R. D. Carmichael, Representation of holomorphic functions in tubes and boundary values,
Тр. МИАН, 1994, том 203, 215–234

Труды Математического института РАН
1994, том 203

UDC 517.55

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Representation of holomorphic functions in tubes and boundary values

AMS subject classifications: 46F12, 46F20, 32A07, 32A10.
Key words and phrases: ultradistributions, Cauchy integral, Fourier-Laplace transform, norm growth.

The analysis in this paper has been motivated by work of V.S. Vladimirov. For the ultradistributions of Beurling type $\mathcal{D}'((M_p), L^s)$ and of Roumieu type $\mathcal{D}'(\{M_p\}, L^s)$, where M_p, $p = 0, 1, 2, \ldots$, is a sequence of positive real numbers, a norm growth of the Cauchy integral of elements in $\mathcal{D}'(*, L^s)$, where $*$ is either (M_p) or $\{M_p\}$, is obtained. This Cauchy integral is a holomorphic function in the tube $T^C = \mathcal{R}^n + iC$, where C is a regular cone in $\mathcal{R}^n$. Holomorphic functions in tubes $\mathcal{R}^n + iC$, which are motivated by this norm growth, are defined and are shown to be representable by Fourier–Laplace transforms and Cauchy integrals. Certain of these holomorphic functions obtain ultradistributional boundary values. Problems for future research are stated.

1. Introduction

V.S. Vladimirov has done fundamental work in the study of distributional boundary value properties of holomorphic functions of several complex variables and the representation of the holomorphic functions in terms of integrals defined by the boundary values. Much of his early and subsequent analysis on these topics is contained in his classic books [17] and [18].

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Of particular interest to this author have been Vladimirov's definitions of spaces of holomorphic functions which generalize the Hardy H^p spaces corresponding to tube domains; see, for example, the works [20] and [18, chapter 10]. This analysis of Vladimirov has motivated the author to similarly consider holomorphic functions in tubes which satisfy norm growth conditions and to obtain distributional boundary value properties and representation results of these holomorphic functions with the representations being in terms of Fourier–Laplace, Cauchy, and Poisson integrals. We refer to [1–5] for some of our analysis of this type.

Recently we have shown in [8] that the Cauchy and Poisson kernel functions are elements in the ultradifferentiable spaces $\mathcal{D}(*, L^s)$, $2 \leq s \leq \infty$, where $*$ is either (M_p) or $\{M_p\}$ and M_p, $p = 0, 1, 2, 3, \ldots$ is a sequence which defines ultradistributions. This allows the Cauchy and Poisson integrals of elements in $\mathcal{D}'(*, L^s)$ to be formed, and we have obtained properties of these integrals. In this paper we continue analysis of this type by presenting recent and new results relating holomorphic functions to the ultradistributions $\mathcal{D}'(*, L^s)$.

As we show in this paper the Cauchy integral of elements in $\mathcal{D}'(*, L^s)$, $2 \leq$ $\leq s \leq \infty$, satisfies a certain norm growth estimate. We are led to consider holomorphic functions in tubes which satisfy a norm growth estimate that is similar to that of the Cauchy integral and which leads to an association of these holomorphic functions with the ultradistributions $\mathcal{D}'(*, L^s)$. We show that these holomorphic functions can be represented by Fourier-Laplace integrals and Cauchy integrals and in certain instances obtain boundary values in $\mathcal{D}'(*, L^s)$; again our research effort concerning these particular functions is guided by similar analysis of Vladimirov for other spaces. We also present questions for future research.

All notation in this paper is exactly that of [8]. The definitions of a cone C in $\mathcal{R}^n$ (with vertex at the origin $\overline{0} = (0, 0, ..., 0)$ in $\mathcal{R}^n$), projection of a cone denoted $\mathrm{pr}\,(C)$, compact subcone C' of C, and dual cone $C^* = \{t \in \mathcal{R}^n : \langle t, y \rangle \geq 0$ for all $y \in C\}$ are given in [17]. A regular cone is an open convex cone in $\mathcal{R}^n$ such that $\overline{C}$ does not contain any entire straight line. $\hat{\varphi}(x) = \mathcal{F}[\varphi(t); x]$ and $\mathcal{F}^{-1}[\varphi(t); x]$ will denote the Fourier and inverse Fourier transform of $\varphi \in L^r$, $1 \leq r \leq 2$, respectively, and are the same as in [8]. In this paper M_p, $p = 0, 1, 2, 3, \ldots$, is a sequence of positive real numbers which satisfy some of the conditions (M.1), (M.2), (M.3), (M.2)$'$, or (M.3)$'$ given in [8, p. 87]. For such a sequence M_p the associated functions $M(\rho)$ and $M^*(\rho)$ are defined in [8, p. 88].

2. Cauchy integral of ultradistributions

Let M_p, $p = 0,1,2,\ldots$, be a sequence of positive real numbers and α be any n-tuple of nonnegative integers. $\mathcal{D}((M_p), L^s)$ (respectively, $\mathcal{D}(\{M_p\}, L^s)$), $1 \leq s \leq \leq \infty$, is the set of all complex valued infinitely differentiable functions $\varphi(t)$ such that there is a constant $N > 0$ for which

$$\|D_t^\alpha \varphi(t)\|_{L^s} \leq N h^{|\alpha|} M_{|\alpha|}, \qquad |\alpha| = 0,1,2,\ldots, \tag{2.1}$$

for all $h > 0$ (respectively, for some $h > 0$) where the left side of (2.1) is the L^s norm. $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'(\{M_p\}, L^s)$ are the spaces of continuous linear forms, ultradistributions, on $\mathcal{D}((M_p), L^s)$ and $\mathcal{D}(\{M_p\}, L^s)$, respectively. The convergence of sequences in $\mathcal{D}((M_p), L^s)$ and in $\mathcal{D}(\{M_p\}, L^s)$ and properties of these spaces are stated in [8, section 3] as characterization results for $\mathcal{D}'(*, L^s)$ where $*$ is either (M_p) or $\{M_p\}$.

