A. Borichev, On convolution equations with restrictions on supports,
Algebra i Analiz, 2002, Volume 14, Issue 2, 1–10

Алгебра и анализ
Том 14 (2002), вып. 2

ON CONVOLUTION EQUATIONS WITH RESTRICTIONS ON SUPPORTS

© Alexander Borichev

Let X be a Banach space of sequences $\{a_n\}_{n\in\mathbb{Z}}$, and let X^* be its dual space. Denote

$$X_\pm = \big\{\{a_n\}_{n\in\mathbb{Z}} \in X : a_n = 0 \text{ for } n \lessgtr 0\big\}.$$

Suppose that the shift operator S, $S\{a_n\}_{n\in\mathbb{Z}} = \{a_{n-1}\}_{n\in\mathbb{Z}}$, and its inverse S^{-1} act continuously on X. We are interested in the structure of (closed) biinvariant (that is, S, S^{-1}-invariant) subspaces of X. By the Hahn–Banach theorem, X has proper biinvariant subspaces if and only if the convolution equation

$$u * v = \{\langle S^n u, v\rangle\}_{n\in\mathbb{Z}} = 0 \tag{1}$$

has solutions $u \in X\setminus\{0\}$, $v \in X^*\setminus\{0\}$. Finding solutions of this equation becomes much more difficult if some restrictions are imposed on the supports of u and v. For example, an interesting question is whether (1) has solutions $u \in X_+\setminus\{0\}$, $v \in (X^*)_-\setminus\{0\}$. In this case, for every Banach space of sequences $\widetilde{X}$ with $\widetilde{X}_+ = X_+$, u generates a proper biinvariant subspace $\widetilde{E}$ of $\widetilde{X}$, and $\widetilde{E}\cap X_+$ is a proper S-invariant subspace of X_+.

Consider a weight σ, that is, a function $\sigma : \mathbb{Z}_+ \to (0, +\infty)$ such that

$$0 < \inf_{n\geqslant 0} \frac{\sigma(n+1)}{\sigma(n)} \leqslant \sup_{n\geqslant 0} \frac{\sigma(n+1)}{\sigma(n)} < \infty,$$

and $\sigma(0) = 1$. We set $\sigma(-n) = 1/\sigma(n)$, $n > 0$, and define

$$\ell^2_\sigma(\mathbb{Z}) = \Big\{\{a_n\}_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |a_n|^2\sigma^2(n) < \infty\Big\}, \quad \ell^2_\sigma(\mathbb{Z}_\pm) = \big(\ell^2_\sigma(\mathbb{Z})\big)_\pm.$$

In [12], Jean Esterle produced solutions of the convolution equation

$$u * v = 0, \quad u \in \ell^2_\sigma(\mathbb{Z}_+)\setminus\{0\}, \ v \in \ell^2_\sigma(\mathbb{Z}_-)\setminus\{0\}, \tag{2}$$

for some weights σ of arbitrarily slow growth. These weights σ are monotonic, but not too far more regular than that.

This work is part of the research project of the network "Analysis and Operators", Contract HPRN-CT-2000-00116, supported by the European Commission.

A. In the first part of the present paper we show that some mild regularity conditions on σ guarantee the absence of solutions to equation (2).

Clearly, (2) has no solutions in the usual ℓ^2 case where $\sigma \equiv 1$.

A1. *Suppose that $\ell^2_\sigma(\mathbb{Z}_+)$ is a convolution Banach algebra, that is, for some $c > 0$ we have $\|u * v\| \leqslant c\|u\|\|v\|$, $u, v \in \ell^2_\sigma(\mathbb{Z}_+)$. (This is so, for instance, if the weight σ satisfies the condition*

$$\sup_{n\geqslant 0} \sum_{0\leqslant k\leqslant n} \left(\frac{\sigma(n)}{\sigma(k)\sigma(n-k)}\right)^2 < \infty.)$$

Then equation (2) has no solutions.

