proposition 2 (i) below).

Proposition 1.

(i) *Suppose that $(S_1(t))_{t\in[0,\tau)}$ is an (A, k_1, B)-regularized C_1-pseudoresolvent family and $(S_2(t))_{t\in[0,\tau)}$ is an (A, k_2, B)-regularized C_2-uniqueness family. Then $C_2(k_2 * S_1)(t)x = (k_1 * S_2)(t)C_1x$, $t \in [0, \tau)$, $x \in \overline{Y}^X$.*

(ii) *Let $(S(t))_{t\in[0,\tau)}$ be an (A, k, B)-regularized C-pseudoresolvent family. Assume that Y has the Radon — Nikodym property. Then $(S(t))_{t\in[0,\tau)}$ is an (A, k, B)-regularized C-resolvent family. Furthermore, if Y is reflexive, then $S(t)(Y) \subseteq Y$, $t \in [0, \tau)$, and the mapping $t \mapsto S(t)y$, $t \in [0, \tau)$, is weakly continuous in Y for all $y \in Y$.*

Proposition 2.

(i) *Assume that $(V(t))_{t\in[0,\tau)}$ is an (A, k, B)-regularized C-uniqueness family, $f \in C([0, \tau) : X)$ and $u(t)$ is a mild solution of (1). Then $(kC * u)(t) = (V * f)(t)$, $t \in [0, \tau)$, and mild solutions of (1) are unique provided in addition that $k(t)$ is a kernel on $[0, \tau)$.*

(ii) *Assume $n \in \mathbb{N}$, $(S(t))_{t\in[0,\tau)}$ is an $(n-1)$-times integrated (A, B)-regularized C-pseudoresolvent family, $C^{-1}f \in C^{n-1}([0, \tau) : X)$ and $f^{(i)}(0) = 0$, $0 \le i \le n-1$. Then the following assertions hold:*

(a) *Let $(C^{-1}f)^{(n-1)} \in AC_{\text{loc}}([0, \tau) : Y)$ and $(C^{-1}f)^{(n)} \in L^1_{\text{loc}}([0, \tau) : Y)$. Then the function $t \mapsto u(t)$, $t \in [0, \tau)$, given by*

$$u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds = \int_0^t dU(s)(C^{-1}f)^{(n)}(t-s)$$

is a strong solution of (1). Moreover, $u \in C([0, \tau) : Y)$.

(b) *Let $(C^{-1}f)^{(n)} \in L^1_{\text{loc}}([0, \tau) : X)$ and $\overline{Y}^X = X$. Then the function of the form $u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds$, $t \in [0, \tau)$, is a mild solution of (1).*

(c) *Let* $C^{-1}g \in W^{n,1}_{loc}([0,\tau) : \overline{Y}^X)$, $a \in L^1_{loc}([0,\tau))$, $f(t) = (g_n * a * g^{(n)})(t)$, $t \in [0,\tau)$, *and let* $(S(t))_{t\in[0,\tau)}$ *be a-regular. Then* $u(t) = \int_0^t S(t-s)(a * (C^{-1}g)^{(n)})(s)\,ds$, $t \in [0,\tau)$, *is a strong solution of* (1).

The uniqueness of solutions in (a), (b) *or* (c) *holds provided that for each* $y \in Y \cap D(B)$ *we have* $S(t)By = BS(t)y$, $t \in [0,\tau)$.

(iii) *Let* $(S(t))_{t\in[0,\tau)}$ *be an* (A,k,B)*-regularized* C*-resolvent family. Put* $u(t;y) := S(t)y$, $t \in [0,\tau)$, $y \in Y$. *Then* $u(t;y)$ *is a strong solution of* (2), *and* (2) *is* (kC)*-well posed if* $k(t)$ *is a kernel on* $[0,\tau)$ *and* $S(t)By = BS(t)y$, $t \in [0,\tau)$, $y \in Y \cap D(B)$.

Before we clarify the Hille — Yosida type theorem for (A,k,B)-regularized C-pseudoresolvent families, it should be observed that there exists a great number of statements from [2] which can be reconsidered in degenerate case; without going into details, we only want to capture our readers' attention to the assertions of Proposition 1 (iii) (b), (c), Proposition 3 (ii), (iii), Proposition 4, Remark 1, Theorem 2, Remark 2, Proposition 5 from [2]. It is also worth noting that the class of (A,k,B)-regularized C-uniqueness families can be characterized through the vector-valued Laplace transform and that we need the condition $S(t)By = BS(t)y$, $t \geq 0$, $y \in Y \cap D(B)$, see the formulation of Theorem 1, so as to prove the injectiveness of operator $B - \tilde{A}(\lambda)$ for $\mathrm{Re}(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$ (in Theorem 4 below, this condition will not be used).