Let C be a regular cone in $\mathcal{R}^n$. The Cauchy kernel function corresponding to the tube $T^C = \mathcal{R}^n + iC$ is

$$K(z-t) = \int_{C^*} \exp(2\pi i \langle z-t, \eta \rangle)\, d\eta, \qquad z \in T^C, \quad t \in \mathcal{R}^n. \tag{2.2}$$

In [8, Theorem 4.1] we have proved that $K(z-t) \in \mathcal{D}(*, L^s)$, $2 \leq s \leq \infty$, as a function of $t \in \mathcal{R}^n$ for $z \in T^C$ where $*$ is either (M_p) or $\{M_p\}$ for the sequence M_p satisfying (M.1) and (M.3)$'$.

Let $U \in \mathcal{D}'(*, L^s)$, $2 \leq s \leq \infty$, where $*$ is either (M_p) or $\{M_p\}$, and let the sequence M_p satisfy (M.1) and (M.3)$'$. We now form

$$C(U; z) = \langle U_t, K(z-t) \rangle, \qquad z \in T^C; \tag{2.3}$$

the Cauchy integral of U, which is a well defined function of $z \in T^C$. Properties of this Cauchy integral are obtained in [8, section 5] including the fact that $C(U; z)$ is holomorphic in T^C. In [8, Theorem 5.1] a pointwise growth estimate for this Cauchy integral was obtained. We now prove a norm growth estimate for this Cauchy integral. Before doing so we state the following needed lemma which is proved in [9, Lemma 3.1].

L e m m a 2.1. *Let C be a regular cone in $\mathcal{R}^n$. For any n-tuple α of non-negative integers we have*

$$(t^\alpha I_{C^*}(t) e^{-2\pi \langle y,t \rangle}) \in L^r \tag{2.4}$$

for all r, $1 \le r \le \infty$, for $y \in C$ where $I_{C^*}(t)$ is the characteristic function of the dual cone C^* of C.

Using Lemma 2.1 we obtain a norm growth estimate on the Cauchy integral defined in (2.3); and this growth will motivate the growth on holomorphic functions defined later in this paper.

Theorem 2.1. *Let C be a regular cone in $\mathcal{R}^n$, and let the sequence M_p, $p = 0, 1, 2, \ldots$, satisfy* (M.1) *and* (M.3)′. *If $U \in \mathcal{D}'((M_p), L^s)$, $2 \le s < \infty$, then for each compact subcone $C' \subset C$ there exists a constant $T = T(C') > 0$ depending on $C' \subset C$ such that*

$$\|C(U;z)\|_{L^s} = \Big(\int_{\mathcal{R}^n} |C(U;x+iy)|^s\,dx\Big)^{1/s} \le$$

$$\le \begin{cases} K(U)\exp(M^*(T/|y|)), & y \in C' \subset C, \text{ if } s = 2, \\ K(U,C',s,r,n)|y|^{-n(s-r)/rs}\exp(M^*(T/|y|)), & \\ \qquad y \in C' \subset C, & \text{if } 2 < s < \infty, \end{cases} \tag{2.5}$$

where $K(U)$ is a constant depending on U if $s = 2$ and $K(U, C', s, r, n)$ is a constant depending on U, C', s, r, and n if $2 < s < \infty$. If $U \in \mathcal{D}'(\{M_p\}, L^s)$, $2 \le s < \infty$, then for each compact subcone $C' \subset C$ and arbitrary constant $T > 0$, which may or may not depend on $C' \subset C$, the inequality (2.5) *holds where $K(U)$ is a constant depending on U if $s = 2$ and $K(U, C', s, r, n)$ is a constant depending on U, C', s, r, and n if $2 < s < \infty$.*

Proof. We prove (2.5) for the case that $U \in \mathcal{D}'(\{M_p\}, L^s)$, $2 \le s < \infty$. (The proof of (2.5) for the case $U \in \mathcal{D}'((M_p), L^s)$, $2 \le s < \infty$, is given in [9].) Using the representation theorem [8, Theorem 3.2] and Fubini's theorem we obtain

$$C(U;z) = \sum_{|\alpha|=0}^{\infty} \langle D_t^\alpha g_\alpha(t), K(z-t)\rangle = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} \langle g_\alpha(t), D_t^\alpha K(z-t)\rangle =$$

$$= \sum_{|\alpha|=0}^{\infty} \int_{\mathcal{R}^n} g_\alpha(t) \int_{\mathcal{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z-t,\eta\rangle}\,d\eta dt =$$

$$= \sum_{|\alpha|=0}^{\infty} \int_{\mathcal{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle}\mathcal{F}^{-1}[g_\alpha(t);\eta]\,d\eta, \tag{2.6}$$

where each $g_\alpha(t) \in L^r$, $1/r + 1/s = 1$, and $I_{C^*}(\eta)$ is the characteristic function of C^*. Since each $g_\alpha(t) \in L^r$, $1 < r \leq 2$, for each s, $2 \leq s < \infty$, $1/r + 1/s = 1$, then $\mathcal{F}^{-1}[g_\alpha(t); \eta] \in L^s$, $2 \leq s < \infty$. If $s = 2$ then $r = 2$, and by Lemma 2.1 each of the integrals in the last term in (2.6) is in $L^1 \cap L^2$. For the case $s = 2$ we have using Parseval's equality that

$$\left\| \int_{\mathcal{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]\, d\eta \right\|_{L^2} =$$

$$= \|\mathcal{F}[\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]; x]\|_{L^2} =$$

$$= \|\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]\|_{L^2} \leq$$

$$\leq \left(\sup_{\eta \in \mathcal{R}^n} |\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle}|^2 \int_{\mathcal{R}^n} |\mathcal{F}^{-1}[g_\alpha(t); \eta]|^2\, d\eta \right)^{1/2}. \tag{2.7}$$

Now let $s > 2$ and note that by Hölder's inequality

$$\int_{\mathcal{R}^n} |I_{C^*}(\eta)\eta^\alpha e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]|^r\, d\eta \leq$$

$$\leq \left(\int_{\mathcal{R}^n} (|\mathcal{F}^{-1}[g_\alpha(t); \eta]|^r)^{s/r}\, d\eta \right)^{r/s} \| \, |I_{C^*}(\eta)\eta^\alpha e^{-2\pi\langle y,\eta\rangle}|^r \|_{L^{s/(s-r)}}, \tag{2.8}$$

where both terms on the right of (2.8) are finite since $\mathcal{F}^{-1}[g_\alpha(t); \eta] \in L^s$ and because of Lemma 2.1. Thus by (2.8) and Lemma 2.1, each of the integrals in the last term in (2.6) is in $L^1 \cap L^r$ for the case $s > 2$, $1/r + 1/s = 1$. By the Parseval inequality and (2.8) we then have for the case $s > 2$ that

$$\left\| \int_{\mathcal{R}^n} I_{C^*}(\eta)\eta^\alpha e^{2\pi i\langle z,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]\, d\eta \right\|_{L^s} =$$

$$= \|\mathcal{F}[\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]; x]\|_{L^s} \leq$$

$$\leq \|\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]\|_{L^r} \leq$$

$$\leq \left(\int_{\mathcal{R}^n} (|\mathcal{F}^{-1}[g_\alpha(t); \eta]|^r)^{s/r}\, d\eta \right)^{1/s} \times$$

$$\times \left(\int_{\mathcal{R}^n} |I_{C^*}(\eta)\eta^\alpha e^{-2\pi\langle y,\eta\rangle}|^{rs/(s-r)}\, d\eta \right)^{(s-r)/sr}. \tag{2.9}$$