Proof. Clearly, the weight function σ satisfies the inequality

$$\sigma(n) \leqslant c\sigma(k)\sigma(n-k), \quad 0 \leqslant k \leqslant n,\ n \geqslant 0.$$

Hence, $\sigma(n) \leqslant \|S^n\|_{\ell^2_\sigma(\mathbb{Z}_+)} \leqslant c\sigma(n)$, and the following limit exists:

$$0 < \delta = \lim_{n\to+\infty} \sigma^{-1/n}(n) < \infty.$$

Without loss of generality, we assume that $\delta = 1$. For every $u \in \ell^2_\sigma(\mathbb{Z}_+)$, the function $u(z) = \sum_{n\geqslant 0} u_n z^n$ is analytic in the unit disk $\mathbb{D}$, and is continuous up to the boundary of $\mathbb{D}$. The space of maximal ideals of $\ell^2_\sigma(\mathbb{Z}_+)$ coincides with $\overline{\mathbb{D}}$ (see, e.g., [20, Corollary 1, p. 94]).

The space $\ell^2_\sigma(\mathbb{Z}_+)$ has the (restricted) division property: for some ρ with $0 < \rho = \rho(\sigma) \leqslant 1$, if $u \in \ell^2_\sigma(\mathbb{Z}_+)$ and $u(\lambda) = 0$ for some $\lambda \in \rho\mathbb{D}$, then there exists $u_\lambda \in \ell^2_\sigma(\mathbb{Z}_+)$ such that $(z-\lambda)u_\lambda(z) = u(z)$. Furthermore, if u and v satisfy (2), and $u(\lambda) = 0$ for some $\lambda \in \rho\mathbb{D}$, then $u_\lambda * v = 0$. Indeed, the relation $(S-\lambda)u_\lambda * v = 0$ implies that $u_\lambda * v = \{c\lambda^{-n}\}_{n\in\mathbb{Z}}$. Suppose that $c \neq 0$. Since

$$|(u_\lambda * v)_n| = |\langle S^n u_\lambda, v\rangle| \leqslant \|S\|^n \|u_\lambda\|\|v\|,$$

we get $|c\lambda^{-n}| \leqslant \|S\|^n \leqslant c\sigma(n)$, which is impossible for large n because $|\lambda| < 1$.

Now we fix u and v satisfying (2) and consider the S-invariant subspace (ideal) I of $\ell^2_\sigma(\mathbb{Z}_+)$ consisting of all w such that $w * v = 0$. Denote by $Z(I)$ the set of common zeros of I in $\overline{\mathbb{D}}$. Our previous remark shows that $Z(I) \cap \rho\mathbb{D} = \varnothing$. Furthermore, $Z(I) \cap \mathbb{T}$ is a proper closed subset of zero Lebesgue measure of $\mathbb{T}$, where $\mathbb{T} = \partial\mathbb{D}$.

We denote by $I^\perp$ the set of all $w \in \ell^2_\sigma(\mathbb{Z}_-)$ vanishing on I. For every element $w \in I^\perp$ we define its analytic transform

$$\widehat{w}(\lambda) = \langle (z - \lambda + I)^{-1}, w\rangle, \quad \lambda \in \widehat{\mathbb{C}} \setminus Z(I),$$

where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\widehat{w}(\infty) = 0$. By [10, Theorem 2.4], $\widehat{w}$ is well defined and analytic in the (connected) domain $\widehat{\mathbb{C}} \setminus Z(I)$. If $w \neq 0$, then $\widehat{w} \not\equiv 0$. Furthermore,

$$S^{-k}v \in I^{\perp}, \qquad k \geqslant 0,$$
$$\widehat{S^{-k}v}(\lambda) = \widehat{v}(\lambda)\lambda^{-k}, \quad k \geqslant 0, \ \lambda \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Hence, $\lambda^k \widehat{S^{-k}v}(\lambda) = \widehat{v}(\lambda)$, $k \geqslant 0$, $\lambda \in \mathbb{D} \setminus Z(I)$, and $\widehat{v}$ vanishes at zero with all its derivatives. Thus, $v = 0$. Our assertion is proved. •

A2. *Suppose that σ is logarithmically concave, i.e., $\sigma(n-1)\sigma(n+1) \leqslant \sigma^2(n)$, and that $\lim_{n\to+\infty} \sigma(n) = \infty$. Then equation* (2) *has no solutions.*

Proof. Fix u and v satisfying (2), $u(0) \neq 0$, and consider the S^{-1}-invariant subspace E of $\ell^2_\sigma(\mathbb{Z}_-)$ consisting of all w such that $u * w = 0$. Next, consider the compression T of S on $\ell^2_\sigma(\mathbb{Z}_-)$. If $w \in E$, then $\langle u, T^n w\rangle = 0$. Hence, E generates a proper T-invariant subspace E_1 of $\ell^2_\sigma(\mathbb{Z}_-)$. Finally, $u(0) \neq 0$ implies that $e_0 = \{\delta_{0n}\}_{n\in\mathbb{Z}} \notin E_1$, where $\delta_{0n} = 0$ if $n \neq 0$, $\delta_{0n} = 1$ if $n = 0$.