Theorem 1. *Assume* $A \in L^1_{loc}([0,\tau) : L(Y,X))$, $a \in L^1_{loc}[0,\tau)$, $a \neq 0$, $a(t)$ *and* $k(t)$ *satisfy* (P1), $\epsilon_0 \geq 0$ *and*

$$\int_0^\infty e^{-\epsilon t}\big\|A(t)\big\|_{L(Y,X)}\,dt < \infty,\ \epsilon > \epsilon_0. \tag{6}$$

(i) *Let* $(S(t))_{t\geq 0}$ *be an* (A,k,B)*-regularized* C*-pseudoresolvent family such that* $S(t)By = BS(t)y$, $t \geq 0$, $y \in Y \cap D(B)$ *and there exists* $\omega \geq 0$ *with*

$$\sup_{t>0} e^{-\omega t}\Big(\big\|S(t)\big\|_{L(X)} + \big\|BS(t)\big\|_{L(X)} + \sup_{0<s<t}(t-s)^{-1}\big\|U(t)-U(s)\big\|_{L(Y)}\Big) < \infty. \tag{7}$$

Put $\omega_0 := \max(\omega, abs(k), \epsilon_0)$ *and* $H(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)x\,dt$, $x \in X$, $\mathrm{Re}(\lambda) > \omega_0$. *Then the following holds:*

(N1) $(\tilde{A}(\lambda))_{\mathrm{Re}(\lambda)>\epsilon_0}$ *is analytic in* $L(Y,X)$, $\mathrm{R}(C_{|Y}) \subseteq \mathrm{R}(B - \tilde{A}(\lambda))$, $\mathrm{Re}(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$, *and* $B - \tilde{A}(\lambda)$ *is injective,* $\mathrm{Re}(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$.

(N2) $H(\lambda)y = \lambda\tilde{U}(\lambda)y$, $y \in Y$, $\mathrm{Re}(\lambda) > \omega_0$, $(B - \tilde{A}(\lambda))^{-1}C_{|Y} \in L(Y)$, $\mathrm{Re}(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$, $(H(\lambda))_{\mathrm{Re}(\lambda)>\omega_0}$ *is analytic in both spaces,* $L(X)$ *and* $L(Y)$, $H(\lambda)C = CH(\lambda)$, $\mathrm{Re}(\lambda) > \omega_0$, $H(\lambda)By = BH(\lambda)y$, $\mathrm{Re}(\lambda) > \omega_0$, $y \in Y \cap D(B)$, *and for every* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) > \omega_0$ *and* $\tilde{k}(\lambda) \neq 0$, *the following holds:*

$$(B - \tilde{A}(\lambda))H(\lambda)y = \tilde{k}(\lambda)Cy,\ \ y \in Y,$$

$$H(\lambda)(B - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ \ y \in Y \cap D(B).$$

(N3)
$$\sup_{n\in\mathbb{N}_0}\ \sup_{\lambda>\omega_0,\tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!}\Bigg(\Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(X)} + \Big\|\frac{d^n}{d\lambda^n}BH(\lambda)\Big\|_{L(X)} + \Big\|\frac{d^n}{d\lambda^n}H(\lambda)\Big\|_{L(Y)}\Bigg) < \infty.$$

(ii) *Assume that* (N1)–(N3) *hold. Then there exists an exponentially bounded* (A, Θ, B)*-regularized* C*-resolvent family* $(S_1(t))_{t\geq 0}$ *satisfying* $S_1(t)By = BS_1(t)y$, $t \geq 0$, $y \in Y \cap D(B)$.

(iii) *Assume that* (N1)–(N3) *hold,* $B^{-1} \in L(X)$ *and* $\overline{Y}^X = X$*. Then there exists an exponentially bounded* (A, k, B)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geq 0}$ *such that* (7) *holds and* $S(t)By = BS(t)y$, $t \geq 0$, $y \in Y \cap D(B)$.