From [17, Lemma 2, p. 223], given a compact subcone $C' \subset C$ there is a $\delta = \delta(C') > 0$ such that

$$\langle y, t\rangle \geq \delta|y||t|, \qquad y \in C', \quad t \in C^*. \tag{2.10}$$

Using (2.10) and the estimates [9, (3.4) and (3.5)] we have

$$\sup_{\eta \in \mathcal{R}^n} |\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle}| \leq \sup_{\eta \in C^*} (|\eta|^{|\alpha|} \exp(-2\pi\delta|y||\eta|)) \leq$$

$$\leq \begin{cases} 1, & |\alpha| = 0, \\ \left(\dfrac{|\alpha|}{2\pi\delta|y|}\right)^{|\alpha|}, & |\alpha| = 1, 2, \ldots, \end{cases} \tag{2.11}$$

for $y \in C' \subset C$ and $\delta = \delta(C') > 0$.

For the case $s = 2$, we use (2.6), (2.7), (2.11), and the Parseval equality to obtain for $y \in C' \subset C$ that

$$\|C(U; z)\|_{L^2} \leq \sum_{|\alpha|=0}^{\infty} \left\| \int_{\mathcal{R}^n} I_{C^*}(\eta) \eta^\alpha e^{2\pi i\langle z,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]\, d\eta \right\|_{L^2} \leq$$

$$\leq \sum_{|\alpha|=0}^{\infty} \left(\sup_{\eta \in \mathcal{R}^n} |\eta^\alpha I_{C^*}(\eta) e^{-2\pi\langle y,\eta\rangle}|^2 \int_{\mathcal{R}^n} |\mathcal{F}^{-1}[g_\alpha(t); \eta]|^2\, d\eta \right)^{1/2} \leq$$

$$\leq \sum_{|\alpha|=0}^{\infty} \left(\frac{|\alpha|}{2\pi\delta|y|}\right)^{|\alpha|} \|g_\alpha(t)\|_{L^2}, \tag{2.12}$$

where we are using the convention that $|\alpha|^{|\alpha|} = 1$ if $\alpha = \bar{0}$. For $s > 2$ and $C' \subset C$ let us write $\delta = \delta(C') > 0$ in (2.10) as $\delta = \delta_1 + \delta_2$ where $\delta_1 = \delta_1(C') > 0$ and $\delta_2 = \delta_2(C') > 0$. Now using (2.6), (2.9), Parseval's inequality, (2.10) with $\delta = \delta_1 + \delta_2$, (2.11), and [15, Theorem 32, p. 39] we have for $y \in C' \subset C$

$$\|C(U; z)\|_{L^s} \leq \sum_{|\alpha|=0}^{\infty} \left\| \int_{\mathcal{R}^n} I_{C^*}(\eta) \eta^\alpha e^{2\pi i\langle z,\eta\rangle} \mathcal{F}^{-1}[g_\alpha(t); \eta]\, d\eta \right\|_{L^s} \leq$$

$$\leq \sum_{|\alpha|=0}^{\infty} \|\mathcal{F}^{-1}[g_\alpha(t); \eta]\|_{L^s} \|I_{C^*}(\eta) \eta^\alpha e^{-2\pi\langle y,\eta\rangle}\|_{L^{rs/(s-r)}} \leq$$

$$\leq \sum_{|\alpha|=0}^{\infty} \|g_\alpha(t)\|_{L^r} \big(\sup_{\eta \in \mathcal{R}^n} |I_{C^*}(\eta) \eta^\alpha \exp(-2\pi\delta_1|y||\eta|)| \big) \times$$

$$\times\Big(\int\limits_{C^*}\exp(-2\pi\delta_2|y||\eta|rs/(s-r))\,d\eta\Big)^{(s-r)/rs}\le$$

$$\le\Omega_n\Big(\int\limits_0^\infty w^{n-1}\exp(-2\pi\delta_2|y|wrs/(s-r))\,dw\Big)^{(s-r)/rs}\times$$

$$\times\sum_{|\alpha|=0}^\infty\|g_\alpha(t)\|_{L^r}\Big(\frac{|\alpha|}{2\pi\delta_1|y|}\Big)^{|\alpha|}=$$

$$=\Omega_n\left((n-1)!(2\pi\delta_2|y|rs/(s-r))^{-n}\right)^{(s-r)/rs}\times$$

$$\times\sum_{|\alpha|=0}^\infty\|g_\alpha(t)\|_{L^r}\Big(\frac{|\alpha|}{2\pi\delta_1|y|}\Big)^{|\alpha|},\tag{2.13}$$

where Ω_n is the surface area of the unit sphere in $\mathcal{R}^n$.

