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The principal series of representations of the group of p-adic quaternions in spaces over non-Archimedean fields

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1. Let K be a complete non-Archimedean valued extension of the field $\mathbf{Q}_p$ of p-adic numbers, $p \neq 2$; the norm on K continuing that on $\mathbf{Q}_p$: $\mathbf{N} = \{0, 1, 2, \ldots\}$. We fix ε, a non-square unit of the ring of p-adic integers $\mathbf{Z}_p$; $\mathbf{Q}_p(\sqrt{\varepsilon})$ is a quadratic extension of the field $\mathbf{Q}_p$. By K^* and $\mathbf{Z}_p(\sqrt{\varepsilon})^*$ we denote the multiplicative groups of the field K and the ring of integers $\mathbf{Z}_p(\sqrt{\varepsilon})$ of the field $\mathbf{Q}_p(\sqrt{\varepsilon})$. For a compact set X we denote by $C(X, K)$ the Banach space of continuous functions on X with values in K, equipped with the sup norm. For $z = x + y\sqrt{\varepsilon} \in \mathbf{Q}_p(\sqrt{\varepsilon})$ we put $\bar{z} = x - y\sqrt{\varepsilon}$. Compact Lie groups G over $\mathbf{Q}_p$ are p-non-regular, that is, there are no non-trivial continuous linear translation-invariant functionals on the space $C(G, K)$ (see [1]). Because of the absence of invariant integration, the representation theory of p-non-regular groups is not yet well developed as compared with that of p-regular groups. It is known that for every p-non-regular group there exist representations that are not completely reducible; a compact group may have infinite-dimensional irreducible representations [2]; there are papers on representations of the groups $\mathbf{Z}_p$ and Aff $\mathbf{Z}_p$. It is difficult to give a satisfactory explanation of the peculiarities of the non-regular case without considering new examples.

As is known, over the field $\mathbf{Q}_p$ there exist up to isomorphism exactly two quaternion algebras: the algebra of all (2×2)-matrices over $\mathbf{Q}_p$ and the division ring of p-adic quaternions. In [3] there is a complete description of the principal series of representations of the groups $GL(2, \mathbf{Z}_p)$ and $GL(2, \mathbf{Q}_p)$. With the division ring of quaternions one can connect several groups of a similar structure. In this note we consider the principal series of representations of one of these groups, namely PG, the factor group of the group G of unimodular quaternions by its centre. We remark that the complex representations of the group of p-adic quaternions are studied in [4], precisely using the group PG as an example.

2. It is obvious that a description of the representations of PG is equivalent to that of the representations $T(g)$ of the group G for which $T(g) \equiv 1$ on the centre of G. We use the realization of G as the group of matrices of the form $\begin{pmatrix} a & b \\ p\bar{b} & \bar{a} \end{pmatrix}$, where $a, b \in \mathbf{Q}_p(\sqrt{\varepsilon})$, $|a| = 1$, $|b| \leqslant 1$. Now G acts transitively on the set $S = \{(z_1, z_2) \in \mathbf{Q}_p(\sqrt{\varepsilon})^2 \mid |z_1| = 1, |z_2| \leqslant 1\}$. In the space $C(S, K)$ there arises an isometric quasiregular representation of G: $\left[T\begin{pmatrix} a & b \\ p\bar{b} & \bar{a} \end{pmatrix} f\right](z_1, z_2) = f(az_1 + p\bar{b}z_2,\ bz_1 + \bar{a}z_2)$. By T_π we denote the restriction of T to the space of homogeneous functions $C_\pi = \{f \in C(S, K) \mid f(tz_1, tz_2) = \pi(t) f(z_1, z_2)$ for $(z_1, z_2) \in S,\ t \in \mathbf{Z}_p(\sqrt{\varepsilon})^*\}$, where $\pi(t) \in \mathrm{Hom}(\mathbf{Z}_p(\sqrt{\varepsilon})^*, K^*)$. Passing from the $f(z_1, z_2)$ to the functions $\varphi(z) = f(1, z)$ we obtain a realization of T_π in the space $C(\mathbf{Z}_p(\sqrt{\varepsilon}), K)$: $\left[T_\pi\begin{pmatrix} a & b \\ p\bar{b} & \bar{a} \end{pmatrix} \varphi\right](z) = \pi(a + p\bar{b}z)\,\varphi(b + \bar{a}z/a + p\bar{b}z)$.