(iv) *Assume* $(S(t))_{t\geq 0}$ *is an* (A, k, B)*-regularized* C*-pseudoresolvent family satisfying* (7) *with some* $\omega \geq 0$*. Let* $\omega' \geq \max(\omega, \operatorname{abs}(a), \operatorname{abs}(k), \epsilon_0)$*. Then* $(S(t))_{t\geq 0}$ *is* a*-regular and* $\sup_{t\geq 0} e^{-\omega' t}\|a * S(t)\|_{L(\overline{Y}^X, Y)} < \infty$ *iff there exists a number* $\omega_1 \geq \omega'$ *such that*

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_1, \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega')^{n+1}}{n!} \Big\| \frac{d^n}{d\lambda^n}\big(\tilde{a}(\lambda)H(\lambda)\big)\Big\|_{L(\overline{Y}^X, Y)} < \infty.$$

The hyperbolic perturbation results for non-scalar Volterra equations have been studied in [1, Theorem 6.1, p. 159] and [2, Theorem 3]. It is worth noting that the above-mentioned results can be generalized to degenerate non-scalar Volterra equations. Speaking-matter-of-factly, the following theorem holds good (the proof can be deduced by slightly modifying the corresponding proof of [1, Theorem 6.1], with $K_0 = S*C^{-1}BD_0$ and $K_1 = S * C^{-1}BD_1$).

Theorem 2. *Assume* $L^1_{\mathrm{loc}}([0, \tau)) \ni a$ *is a kernel on* $[0, \tau)$*,* $C(Y) \subseteq Y$*,* $\overline{Y}^X = X$*,* $CB \subseteq BC$*,*

$$D(t)y = D_0(t)y + \big(a * D_1\big)(t)y,\ t \in [0, \tau),\ y \in Y,$$

where $(D_0(t))_{t\in[0,\tau)} \subseteq L(Y)$*,* $(BD_0(t))_{t\in[0,\tau)} \subseteq L(X, [R(C)])$*,* $(BD_1(t))_{t\in[0,\tau)} \subseteq L(Y, [R(C)])$*,*

(i) $C^{-1}BD_0(\cdot)y \in BV_{\mathrm{loc}}([0, \tau) : Y)$ *for all* $y \in Y$*,* $C^{-1}BD_0(\cdot)x \in BV_{\mathrm{loc}}([0, \tau) : X)$ *for all* $x \in X$*,*

(ii) $C^{-1}BD_1(\cdot)y \in BV_{\mathrm{loc}}([0, \tau) : X)$ *for all* $y \in Y$*, and*

(iii) $CBD(t)y = BD(t)Cy$*,* $y \in Y$*,* $t \in [0, \tau)$*.*

Then the existence of an a*-regular* (A, B)*-regularized* C*-(pseudo)resolvent family* $(S(t))_{t\in[0,\tau)}$ *is equivalent with the existence of an* a*-regular* $(A + BD, B)$*-regularized* C*-(pseudo)resolvent family* $(R(t))_{t\in[0,\tau)}$*.*

Theorem 2 can be applied to abstract degenerate non-scalar Volterra equations involving abstract differential operators. For example, let $1 < p < \infty$, let $X := L^p(\mathbb{R}^n)$, and let $0 < \alpha < 2$. Then it is clear that the operators $\partial/\partial x_j$, acting with their maximal distributional domains, are commuting generators of bounded C_0-groups on X; set $A := (-i\partial/\partial x_1, \cdot\cdot\cdot, -i\partial/\partial x_n)$. Suppose that $P_1(x)$ and $P_2(x)$ are non-zero complex polynomials satisfying $P_2(x) \neq 0$, $x \in \mathbb{R}^n$, and

$$\sup_{x\in\mathbb{R}^n} \mathrm{Re}\left(\left(\frac{P_1(x)}{P_2(x)}\right)^{1/\alpha}\right) \leq \omega.$$

Define the strongly continuous operator family $(G_\alpha(t))_{t\geq 0} \subseteq L(X)$ and the operators $\overline{P_i(A)}$, $i = 1, 2$, in the usual way. Set $Y := D(\overline{P_1(A)}) \cap D(\overline{P_2(A)})$, $\|f\| := \|f\|_X +$

$\|\overline{P_1(A)}f\|_X + \|\overline{P_2(A)}f\|_X$ ($f \in Y$), $A(t) := g_\alpha(t)\overline{P_1(A)}_{|Y}$ ($t > 0$) and $C := \overline{P_2(A)}G_\alpha(0)$. Then $(Y, \|\cdot\|_Y)$ is a Banach space continuously embedded in X, $\overline{Y}^X = X$, $C\overline{P_2(A)} \subseteq \overline{P_2(A)}C$, $C(Y) \subseteq Y$, and a simple analysis shows that $(G_\alpha(t))_{t\geq 0}$ is an $(A, \overline{P_2(A)})$-resolvent C-regularized resolvent family obeying the property that for each $f \in Y$ the mapping $t \mapsto \int_0^t G_\alpha(s)f\,ds$, $t \geq 0$, is continuously differentiable in Y. Therefore, Theorem 2 can be applied with the regularizing operator C being in general the non-identity operator on X.