Without loss of generality, we assume that $\lim_{n\to\infty} \sigma^{1/n}(n) = 1$. Using a discrete version of [5, Proposition B.1] (see also [11, Lemma 5.2]) and replacing σ by an equivalent weight $\widetilde{\sigma}$, $0 < c_1 \leqslant \widetilde{\sigma}(n)/\sigma(n) \leqslant c_2 < \infty$, we find a continuous positive integrable function φ on $[0,1)$ such that

$$2\int_0^1 r^{2n+1}\varphi(r)\,dr = \widetilde{\sigma}^2(-n), \quad n \geqslant 0.$$

Then $\ell^2_{\widetilde{\sigma}}(\mathbb{Z}_-)$ is isometrically isomorphic to the weighted Bergman space B,

$$B = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \varphi(|z|)\, dm_2(z) < \infty \Big\}.$$

Furthermore, E becomes a subspace of B invariant under multiplication by z, and E_1 becomes a subspace of B invariant under the backward shift operator $f \mapsto (f - f(0))/z$, $1 \notin E_1$.

Applying [1, Theorem 4.8], we conclude that E_1 is a subset of the Nevanlinna class. Take $f \in E\setminus\{0\}$, $f = f_1/f_2$, $f_1, f_2 \in H^\infty$. Then $f_1 \in E\setminus\{0\}$. Since $\|z^n\|_B \to 0$ as $n \to \infty$, there exists a function $g \in B$ with nontangential boundary values nowhere on $\mathbb{T}$. Then $f_1 g \in E \subset E_1$ has nontangential boundary values almost nowhere on $\mathbb{T}$, and, hence, does not belong to the Nevanlinna class. This completes the proof. •

A3. In the situations described in subsections A and B, the space $X = \ell^2_\sigma(\mathbb{Z}_+)$ satisfies the following *index* 1 *property*: for every proper S-invariant subspace E of X, the index of E (that is, the dimension of E/SE) is equal to 1. On the other hand, in the situation considered by Esterle in [12, Theorem 4.10] (see also Theorem 4.2 there), both $\ell^2_\sigma(\mathbb{Z}_+)$ and $\ell^2_\sigma(\mathbb{Z}_-)$ do not satisfy this property.

Question: Suppose that X_+ satisfies the index 1 property. Does equation (1) have solutions $u \in X_+ \setminus \{0\}$, $v \in (X^*)_- \setminus \{0\}$?

B. In the second part of this paper we consider the equation

$$u * v = 0, \quad u \in X_+ \setminus \{0\},\ v \in X^* \setminus \{0\}. \tag{3}$$

If u and v satisfy (3), then u generates a biinvariant subspace $E \subset X$ such that $E_1 = E \cap X_+$ is a proper S-invariant subspace of X, and E is generated by E_+. In the terminology of [13], such subspaces are called *analytic* subspaces.

For some weighted ℓ^2 spaces of sequences with asymmetric weights, every biinvariant subspace is analytic [13, 14]. For a short historical survey of related results on translation invariant subspaces, see [19].

Suppose that $X = \ell^2_\sigma(\mathbb{Z})$ for a weight function σ (and, consequently, $X = X^*$). If $\sigma \equiv 1$, then equation (3) has no solutions. If σ decays sufficiently fast,

$$\frac{n}{\log^\alpha n} \leqslant \log \frac{1}{\sigma(n)} = o(n), \quad n \to +\infty,$$

for some $0 < \alpha < \infty$, then equation (3) has solutions (see [6, Theorem 1.3]). For weights σ decreasing polynomially, $\sigma(n) = (n+1)^{-A}$, $n \geqslant 0$, for some $A > 0$, the existence of solutions of (3) is an open problem (see, e.g., [15, Section 8.8.11]).