The condition [8, (3.12)] on the sequence $\{g_\alpha(t)\}$ in [8, Theorem 3.2, p. 90], and hence on the sequence $\{g_\alpha(t)\}$ in this theorem, holds for all $k>0$ and is obviously equivalent to the condition

$$\sup_\alpha(k^{|\alpha|}\,M_{|\alpha|}\,\|g_\alpha\|_{L^r})<\infty\tag{2.14}$$

for all $k>0$. For the case $s=2$ in this theorem and for all $k>0$ we return to (2.12) and use (2.14) to obtain

$$\|C(U;z)\|_{L^2}\le\sup_\alpha(k^{|\alpha|}M_{|\alpha|}\|g_\alpha\|_{L^2})\times$$

$$\times\sum_{|\alpha|=0}^\infty\Big(\frac{1}{2}\Big)^{|\alpha|}(k\pi\delta)^{-|\alpha|}|y|^{-|\alpha|}|\alpha|^{|\alpha|}(M_{|\alpha|})^{-1}.\tag{2.15}$$

Recall from [8, p. 97] that

$$|\alpha|^{|\alpha|}\le e^{|\alpha|}|\alpha|,\qquad |\alpha|=1,2,3,...,\tag{2.16}$$

and recall our convention that $|\alpha|^{|\alpha|}=1$ if $|\alpha|=0$; using this in (2.15) and putting $T=e/k\pi\delta$ we have from (2.15)

$$\|C(U;z)\|_{L^2}\le\sup_\alpha(k^{|\alpha|}\,M_{|\alpha|}\|g_\alpha\|_{L^2})\times$$

$$\times\sum_{|\alpha|=0}^\infty\Big(\frac{1}{2}\Big)^{|\alpha|}\Big(\frac{T}{|y|}\Big)^{|\alpha|}\frac{|\alpha|!}{M_{|\alpha|}}\le$$

$$\leq \sup_{\alpha} (k^{|\alpha|} M_{|\alpha|} \|g_\alpha\|_{L^2}) \Big(\sum_{|\alpha|=0}^{\infty} \Big(\frac{1}{2}\Big)^{|\alpha|} \Big) (M_0)^{-1} \exp(M^*(T/|y|)) \tag{2.17}$$

for $y \in C' \subset C$ where we have used the calculation [8, (5.12)]. (2.17) holds for all $k > 0$; thus $T = e/k\pi\delta > 0$ is arbitrary and may or may not depend on $C' \subset C$ given the choice of the arbitrary $k > 0$. (2.17) thus proves (2.5) for the case that $U \in \mathcal{D}'(\{M_p\}, L^2)$, where

$$K(U) = (M_0)^{-1} \sup_{\alpha} (k^{|\alpha|} M_{|\alpha|} \|g_\alpha\|_{L^2}) \Big(\sum_{|\alpha|=0}^{\infty} \Big(\frac{1}{2}\Big)^{|\alpha|} \Big).$$

For $s > 2$ we return to (2.13) and proceed using (2.14) and (2.16) to obtain (2.5) for the case $U \in \mathcal{D}'(\{M_p\}, L^s)$, $2 < s < \infty$, similarly as we did for the case $U \in \mathcal{D}'(\{M_p\}, L^2)$, where

$$K(U, C', s, r, n) = \Omega_n((n-1)!(2\pi\delta_2 rs/(s-r))^{-n})^{(s-r)/rs}(1/M_0) \times$$
$$\times \sup_{\alpha} (k^{|\alpha|} M_{|\alpha|} \|g_\alpha\|_{L^r}) \Big(\sum_{|\alpha|=0}^{\infty} \Big(\frac{1}{2}\Big)^{|\alpha|} \Big)$$

and $T = e/k\pi\delta_1 > 0$ is arbitrary and may or may not depend on $C' \subset C$ since $k > 0$ is arbitrary.

Under the assumption $U \in \mathcal{D}'((M_p), L^s)$, $2 \leq s < \infty$, the stated conclusions in (2.5) are obtained using the characterization result [8, Theorem 3.1] together with [8, (3.11), p. 90] and proceeding with analysis similar to that in the case for $U \in \mathcal{D}'(\{M_p\}, L^s)$ given previously in this proof; see the proof of [9, Theorem 3.1]. For $U \in \mathcal{D}'((M_p), L^s)$, the constant $T = T(C') > 0$ in (2.5) depends on the compact subcone $C' \subset C$. As in the above proof, T is chosen to be $T = e/k\pi\delta$ for the case $U \in \mathcal{D}'((M_p), L^2)$ and $T = e/k\pi\delta_1$ for the cases $U \in \mathcal{D}'((M_p), L^s)$, $2 < s < \infty$, where now $k > 0$ is fixed from [8, (3.9) and (3.11), p. 90]; thus the constant T is fixed and depends on $C' \subset C$ because of the δ for the case $s = 2$ and the δ_1 for the cases $2 < s < \infty$. The proof of Theorem 2.1 is complete.

3. Holomorphic functions in tubes

We assume that the sequence M_p, $p = 0, 1, 2, \ldots$, satisfies (M.1) and (M.3)$'$ throughout this section, which ensures that $M^*(\rho)$, $\rho > 0$, is finite ([8, Lemma 2.1, p. 88] and [12, lines 7–9, p. 94]).

Let B be a proper open subset of $\mathcal{R}^n$. Let $d(y)$ denote the distance from $y \in B$ to the complement of B in $\mathcal{R}^n$. Let $f(z)$ be defined in $T^B = \mathcal{R}^n + iB$ and satisfy

$$\|f(x+iy)\|_{L^r} = \left(\int_{\mathcal{R}^n} |f(x+iy)|^r \, dx \right)^{1/r} \leq$$

$$\leq K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \qquad y \in B, \tag{3.1}$$

where $K > 0$, $T > 0$, $m \geq 0$, and $q \geq 0$ are all independent of $y \in B$ and M^* is the associated function of the sequence M_p, $p = 0, 1, 2, \ldots$, defined in [8, p. 88].

If $B = C$, an open connected cone in $\mathcal{R}^n$, we have from the formula

$$d(y) = \inf_{t \in \mathrm{pr}(C^*)} \langle t, y \rangle, \qquad y \in C,$$

given in [18, p. 159] that $d(y) \leq |y|$, $y \in C$. From this inequality we have for $B = C$ that the term $(1 + (d(y))^{-m})^q$ in (3.1) is a generalizing factor in this growth as opposed to a similar term with $d(y)$ replaced by $|y|$. Further the right side of the growth (3.1) allows for divergence to ∞ as y approaches any point on the boundary of C and not just as y approaches $\bar{0}$ as would be the case if $|y|$ were in place of $d(y)$ in (3.1). Because of the norm growth obtained on the Cauchy integral in Theorem 2.1 and because of the generalizing nature of $d(y)$ in (3.1) as opposed to $|y|$ we are led to consider holomorphic functions in tubes which satisfy (3.1). We show that for the cases $1 < r \leq 2$, holomorphic functions in tubes T^B which satisfy (3.1) can be represented by Fourier–Laplace integrals. We need several lemmas which we now state; proofs are contained in [7].