We continue by introducing the following definition (cf. [2, Definition 3 (i)] for non-degenerate case).

Definition 4. Let $k \in C([0,\infty))$, $k \neq 0$, $A \in L^1_{\text{loc}}([0,\infty) : L(Y,X))$, $\alpha \in (0,\pi]$, and let $(S(t))_{t\geq 0} \subseteq L(X,[D(B)])$ be a (weak) (A,k,B)-regularized C-(pseudo)resolvent family. Then it is said that $(S(t))_{t\geq 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α, if there exists an analytic function $\mathbf{S} : \Sigma_\alpha \to L(X,[D(B)])$ satisfying $\mathbf{S}(t) = S(t)$, $t > 0$, $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}(z)x = S(0)x$ and $\lim_{z\to 0, z\in\Sigma_\gamma} B\mathbf{S}(z)x = BS(0)x$ for all $\gamma \in (0,\alpha)$ and $x \in X$. We say that $(S(t))_{t\geq 0}$ is an exponentially bounded, analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, resp. bounded analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, of angle α, if $(S(t))_{t\geq 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α and for each $\gamma \in (0,\alpha)$ there exist $M_\gamma > 0$ and $\omega_\gamma \geq 0$, resp. $\omega_\gamma = 0$, such that $\|\mathbf{S}(z)\|_{L(X)} + \|B\mathbf{S}(z)\|_{L(X)} \leq M_\gamma e^{\omega_\gamma |z|}$, $z \in \Sigma_\gamma$. Since no confusion seems likely, we shall identify $S(\cdot)$ and $\mathbf{S}(\cdot)$ in the sequel.

The most important properties of exponentially bounded, analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent families are collected in the subsequent theorems.

Theorem 3.

(i) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), $\omega \geq \max(\text{abs}(k), \epsilon_0)$, (6) *holds,* $(S(t))_{t\geq 0}$ *is a weak analytic* (A,k,B)*-regularized* C*-pseudoresolvent family of angle* $\alpha \in (0,\pi/2]$ *and*

$$\sup_{z\in\Sigma_\gamma}\Big[\big\|e^{-\omega z}S(z)\big\|_{L(X)} + \big\|e^{-\omega z}BS(z)\big\|_{L(X)}\Big] < \infty \text{ for all } \gamma \in (0,\alpha). \tag{8}$$

Then there exists an analytic mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X,[D(B)])$ *such that:*

(a) $BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\text{Re}(\lambda) > \omega$, $\tilde{k}(\lambda) \neq 0$; $H(\lambda)C = CH(\lambda)$, $\text{Re}(\lambda) > \omega$,

(b) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\Big[\big\|(\lambda-\omega)H(\lambda)\big\|_{L(X)} + \big\|(\lambda-\omega)BH(\lambda)\big\|_{L(X)}\Big] < \infty$ *for all* $\gamma \in (0,\alpha)$,

(c) *there exists an operator* $E \in L(X,[D(B)])$ *such that* $BEx = k(0)Cx$, $x \in X$ *and* $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda H(\lambda)x = Ex$, $x \in X$, *and*

(d) $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda BH(\lambda)x = k(0)Cx$, $x \in X$.

(ii) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), (6) *holds,* $\omega \geq \max(\text{abs}(k), \epsilon_0)$, $\alpha \in (0,\pi/2]$, *there exists an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X,[D(B)])$ *such that* (a), (b) *and* (c) *of the item* (i) *hold and that, in the case* $\overline{Y}^X \neq X$, (d) *also holds. Then there exists a weak analytic* (A,k,B)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geq 0}$ *of angle* α *such that* (8) *holds.*

Theorem 4.