Finally, we consider growing weights σ. As in part A1, we deal with the case where $\ell^2_\sigma(\mathbb{Z}_+)$ is a Banach algebra.

B1. Theorem. *Let $\ell^2_\sigma(\mathbb{Z}_+)$ be a convolution Banach algebra, and let $\lim_{n\to+\infty} \sigma(n)^{1/n} = 1$. Suppose that either*

$$1 < \liminf_{n\to+\infty} \frac{\log \sigma(n)}{\log n} \leqslant \limsup_{n\to+\infty} \frac{\log \sigma(n)}{\log n} < \infty, \tag{I}$$

or

$$\lim_{n\to+\infty} \frac{\log \sigma(n)}{\log n} = \infty. \tag{II}$$

In the second case we assume that σ extends to a smooth function on $\mathbb{R}_+$ such that the functions φ_k, $\varphi_k(t) = \log[\sigma(t)/t^k]$, $k \geqslant 0$, are concave, and the function ψ, $\psi(t) = \log \sigma(\exp t)$, is convex for large t.

Then equation (3) *has no solutions.*

Proof. Arguing as before, we fix u and v satisfying (3), and consider the S-invariant subspace (ideal) I of $\ell^2_\sigma(\mathbb{Z}_+)$ consisting of all w such that $w * v = 0$. As in Subsection A1, $Z(I) \cap \mathbb{T}$ is a proper closed subset of $\mathbb{T}$. Denote by v^k the element of $\ell^2_\sigma(\mathbb{Z}_-)$ given by

$$(v^k)_n = v_{k+n}, \quad n \leqslant 0.$$

Then $v^k \in I^\perp$, $k \in \mathbb{Z}$. We define the analytic transform as in A1, obtaining

$$\widehat{v^k}(\lambda) = \widehat{v^{k-1}}(\lambda)\lambda^{-1} + v_k \lambda^{-1}, \quad k \in \mathbb{Z},\ \lambda \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Let

$$\widehat{v^0}(\lambda) = \sum_{n\geqslant 0} a_n\lambda^n, \quad \lambda \in \rho\mathbb{D}.$$

Then by induction we obtain

$$\widehat{v^k}(\lambda) = \sum_{n\geqslant 0} a_{n+k}\lambda^n, \quad \lambda \in \rho\mathbb{D},$$
$$a_k = -v_{k+1},$$

and, as a result,

$$\widehat{v^0}(\lambda) = -\sum_{n\geqslant 0} v_{n+1}\lambda^n, \quad \lambda \in \rho\mathbb{D}.$$

Since the power series on the right converges in $\mathbb{D}$, we see that $\widehat{v^0}$ is analytic in $\mathbb{D}$. Note that by definition,

$$\widehat{v^0}(\lambda) = -\sum_{n\geqslant 0} v_{-n}\lambda^{-n-1}, \quad \lambda \in \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}.$$

If the class $\ell^2_\sigma(\mathbb{Z}_+)$ is quasianalytic (i.e., the conditions $u \in \ell^2_\sigma(\mathbb{Z}_+)$ and $u^{(k)}(\lambda) = 0$ $(k = 0, 1, \dots)$ for some $\lambda \in \overline{\mathbb{D}}$ imply $u = 0$), then $Z(I)$ is a finite subset of $\mathbb{D}$, and the results of Domar [9] show that the ideal I is determined by its zero set if the multiplicities are taken into account. Accordingly, $I^\perp$ is finite-dimensional. Hence, $v^0|\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$ is a finite linear combination of elementary fractions $1/(z-\lambda)^k$, $\lambda \in \mathbb{T}$, and v^0 cannot be smooth on $\overline{\mathbb{D}}$.

Consider

$$\mathcal{H} = \Big\{\sum_{n\geqslant 0} a_n z^n,\ z \in \overline{\mathbb{D}} : \{a_n\} \in \ell^2_\sigma(\mathbb{Z}_+)\Big\},$$
$$\mathcal{G} = \Big\{\sum_{n\geqslant 0} a_{-n} z^{-n-1},\ z \in \widehat{\mathbb{C}}\setminus\overline{\mathbb{D}} : \{a_n\} \in \ell^2_\sigma(\mathbb{Z}_-)\Big\}.$$

To complete the proof of our claim, it remains to establish the following result.