L e m m a 3.1. *Let B be a proper open connected subset of $\mathcal{R}^n$. Let $1 \leq s < < \infty$. Let $g(t)$ be a measurable function on $\mathcal{R}^n$ which satisfies*

$$\|e^{-2\pi\langle y,t\rangle} g(t)\|_{L^s} \leq K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \qquad y \in B, \tag{3.2}$$

where $K > 0$, $T > 0$, $m \geq 0$, and $q \geq 0$ are independent of $y \in B$. Then

$$f(z) = \int_{\mathcal{R}^n} g(t) e^{2\pi i \langle z,t \rangle} \, dt, \qquad z \in T^B, \tag{3.3}$$

is a holomorphic function of $z \in T^B = \mathcal{R}^n + iB$.

In the following lemma supp (g) denotes the support of the function g.

L e m m a 3.2. *Let C be an open connected cone in $\mathcal{R}^n$. Let $1 \leq s < \infty$. Let $g(t)$ be a measurable function on $\mathcal{R}^n$ such that* (3.2) *holds for $y \in C$. Then* $\operatorname{supp}(g) \subseteq C^*$ *almost everywhere.*

L e m m a 3.3. *Let B denote an open connected subset of $\mathcal{R}^n$ which does not contain $\overline{0}$. Let $1 < r \leq 2$. Let $f(z)$ be holomorphic in T^B and satisfy* (3.1). *Then for all y and y' in B we have*

$$e^{2\pi\langle y,t\rangle} h_y(t) = e^{2\pi\langle y',t\rangle} h_{y'}(t)$$

for almost every $t \in \mathcal{R}^n$, where

$$h_y(t) = \mathcal{F}^{-1}[f(x+iy);t], \qquad y \in B,$$

is the L^s, $1/r + 1/s = 1$, inverse Fourier transform of $f(x+iy)$, $y \in B$.

We now show that holomorphic functions in tubes which satisfy (3.1) for $1 < r \leq 2$ have Fourier–Laplace integral representations.

T h e o r e m 3.1. *Let B denote an open connected subset of $\mathcal{R}^n$ which does not contain $\overline{0} \in \mathcal{R}^n$. Let $1 < r \leq 2$. Let $f(z)$ be holomorphic in $T^B = \mathcal{R}^n + iB$ and satisfy* (3.1). *Then there exists a measurable function $g(t)$, $t \in \mathcal{R}^n$, such that* (3.2) *holds, $1/r + 1/s = 1$, where $K > 0$, $T > 0$, $m \geq 0$, and $q \geq 0$ are independent of $y \in B$, and*

$$f(z) = \int_{\mathcal{R}^n} g(t)\, e^{2\pi i\langle z,t\rangle}\, dt, \qquad z \in T^B. \tag{3.4}$$

P r o o f. Put

$$g(t) = e^{2\pi\langle y,t\rangle} h_y(t), \qquad y \in B, \tag{3.5}$$

where $h_y(t) = \mathcal{F}^{-1}[f(x+iy);t]$, $y \in B$, is the L^s, $1/r + 1/s = 1$, inverse Fourier transform of $f(x+iy)$, $y \in B$; by the Plancherel Fourier transform theory $h_y(t)$ is an element of L^s since by (3.1) $f(x+iy) \in L^r$ as a function of $x \in \mathcal{R}^n$ for $y \in B$. By Lemma 3.3 $g(t)$ is independent of $y \in B$. From (3.5) we have

$$e^{-2\pi\langle y,t\rangle} g(t) = \mathcal{F}^{-1}[f(x+iy);t], \qquad y \in B. \tag{3.6}$$

Since $f(x+iy) \in L^r$, $1 < r \leq 2$, as a function of $x \in \mathcal{R}^n$ for $y \in B$, then $(e^{-2\pi\langle y,t\rangle} g(t)) \in L^s$, $1/r + 1/s = 1$, $y \in B$, by the Plancherel Fourier transform theory, and

$$\|e^{-2\pi\langle y,t\rangle} g(t)\|_{L^s} \leq \|f(x+iy)\|_{L^r} \leq K(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)),$$

$y \in B$, by the Parseval inequality and (3.1); thus (3.2) is obtained. From (3.6) we have

$$f(x+iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}g(t); x], \qquad z = x+iy \in T^B, \tag{3.7}$$

by the Plancherel Fourier transform theory with the Fourier transform being in L^r. But by (3.2) and Lemma 3.1 the integral on the right side of (3.4), which is the L^1 transform of $(e^{-2\pi\langle y,t\rangle}g(t))$, $y \in B$, is a holomorphic function of $z \in T^B$. (Recall from the proof of Lemma 3.1 given in [7, Lemma 2.1] that $(e^{-2\pi\langle y,t\rangle}g(t)) \in$ $\in L^1$, $y \in B$, since (3.2) is satisfied here.) From the Plancherel Fourier transform theory we know that the L^1 and L^r, $1 < r \leq 2$, Fourier transforms of the same function are equal when both of these transforms exist; hence the desired equality (3.4) follows from (3.7). The proof is complete.

The proof of the following corollary is obtained by combining Theorem 3.1 and Lemma 3.2.

C o r o l l a r y 3.1. *Let C be an open connected cone in $\mathcal{R}^n$. Let $1 < r \leq 2$. Let $f(z)$ be holomorphic in T^C and satisfy* (3.1) *for $y \in C$. Then there exists a measurable function $g(t)$, $t \in \mathcal{R}^n$, such that* (3.2) *holds for $y \in C$,* $\operatorname{supp}(g) \subseteq C^*$ *almost everywhere, and* (3.4) *holds for $z \in T^C$.*

The conclusion (3.2) on $g(t)$ in Corollary 3.1 implies that $(\exp(-2\pi\langle y,t\rangle) \times$ $\times g(t)) \in L^s$, $y \in C$. If $g(t)$ itself were in L^s, it then could be the inverse Fourier transform of some function in L^r, $1 < r \leq 2$, $1/r + 1/s = 1$. If this were the case, we can obtain additional information on the holomorphic function $f(z)$ in Corollary 3.1 as noted in the following corollary.