It is obvious that T_π is trivial on the centre of G if and only if $\pi(t) = 1$ for $t = \bar{t}$. From this condition it is easy to deduce that in a sufficiently small neighbourhood of the element $1 \in \mathbf{Z}_p(\sqrt{\varepsilon})$ we have, in polar coordinates (r, φ), where $r(t) = (\bar{t}t)^{1/2}$, $\varphi(t) = t/r(t)$,

$$\pi(t) = \varphi(t)^\alpha, \quad \text{where } \alpha \in K. \tag{1}$$

We call a character integral if $\alpha \in \mathbf{Z}$. The following formula gives a general form for integral characters:

$$\pi(t) = \mathrm{sign}^{k(p-1)} t\, \varphi^\alpha(t/\mathrm{sign}\, t)\, \gamma_n^{\frac{\log \varphi(t/\mathrm{sign}\, t)}{p\sqrt{\varepsilon}}}, \tag{2}$$

where $k = 0, \ldots, p$, $\alpha \in \mathbf{Z}$, γ_n is a primitive p^n-th root of unity, $\gamma_n \in K$, $\mathrm{sign}\, t = \lim\limits_{n \to \infty} t^{p^{2n}}$.

Fixing $\alpha \in \mathbf{Z}$, $\lambda, \mu \in \mathbf{N}$ such that $\lambda - \mu = \alpha$, we define $H_m^{(\lambda\mu)}$ for $m \in \mathbf{N}$ as the subspace in $C(\mathbf{Z}_p(\sqrt{\varepsilon}), K)$ of all functions that on every sphere of radius $|p|^m$ are given by the formula $P(z, \bar{z})\Omega(z)$, where $P(z, \bar{z})$ is a polynomial in the variables z and $\bar{z}$ whose degree in z and $\bar{z}$ does not exceed λ and μ, respectively, and $\Omega(z) = (1 - p\overline{z\bar{z}})^{\alpha/2-\lambda}$.

3. ***Theorem*** **1.** *For a representation T_π with non-integral characters π there are no finite-dimensional invariant subspaces. Every finite-dimensional invariant subspace of T_π in the case of an integral character π is contained in a certain space $H_m^{(\lambda\mu)}$, where m, $\lambda, \mu \in \mathbf{N}$, $\lambda - \mu = \alpha$, α and π are related by* (1).

Theorem **2.** *For a locally compact field K the representation T_π has no proper closed infinite-dimensional invariant subspaces.*

In the proof of Theorem 1 one uses infinitesimal methods and the rudiments of the theory of locally analytic functions. The methods of proof of Theorem 2 are similar to those used in [3].

The invariant subspace $H_m^{(\lambda\mu)}$ splits into the direct sum of invariant subspaces $C_m^{(\lambda_1\mu_1)}$: $H_m^{(\lambda\mu)} = \oplus C_m^{(\lambda_1\mu_1)}$, where the summation is over all sets $(\lambda_1\mu_1)$ for which $\lambda_1, \mu_1 \in \mathbf{N}$, $\lambda_1 - \mu_1 = \alpha$, $\lambda_1 \leqslant \lambda$, $\mu_1 \leqslant \mu$. The spaces $C_m^{(\lambda_1\mu_1)}$ consist of the functions that on every sphere of radius $|p|^m$ can be represented as

$$\sum_{i=-\mu_1}^{\lambda_1} c_i \sum_{j=0}^{\lambda-i} \binom{\lambda_1}{\lambda_1 - i - j} z^{i+j} \binom{\mu_1}{j} (p\bar{z})^j \Omega_{\lambda_1\mu_1}(z), \qquad c_i \in K.$$

For $n > 0$ and a representation T_π, where the character π is given by (2), all the spaces $C_m^{(\lambda\mu)}$, $\lambda - \mu = \alpha$, are invariant for $m > 0$, while for $n = 0$ the spaces $C_m^{(\lambda\mu)}$, $\lambda - \mu = \alpha$, $m \geqslant 1$, are invariant. The invariant spaces $C_m^{(\lambda\mu)}$ with $m > 0$ are, generally speaking, reducible, and apparently completely reducible. The spaces $C_0^{(\lambda\mu)}$, which are invariant only for the characters $\pi(t) = (t/\bar{t})^{\alpha/2}$ for even $\alpha \in \mathbf{Z}$, $\lambda - \mu = \alpha$, are irreducible; $\dim C_0^{(\lambda\mu)} = \lambda + \mu + 1$. The representations in the spaces $C_0^{(\lambda_1\mu_1)}$ and $C_0^{(\lambda_2\mu_2)}$ corresponding possibly to distinct characters π are equivalent if and only if their dimensions are equal. The formula for the character of T_π in the space $C_0^{(\lambda\mu)}$ is:

$$\operatorname{tr} T_\pi(g)\big|_{C_0^{(\lambda\mu)}} = (\lambda_1\lambda_2)^{-(\lambda+\mu)/2} \frac{\lambda_1^{\lambda+\mu+1} - \lambda_2^{\lambda+\mu+1}}{\lambda_1 - \lambda_2},$$

where λ_1 and λ_2 are the eigenvalues of the matrix $g \in G$.

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