Proposition. *Suppose that the class $\mathcal{H}$ is nonquasianalytic. Then no function $h_1 \in \mathcal{H}\setminus\{0\}$ extends analytically to a function $g \in \mathcal{G}$ across a subset $\mathbb{T}\setminus F$ of the unit circle such that $h_2|F = 0$ for some $h_2 \in \mathcal{H}\setminus\{0\}$.*

Proof. We start with a rather standard argument relating the growth of functions in $\mathcal{G}$ with the rate of decay of functions in $\mathcal{H}$ near their zeros of infinite order in case (II).

Since $\log\sigma$ is concave, for every small $\varepsilon > 0$ the function $t \mapsto \sigma(t)e^{-\varepsilon t}$ attains its maximal value $M(\varepsilon)$ at a unique point t_0 such that $\varepsilon = \sigma'(t_0)/\sigma(t_0)$. Then $M(\varepsilon) \nearrow +\infty$, $\varepsilon t_0 \nearrow +\infty$ as $\varepsilon \to 0$.

First, we prove that

$$\sum_{n\geqslant 0}\sigma^2(n)e^{-2\varepsilon n}\leqslant ct_0M^2(\varepsilon),\quad \varepsilon>0, \tag{4}$$

$$t_0^k\leqslant c(k)M(\varepsilon),\quad k\geqslant 1. \tag{5}$$

Indeed, since the φ_k are concave, it follows that for every k, for sufficiently large t_0, and for $n\geqslant t_0$ we have

$$\varphi_k(n)-\varphi_k(t_0)\leqslant (n-t_0)\varphi_k'(t_0)=(n-t_0)\Big(\varepsilon-\frac{k}{t_0}\Big),$$

and for $k=1$ we obtain

$$\sigma(n)e^{-\varepsilon n}=ne^{\varphi_1(n)-\varepsilon n}\leqslant ne^{\varphi_1(t_0)-\varepsilon t_0}e^{-(n-t_0)/t_0},$$

$$\sum_{n\geqslant t_0}\sigma^2(n)e^{-2\varepsilon n}\leqslant\sum_{n\geqslant t_0}\Big(\frac{n}{t_0}\Big)^2M^2(\varepsilon)e^{-2(n-t_0)/t_0}\leqslant ct_0M^2(\varepsilon).$$

In a similar way, for $n<t_0$ we have

$$\varphi_k(t_0)-\varphi_k(n)\geqslant (t_0-n)\varphi_k'(t_0),$$

and we conclude that

$$\sum_{n<t_0}\sigma^2(n)e^{-2\varepsilon n}\leqslant ct_0M^2(\varepsilon)$$

and that

$$\log\sigma(t_0)\geqslant c(k)+k\log t_0+t_0(\log\sigma)'(t_0)=c(k)+k\log t_0+t_0\varepsilon,$$
$$M(\varepsilon)=\sigma(t_0)e^{-\varepsilon t_0}\geqslant c(k)t_0^k.$$

This proves relations (4) and (5).

Since ψ is convex, for large a the function $t\mapsto t^a/\sigma(t)$ attains its maximal value at a unique point t_1 such that $a=t_1\sigma'(t_1)/\sigma(t_1)$. Therefore,

$$\sum_{n\geqslant 1}\frac{n^{2a-2}}{\sigma^2(n)}\leqslant c\frac{t_1^{2a}}{\sigma^2(t_1)}.$$

Put $a=\varepsilon t_0$. Then $t_1=t_0$. Choosing $s\in\mathbb{Z}_+$ such that $s\leqslant a-1<s+1$ and using the Stirling formula, we obtain

$$\begin{aligned}\min_{k\in\mathbb{Z}_+}\left[\frac{\varepsilon^k}{k!}\Big(\sum_{n\geqslant 1}\frac{n^{2k}}{\sigma^2(n)}\Big)^{1/2}\right]&\leqslant\frac{\varepsilon^s}{s!}\Big(\sum_{n\geqslant 1}\frac{n^{2a-2}}{\sigma^2(n)}\Big)^{1/2}\\ &\leqslant c\frac{e^s\varepsilon^s}{s^s}\frac{t_0^a}{\sigma(t_0)}=c\frac{(\varepsilon e)^{s-a}e^{\varepsilon t_0}(\varepsilon t_0)^a}{s^s\sigma(t_0)}=c\frac{e^{s-a}}{M(\varepsilon)}\Big(\frac{a}{s}\Big)^s\Big(\frac{a}{\varepsilon}\Big)^{a-s}\\ &\leqslant c\frac{t_0^2}{M(\varepsilon)}.\end{aligned} \tag{6}$$