C o r o l l a r y 3.2. *Let C be a regular cone in $\mathcal{R}^n$. Let $1 < r \leq 2$. Let $f(z)$ be holomorphic in T^C and satisfy* (3.1) *for $y \in C$. Let the function $g(t)$ of Corollary* 3.1 *be the inverse Fourier transform of a function $h(t) \in L^r$, $1 < r \leq 2$. Then $g(t) \in L^s$, $1/r + 1/s = 1$, and*

$$f(z) = \int\limits_{\mathcal{R}^n} h(t)K(z-t)\,dt, \qquad z \in T^C. \tag{3.8}$$

P r o o f. By assumption the obtained function $g(t)$ of Corollary 3.1 satisfies $g(\eta) = \mathcal{F}^{-1}[h(t);\eta]$, $\eta \in \mathcal{R}^n$, for $h(t) \in L^r$, $1 < r \leq 2$, here. By the Plancherel theory we then have that $g(\eta) \in L^s$, $1/r + 1/s = 1$. Note that the Cauchy integral in (3.8) is well defined here because of the properties of $K(z-t)$ as given in [1, Lemma 2.1] or [8, Theorem 4.1]. Using the definition of the Cauchy kernel $K(z-t)$, $t \in \mathcal{R}^n$, $z \in T^C$, given in (2.2), Fubini's theorem, and the representation (3.4) obtained in the conclusion of Corollary 3.1 we obtain for $z \in T^C$

$$\int_{\mathcal{R}^n} h(t)K(z-t)\,dt = \lim_{k\to\infty} \int_{|t|\le k} h(t) \int_{C^*} \exp(2\pi i\langle z-t,\eta\rangle)\,d\eta\,dt =$$

$$= \lim_{k\to\infty} \int_{C^*} \exp(2\pi i\langle z,\eta\rangle) \int_{|t|\le k} h(t)\exp(-2\pi i\langle t,\eta\rangle)\,dt\,d\eta =$$

$$= \int_{C^*} g(\eta)\exp(2\pi i\langle z,\eta\rangle)\,d\eta = f(z),$$

where we recall from the conclusion of Corollary 3.1 that $\mathrm{supp}\,(g) \subseteq C^*$ almost everywhere here. This proves (3.8).

The following result is a dual theorem to Theorem 3.1. The proof is easily obtained from the stated assumption (3.9) by using Lemma 3.1 and the Parseval inequality for Fourier transforms.

Theorem 3.2. *Let B be a proper open connected subset of $\mathcal{R}^n$. Let $1 < r \le 2$. Let $g(t)$ be a measurable function on $\mathcal{R}^n$ which satisfies*

$$\|e^{-2\pi\langle y,t\rangle} g(t)\|_{L^r} \le K(1+(d(y))^{-m})^q \exp(M^*(T/|y|)), \qquad y \in B, \qquad (3.9)$$

where $K > 0$, $T > 0$, $m \ge 0$, and $q \ge 0$ are independent of $y \in B$. Then

$$f(z) = \int_{\mathcal{R}^n} g(t)e^{2\pi i\langle z,t\rangle}\,dt, \qquad z \in T^B,$$

is a holomorphic function of $z \in T^B$ and satisfies

$$\|f(x+iy)\|_{L^s} \le K(1+(d(y))^{-m})^q \exp(M^*(T/|y|)), \qquad y \in B.$$

For the case that B is an open connected subset of R^n which does not contain $\bar{0} \in \mathcal{R}^n$ and $r = 2$, Theorem 3.2 is a converse result to Theorem 3.1.

4. Boundary values

Let C be a regular cone in $\mathcal{R}^n$. For the sequence of positive real numbers M_p, $p = 0,1,2,\ldots$, put $m_p^* = m_p/p$, where $m_p = M_p/M_{p-1}$, $p = 1,2,3,\ldots$ Petzsche

[13] has obtained properties of ultradistributions under the assumption that m_p^* is a nondecreasing sequence. (For example $M_p = (p!)^s$, $s > 1$, is a sequence for which m_p^* is nondecreasing.) We consider such sequences M_p in this section.

Specifically, we make the following assumptions on M_p, $p = 0,1,2,3,\ldots$, throughout this section. We assume that M_p satisfies (M.1), (M.2), and (M.3)'. We further assume that for M_p, the sequence m_p^* is nondecreasing and the sequence $M_p/p!$ satisfies (M.1).

We consider functions $f(z)$ which are holomorphic in $T^C = \mathcal{R}^n + iC$ and which satisfy

$$\|f(x+iy)\|_{L^r} \leq K \exp(M^*(T/|y|)), \qquad y \in C, \tag{4.1}$$

for $1 < r \leq 2$, where $K > 0$ and $T > 0$ are constants which are independent of $y \in C$ and M^* is the associated function of the sequence M_p, $p = 0,1,2,\ldots$, defined in [8, p. 88]; thus the norm growth which we are considering in this section is that in (3.1) with $m = 0$ or $q = 0$ there. We show that holomorphic functions $f(z)$ in T^C which satisfy (4.1) with $1 < r \leq 2$ have ultradistributional boundary values. The proof of this result is like that of [9, Theorem 5.1]; see [9] for details.

T h e o r e m 4.1. *Let $f(z)$ be holomorphic in T^C and satisfy* (4.1) *with* $1 < < r \leq 2$. *Then there exists an element $U \in \mathcal{D}'((M_p), L^1)$ such that*

$$\lim_{\substack{y \to \bar{0} \\ y \in C}} f(x+iy) = U \tag{4.2}$$

in $\mathcal{D}'((M_p), L^1)$.

The proof of Theorem 4.1 uses facts obtained in the results of section 3.

The following result is a dual theorem to Theorem 4.1, and boundary value results are obtained in $\mathcal{D}'((M_p), L^r)$, $1 < r \leq 2$. The proof of the following result, which is contained in [9, section 5], uses properties of the spaces $\mathcal{F}\mathcal{D}((M_p), L^r)$ and $\mathcal{F}'\mathcal{D}((M_p), L^r)$, $1 < r \leq 2$, which are Fourier transform spaces of $\mathcal{D}((M_p), L^r)$ and $\mathcal{D}'((M_p), L^r)$ and which are constructed in [9, section 2].