(i) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), $\omega_0 \geq \max(\text{abs}(k), \epsilon_0)$, (6) *holds*, $\alpha \in (0, \pi/2]$, $(S(t))_{t \geq 0}$ *is an analytic* (A, k, B)*-regularized* C*-resolvent family of angle* α, *the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$, *can be analytically extended to the sector* Σ_α (*we shall denote the analytical extensions of* $U(\cdot)$ *and* $S(\cdot)$ *in the space* $L(Y)$ *by the same symbols*), *and for each* $\gamma \in (0, \alpha)$ *one has:*

$$\sup_{z \in \Sigma_\gamma} \left[\left\| e^{-\omega_0 z} S(z) \right\|_{L(X)} + \left\| e^{-\omega_0 z} B S(z) \right\|_{L(X)} + \sup_{z \in \Sigma_\gamma} \left\| e^{-\omega_0 z} S(z) \right\|_{L(Y)} \right] < \infty. \quad (9)$$

Denote $H(\lambda)x = \int_0^\infty e^{-\lambda t} S(t)x\, dt$, $x \in X$, $\text{Re}(\lambda) > \omega_0$. *Then* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *is an analytic mapping,* $H_{|Y} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(Y)$ *is likewise an analytic mapping, and the following holds:*

(a) $\sup_{\lambda \in \omega_0 + \Sigma_{\frac{\pi}{2}+\gamma}} \left[\left\| (\lambda - \omega_0) H(\lambda) \right\|_{L(X)} + \left\| (\lambda - \omega_0) B H(\lambda) \right\|_{L(X)} + \left\| (\lambda - \omega_0) H(\lambda) \right\|_{L(Y)} \right]$ *is finite for all* $\gamma \in (0, \alpha)$,

(b) $BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\text{Re}(\lambda) > \omega$, $\tilde{k}(\lambda) \neq 0$; $BH(\lambda)y - \tilde{A}(\lambda)H(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\text{Re}(\lambda) > \omega$, $\tilde{k}(\lambda) \neq 0$; $H(\lambda)C = CH(\lambda)$, $\text{Re}(\lambda) > \omega_0$,

(c) *there exists an operator* $E \in L(X, [D(B)])$ *such that* $BEx = k(0)Cx$, $x \in X$, $\lim_{\lambda \to +\infty, \tilde{k}(\lambda) \neq 0} \lambda H(\lambda)x = Ex$, $x \in X$, *and*

(d) $\lim_{\lambda \to +\infty, \tilde{k}(\lambda) \neq 0} \lambda B H(\lambda)x = k(0)Cx$, $x \in X$.

(ii) *Assume* $\alpha \in (0, \pi/2]$, $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1) *and* (6) *holds. Let* $\omega_0 \geq \max(\text{abs}(k), \epsilon_0)$, *and let there exist an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *such that* $H_{|Y} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(Y)$ *is an analytic mapping, as well as that* (a)–(c) *of the item* (i) *of this theorem hold and that, in the case* $\overline{Y}^X \neq X$, (d) *also holds. Then there exists an analytic* (A, k, B)*-regularized* C*-resolvent family* $(S(t))_{t \geq 0}$ *of angle* α *such that* (9) *holds and that the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$, *can be analytically extended to the sector* Σ_α.

Remark 1. If $B^{-1} \in L(X)$, then the condition (i)/(c) in Theorem 3, i. e., the condition (i)/(c) in Theorem 4, automatically holds. Therefore, Theorem 3 and Theorem 4 taken together provide extensions of Theorem 4, Theorem 5 from [2].

Using the argumentation contained in the proofs of Theorem 6, Theorem 7 in [2], we can simply clarify the basic results about differentiability of (A, k, B)-regularized C-pseudoresolvent families in Banach spaces. Although interesting, this theme will not be considered here; let us only mention that the considerations from [2, Example 3] enable one to construct some important examples of (local) (A, k, B)-regularized resolvent families with certain hypoanalytic behaviour.

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АБСТРАКТНЫЕ ВЫРОЖДЕННЫЕ НЕСКАЛЯРНЫЕ УРАВНЕНИЯ ВОЛЬТЕРРА

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В работе исследуются вопросы существования решения абстрактных вырожденных интегро-дифференциальных уравнений Вольтерра. Рассматривается порождение (A, k, B)-регуляризованных семейств C-псевдорезольвент в банаховых пространствах, их аналитические свойства, получены результаты о гиперболических возмущениях.

Ключевые слова: *абстрактное вырожденное дифференциальное уравнение, нескалярное уравнение Вольтерра, вырожденное (A, k)-регуляризованное семейство C-псевдорезольвент, вырожденное дробное семейство резольвент, корректность.*

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