Now we deal with elements of the spaces $\mathcal{G}$ and $\mathcal{H}$ in case (II). By the Cauchy–Schwarz inequality and by (4) and (5), we obtain

$$\begin{aligned}|g(z)| &\leqslant \|g\|_{\mathcal{G}}\Big(\sum_{n\geqslant 0}\sigma^2(n)|z|^{-2n-2}\Big)^{1/2}\\ &\leqslant c\|g\|_{\mathcal{G}}M^2(\log|z|), \quad |z|>1,\ g\in\mathcal{G}. \qquad (7)\end{aligned}$$

If $h\in\mathcal{H}$, $h(z)=\sum_{n\geqslant 0}a_nz^n$, and h vanishes with all its derivatives at a point $\zeta\in\mathbb{T}$, then, by the Taylor formula,

$$|h(z)|\leqslant \min_{k\in\mathbb{Z}_+}\Big[\frac{|z-\zeta|^k}{k!}\sum_{n\geqslant 1}n^k|a_n|\Big], \quad z\in\overline{\mathbb{D}},$$

and the Cauchy–Schwarz inequality together with (5), (6) yields

$$\begin{aligned}|h(z)| &\leqslant \|h\|_{\mathcal{H}}\min_{k\in\mathbb{Z}_+}\Big[\frac{|z-\zeta|^k}{k!}\Big(\sum_{n\geqslant 1}\frac{n^{2k}}{\sigma^2(n)}\Big)^{1/2}\Big]\\ &\leqslant c\|h\|_{\mathcal{H}}M^{-1/2}(|z-\zeta|), \quad z\in\overline{\mathbb{D}}. \qquad (8)\end{aligned}$$

Next, we pass to case (I). We get

$$|g(z)|\leqslant c\|g\|_{\mathcal{G}}\frac{1}{(|z|-1)^c}, \quad 1<|z|<2,\ g\in\mathcal{G}. \qquad (9)$$

Furthermore, $\mathcal{H}$ is continuously embedded in a Lipschitz class on $\overline{\mathbb{D}}$. Therefore, for some $\alpha>0$ and for every $h\in\mathcal{H}$ vanishing at a point $\zeta\in\mathbb{T}$, we get

$$|h(z)|\leqslant c\|h\|_{\mathcal{H}}|z-\zeta|^\alpha, \quad z\in\overline{\mathbb{D}}. \qquad (10)$$

Now we suppose that $g\in\mathcal{G}$, $h_1,h_2\in\mathcal{H}\setminus\{0\}$, h_2 vanishes on a closed subset F of $\mathbb{T}$, and h_1 extends to g across $\mathbb{T}\setminus F$. Without loss of generality, we assume that $|h_1(z)|\leqslant 1$ and $|h_2(z)|\leqslant 1$ for $z\in\mathbb{T}$.

In case (I), we cover F by a sequence of disjoint arcs J_n, $J_n\cap F\neq\varnothing$, for every arc $J_n=\{e^{i\theta}:|\theta-\theta_n|\leqslant\delta_n\}$ we introduce the "square" $Q_n=\{re^{i\theta}:|\theta-\theta_n|\leqslant\delta_n,\ 0\leqslant r-1\leqslant\delta_n\}$, and consider the domain $\Omega=\widehat{\mathbb{C}}\setminus(\overline{\mathbb{D}}\cup\bigcup Q_n)$. The boundary of Ω is a subset of $\mathbb{T}\cup\bigcup\partial Q_n$. Fix $z\in\widehat{\mathbb{C}}\setminus\overline{\mathbb{D}}$. If $\max|J_n|$ is sufficiently small, say, less than $(|z|-1)/2$, then, by the theorem on two constants and by (9),

$$\begin{aligned}\log|g(z)| &\leqslant \int_{\partial\Omega\setminus\mathbb{T}}\log|g(w)|\omega(z,dw,\Omega)=\sum_n\int_{\partial Q_n\setminus\mathbb{T}}\log|g(w)|\omega(z,dw,\Omega)\\ &\leqslant c(z)\sum_n|J_n|\log\frac{1}{|J_n|},\end{aligned}$$

where $\omega(z,dw,\Omega)$ is harmonic measure on $\partial\Omega$ with respect to the point z in the domain Ω, and $|J|$ is the length of the arc J. Furthermore, by (10),

$$\sum_n |J_n| \log \frac{1}{|J_n|} \leqslant -c_1 \int_{\cup J_n} \log(c|h_2(z)|)\, dm(z).$$