T h e o r e m 4.2. *Let $1 < r \leq 2$. Let $g(t)$ be a measurable function on $\mathcal{R}^n$ such that*

$$\|e^{-2\pi\langle y,t\rangle} g(t)\|_{L^r} \leq K \exp(M^*(T/|y|)), \qquad y \in C, \tag{4.3}$$

where $K > 0$ and $T > 0$ are constants which are independent of $y \in C$. Then

$$f(z) = \int_{\mathcal{R}^n} g(t) e^{2\pi i \langle z,t \rangle} \, dt, \qquad z \in T^C, \tag{4.4}$$

is holomorphic in T^C, satisfies (4.1) with L^r replaced by L^s, $1/r + 1/s = 1$, and there is an element $U \in \mathcal{D}'((M_p), L^r)$ such that

$$\lim_{\substack{y \to \bar{0} \\ y \in C}} f(x + iy) = U \tag{4.5}$$

in $\mathcal{D}'((M_p), L^r)$.

The boundary value U obtained in (4.5) is in fact the inverse Fourier transform of $g(t)$ in an ultradistributional sense. In the proof of Theorem 4.2, given in [9, Theorem 5.2], $g \in \mathcal{F}'\mathcal{D}((M_p), L^r)$, $1 < r \leq 2$, and the inverse Fourier transform $U = \mathcal{F}^{-1}[g]$ is defined by the Parseval formula

$$\langle U, \varphi \rangle = \langle g, \check{\psi} \rangle, \tag{4.6}$$

where $\varphi \in \mathcal{D}((M_p), L^r)$, $\psi = \hat{\varphi} \in \mathcal{F}\mathcal{D}((M_p), L^r)$, and $\check{\psi}(x) = \psi(-x)$. The element U defined from g by (4.6) is an element of $\mathcal{D}'((M_p), L^r)$ as shown by the discussion associated with [9, (2.13)]. Using these properties we show in the following corollary of Theorems 4.1 and 4.2 in the case $r = 2$ that the function $f(z)$ of Theorem 4.1 also has a boundary value in $\mathcal{D}'((M_p), L^2)$ and can be recovered by the Cauchy integral of this boundary value.

C o r o l l a r y 4.1. *Let $f(z)$ be holomorphic in T^C and satisfy (4.1) with $r = 2$. Then there is an element $U \in \mathcal{D}'((M_p), L^2)$ such that*

$$\lim_{\substack{y \to \bar{0} \\ y \in C}} f(x + iy) = U \tag{4.7}$$

in $\mathcal{D}'((M_p), L^2)$ and

$$f(z) = \langle U, K(z - t) \rangle, \qquad z \in T^C. \tag{4.8}$$

P r o o f. From Theorem 3.1 and its proof in [7], Corollary 3.1, and the assumption (4.1) here we have the existence of a measurable function $g(t)$ with $\operatorname{supp}(g) \subseteq C^*$ almost everywhere such that (4.3) holds with $r = 2$, and $f(z)$

has the representation (4.4). The existence of $U \in \mathcal{D}'((M_p), L^2)$ for which (4.7) holds now follows from Theorem 4.2; and from the discussion in the paragraph preceding this corollary, $U = \mathcal{F}^{-1}[g]$ as defined in (4.6).

The Cauchy kernel $K(z-t)$, $z \in T^C$, $t \in \mathcal{R}^n$, defined in (2.2) satisfies $K(z-$ $-t) \in \mathcal{D}((M_p), L^2)$ as a function of $t \in \mathcal{R}^n$ for $z \in T^C$ by [8, Theorem 4.1]; thus $\langle U, K(z-t)\rangle$ is a well defined function of $z \in T^C$. Using the fact that $U = \mathcal{F}^{-1}[g]$ here as defined in (4.6), the representation (4.4) where $\operatorname{supp}(g) \subseteq C^*$ almost everywhere, the fact that

$$K(z-t) = \int_{C^*} \exp(2\pi i\langle z-t, \eta\rangle)\, d\eta = \mathcal{F}^{-1}[I_{C^*}(\eta)e^{2\pi i\langle a,\eta\rangle}; t], \qquad z \in T^C,$$

where $I_{C^*}(\eta)$ is the characteristic function of C^*, and (4.6) we have for $z \in T^C$

$$f(z) = \langle g(t), e^{2\pi i\langle z,t\rangle}\rangle = \langle g(t), I_{C^*}(t)e^{2\pi i\langle z,t\rangle}\rangle = \langle U, K(z-t)\rangle$$

which proves (4.8). The proof of Corollary 4.1 is complete.

Boundary value results for $\mathcal{D}'(\{M_p\}, L^r)$ similar to those contained in this section need to be proved; we shall consider this in future investigations.

5. Extension of results to $2 < r < \infty$

In this section we extend Fourier–Laplace integral representation results like we have obtained in section 3 to the cases $2 < r < \infty$ and pose questions concerning the existence of boundary values as in section 4 for these cases. The reader should read [4, section 2] to become familiar with the terms quadrant cone and polygonal cone given there. We use the information given in [4, section 2] here. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be any of the 2^n n-tuples whose components are 0 or 1; $C_\mu = \{y \in \mathcal{R}^n : (-1)^{\mu_j} y_j > 0,\ j = 1, \ldots, n\}$ is called an octant in $\mathcal{R}^n$.

Using analysis similar to that in [4] we obtain Fourier–Laplace integral representations of holomorphic functions in tubes T^C which satisfy (3.1) for $2 < r < \infty$ for C being a quadrant, a quadrant cone, and a polygonal cone. The theorems in this section are proved in [10]. We indicate the technique of the proofs here and pose questions for future research.

We begin with the following result for quadrants.

T h e o r e m 5.1. *Let C be an open convex cone which is contained in or is any of the 2^n quadrants C_μ in $\mathcal{R}^n$. Let $f(z)$ be holomorphic in T^C and satisfy*

(3.1) *for* $2 < r < \infty$. *Then there exists a measurable function* $g(t)$, $t \in \mathcal{R}^n$, *with* $\operatorname{supp}(g) \subseteq C^*$ *almost everywhere such that*

$$\|e^{-2\pi\langle y,t\rangle} g(t)\|_{L^2} \leq M(1 + (d(y))^{-m})^q \exp(M^*(T/|y|)), \qquad y \in C, \tag{5.1}$$

for constants $M > 0$, $T > 0$, $m \geq 0$, *and* $q \geq 0$ *which are independent of* $y \in C$, *and*

$$f(z) = X(z) \int_{\mathcal{R}^n} g(t)\, e^{2\pi i\langle z,t\rangle}\, dt, \qquad z \in T^C, \tag{5.2}$$

where $X(z)$ *is a polynomial in* $z \in T^C$.