Since the integral

$$\int_{\mathbb{T}} \log(|h_2(z)|)\, dm(z)$$

converges, we can find a covering $\{J_n\}$ of F such that

$$-\int_{\cup J_n} \log(c|h_2(z)|)\, dm(z)$$

is arbitrarily small. Hence, $|g(z)| \leqslant 1$ for $z \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and the proof is complete ($g-h_1$ is an $L^2(\mathbb{T})$-function vanishing outside a set of zero measure, hence both g and h_1 are equal to 0).

In case (II), first we assume that

$$\int_0 \log M(t)\, dt < \infty. \tag{11}$$

The set F consists of a countable set of points z_k and a perfect set F_0 of zero measure; h_2 has zeros of infinite order at the points of F_0. As before, we cover F_0 by a sequence of disjoint arcs J_n. Estimates (7) and (8) together with an argument similar to that used in case (I) show that J_n can be chosen in such a way that the expression

$$\begin{aligned}\sum_n \int_{\partial Q_n \setminus \mathbb{T}} \log|g(w)|\omega(z,dw,\Omega) &\leqslant c(z) \sum_n \int_0^{|J_n|} \log M(t)\, dt \\ &\leqslant -c(z) \int_{\cup J_n} \log(c|h_2(z)|)\, dm(z)\end{aligned}$$

is arbitrarily small. Furthermore, we can cover the (countable) set $\{z_k\} \setminus \cup J_n$ by a sequence of disjoint small arcs J'_n such that the sum

$$\sum_n \int_{\partial Q'_n \setminus \mathbb{T}} \log|g(w)|\omega(z,dw,\Omega) \leqslant c(z) \sum_n \int_0^{|J'_n|} \log M(t)\, dt$$

is arbitrarily small. After that, the proof is completed as in case (I).

Finally, if (11) fails, then inequality (8) shows that $\mathcal{H}$ is a quasianalytic class. •

B2. I do not know if an analog of Proposition is true when $\mathcal{H}$ is a quasianalytic class (say, with $F = \{1\}$).

Note that functions $h_1 \in \mathcal{H}$ may extend analytically across $\mathbb{T} \setminus \{1\}$ to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with growth there slightly more rapid than that permitted in $\mathcal{G}$ (see [7, Example 7.1]). On the other hand, if we consider spaces $\widetilde{\mathcal{G}}$ somewhat smaller than $\mathcal{G}$ (and $\mathcal{H}$ still quasianalytic), then the Levinson–Cartwright theorem and a result by the author [3, Theorem 1] imply that $h_1 \in \mathcal{H}$ cannot extend to $g \in \widetilde{\mathcal{G}}$ even across a set of positive measure (extension via nontangential boundary values); see also a related result of Beurling [2, Corollary 4.2, p. 407] for $\widetilde{\mathcal{G}} = H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$.

Under some regularity and growth conditions on M (essentially, if

$$\lim_{\delta \to 0} \frac{\log M(\delta)}{\log(1/\delta)} = \infty, \quad \int_0^\delta \log M(t)\, dt \asymp \delta \log M(\delta) \,),$$

our Proposition follows from a result by Hruščëv (see [17, Theorem 9.1], where no smoothness conditions were imposed on h_1).

The relationship between the space of smooth functions $\mathcal{H}$, the space $\mathcal{G}$, and the class of zero sets $F \subset \mathbb{T}$ of functions in $\mathcal{H}$ such that the claim of our Proposition is fulfilled deserves additional study. In particular, we may need results similar to that proposition when solving equation (3) for weighted ℓ^2 spaces of sequences with asymmetric weights.

Note that for nonquasianalytic (in the closed unit disk) Gevrey classes, the boundary zero sets are described in a rather complicated fashion [16]. Our elementary estimate (8) should be replaced by a much better estimate from [4, Theorem 2.6]. For even more precise estimates of the decay near a zero of infinite order for elements of nonquasianalytic classes on $\mathbb{T}$, see [18].

In the proof of Theorem B1 (case (II)) we could use an argument from Subsection A2 and reduce our problem to an analog of Proposition with $\mathcal{G}$ replaced by $\mathcal{G} \cap N$, where N is the Nevanlinna class in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. A related problem was considered in [8].

References

[1] Aleman A., Richter S., Ross W., *Pseudocontinuations and the backward shift*, Indiana Univ. Math. J. **47** (1998), 223–276.

[2] Beurling A., *The collected works of Arne Beurling*. Vol. 1. *Complex analysis* (L. Carleson, P. Malliavin, J. Neuberger, and J. Wermer, eds.), Contem. Math., Birkhäuser Boston, Inc., Boston, MA, 1989.

[3] Боричев А. А., *Граничные теоремы единственности для почти аналитических функций и асимметричные алгебры последовательностей*, Мат. сб. **136** (1988), №3, 324–340.

[4] Боричев А. А., *Аналитическая квазианалитичность и асимптотически голоморфные функции*, Алгебра и анализ **4** (1992), №2, 70–87.

[5] Borichev A., Hedenmalm H., *Completeness of translates in weighted spaces on the half-line*, Acta Math. **174** (1995), 1–84.

[6] Borichev A., Hedenmalm H., Volberg A., *Large Bergman spaces: invertibility, cyclicity, and subspaces of arbitrary index*, Preprint, 2000.

[7] Боричев А. А., Вольберг А. Л., *Теоремы единственности для почти аналитических функций*, Алгебра и анализ **1** (1989), №1, 146–177.

[8] Bourhim A., El Fallah O., Kellay K., *Comportement radial des fonctions de la classe de Nevanlinna*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 6, 529–532.

[9] Domar Y., *On spectral synthesis in commutative Banach algebras using closed ideals of finite co-dimension*, L'Analyse Harmonique dans le Domaine Complexe (Montpellier, 1972), Lecture Notes in Math., vol. 336, Springer, Berlin, (1973), pp. 63–78.

[10] Domar Y., *On the analytic transform of bounded linear functionals on certain Banach algebras*, Studia Math. **53** (1975), 203–224.

[11] Esterle J., *Singular inner functions and biinvariant subspaces for dissymmetric weighted shifts*, J. Funct. Anal. **144** (1997), 64–104.

[12] Esterle J., *Toeplitz operators on weighted Hardy spaces*, Алгебра и анализ **14** (2002), №2, 92–116.

[13] Esterle J., Volberg A., *Analytic left-invariant subspaces of weighted Hilbert spaces of sequences*, J. Operator Theory **45** (2001), no. 2, 265–301.

[14] Esterle J., Volberg A., *Asymptotically holomorphic functions and translation invariant subspaces of weighted Hilbert spaces of sequences*, Ann. Sci. École Norm. Sup. (4) (to appear).

[15] Hedenmalm H., Korenblum B., Zhu K., *Theory of Bergman spaces*, Grad. Texts in Math., vol. 199, Springer-Verlag, New York, 2000.

[16] Khrushchev S. V., *Sets of uniqueness for the Gevrey classes*, Ark. Mat. **15** (1977), 235–304.

[17] Хрущев С. В., *Проблема одновременной аппроксимации и стирание особенностей интегралов типа Коши*, Тр. Мат. ин-та АН СССР **130** (1978), 124–195.

[18] Matsaev V., Sodin M., *Asymptotics of the Fourier and Laplace transforms in weighted spaces of analytic functions*, Preprint, 2001.

[19] Nikol'skiĭ N., *Featured review on the paper by A. Borichev and H. Hedenmalm. Completeness of translates in weighted spaces on the half-line* (Acta Math. **174** (1995), 1–84), Math. Rev., 96f:43003.

[20] A. L. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory (C. M. Pearcy, ed.), Math. Surveys, No. 13, Amer. Math. Soc., Providence, RI, 1974, pp. 49–128.

University of Bordeaux I
351, cours de la Libération
33405 Talence, France
E-mail: borichev@math.u-bordeaux.fr

Поступило 21 января 2002 г.