We outline the proof. For C being contained in or being the quadrant C_μ, put

$$X(z) = \prod_{j=1}^{n} (1 - i(-1)^{\mu_j} z_j)^{n+2}, \qquad z = x + iy \in T^C. \tag{5.3}$$

We have

$$|1/X(x + iy)| \leq \prod_{j=1}^{n} (1 + x_j^2)^{-1-n/2}, \qquad z \in T^C. \tag{5.4}$$

Put

$$F(z) = f(z)/X(z), \qquad z \in T^C, \tag{5.5}$$

which is holomorphic in T^C. We proceed to show in [10] that $F(z)$ satisfies the norm growth (3.1) with $r = 2$ there. Applying Corollary 3.1 we obtain a function $g(t)$, $t \in \mathcal{R}^n$, with $\operatorname{supp}(g) \subseteq C^*$ almost everywhere such that (5.1) holds, and

$$F(z) = \int_{\mathcal{R}^n} g(t) e^{2\pi i\langle z,t\rangle}\, dt, \qquad z \in T^C. \tag{5.6}$$

(5.2) is now obtained from (5.5) and (5.6) with $X(z)$ being given in (5.3).

We ask if the representation (5.2) can be rewritten in the form $f(z) = \langle V, \exp(2\pi i\langle z,t\rangle)\rangle$, $z \in T^C$, for some ultradistribution V? If so, $g(t)$ will have to possess sufficient properties to allow for $\langle V, \exp(2\pi i\langle z,t\rangle)\rangle$ to be well defined.

If $m = 0$ or $q = 0$ in (5.1) the Fourier-Laplace integral in (5.2) obtains an ultradistributional boundary value as $y = \operatorname{Im}(z) \to \bar{0}$, $y \in C$, by Theorem 4.2 since $g(t)$ satisfies (5.1) with $m = 0$ or $q = 0$. Can this fact be used along with (5.2) to prove that $f(z)$ also obtains an ultradistributional boundary value?

Recall the concept of quadrant cone given in [4, p. 86]. We extend Theorem 5.1 to the case that C is in or is a quadrant cone.

Theorem 5.2. *Let C be an open convex cone that is contained in or is a quadrant cone in $\mathcal{R}^n$. Let $f(z)$ be holomorphic in T^C and satisfy* (3.1) *for* $2 < r < \infty$. *There exists a measurable function $g(t)$, $t \in \mathcal{R}^n$, and a nonsingular linear transformation L such that* $\operatorname{supp}(g) \subseteq (L^{-1}(C))^*$ *almost everywhere,*

$$\|e^{-2\pi(L^{-1}(y),t)}g(t)\|_{L^2} \le M(k+(d(L^{-1}(y)))^{-m})^q \exp(M^*(R/|L^{-1}(y)|)), \qquad y \in C, \tag{5.7}$$

for constants $M > 0$, $R > 0$, $k > 0$, $m \ge 0$, and $q \ge 0$ which are independent of $y \in C$, and

$$f(z) = X(L^{-1}(x) + iL^{-1}(y)) \int_{\mathcal{R}^n} g(t)e^{2\pi i(L^{-1}(x)+iL^{-1}(y),t)}\,dt, \quad x + iy \in \mathcal{R}^n + iC, \tag{5.8}$$

where $X(u + iv)$ is a polynomial, $u + iv = L^{-1}(x) + iL^{-1}(y)$.

Let Γ denote the quadrant cone that C is contained in or is. There exists a nonsingular linear transformation L (with domain and range being $\mathcal{R}^n$) which maps the first quadrant $C_{\bar{0}}$ onto Γ in aone to one manner such that the boundary of $C_{\bar{0}}$ is mapped to the boundary of Γ (see [4, p. 86]). Further, if C is property contained in Γ then $L^{-1}(C)$ is an open convex cone which is contained in $C_{\bar{0}}$, and L maps $L^{-1}(C)$ one to one and onto C with the boundary of $L^{-1}(C)$ being mapped to the boundary of C. ($L^{-1}(C) = C_{\bar{0}}$ if $C = \Gamma$.) For $u + iv \in \mathcal{R}^n + iL^{-1}(C)$ put

$$\begin{gathered} G(u+iv) = f(L(u) + iL(v)) = f(x+iy), \\ u + iv \in \mathcal{R}^n + iL^{-1}(C), \qquad x + iy \in \mathcal{R}^n + iC. \end{gathered} \tag{5.9}$$

$G(u+iv)$ is holomorphic in $\mathcal{R}^n + iL^{-1}(C)$. The proof now proceeds, using Theorem 5.1, by the techniques used in the proof of [4, Lemma 2]; the complete proo is given in [10].

If $m = 0$ or $q = 0$ in (3.1), can we obtain an ultradistributional boundary valut for $f(z)$ in Theorem 5.2 using (5.9) and Theorem 5.1? From the construction of the proof of Theorem 5.2 we could obtain a boundary value if we knew that an ultradistributional boundary value existed in Theorem 5.1 for $m = 0$ or $q = 0$ there.

Now recall the concept of a polygonal cone given in [4, p. 86] and the fact of the intersection property for the quadrant cones C_j, $j = 1, \ldots, m$, whose union is the polygonal cone. Let us extend Theorems 5.1 and 5.2 to the case that C can be a polygonal cone.

For C being a polygonal cone, $C = \cup_{j=1}^{m} C_j$, where the C_j are quadrant cones which have the intersection property noted in [4, p. 86]. Let $f(z)$ be holomorphic in T^C and satisfy (3.1) for $2 < r < \infty$. Now $y \in C$ implies $y \in C_j$ for some $j = 1, \ldots, m$; and note that the distance from y to the boundary of C is greater than or equal to the distance from y to the boundary of C_j. Thus $f(z)$ is holomorphic in $\mathcal{R}^n + iC_j$ and satisfies (3.1) for $y \in C_j$ for each $j = 1, \ldots, m$. Thus by Theorem 5.2 for each quadrant cone C_j there is a nonsingular linear transformation L_j mapping $C_{\bar{0}}$ one to one and onto C_j and a function $g_j(t)$ with $\operatorname{supp}(g_j) \subseteq (L_j^{-1}(C_j))^* = C_{\bar{0}}^*$ almost everywhere and a polynomial X_j such that

$$\|e^{-2\pi\langle L_j^{-1}(y),t\rangle} g_j(t)\|_{L^2} \leq M(k + (d(L_j^{-1}(y)))^{-m})^q \exp(M^*(R/|L_j^{-1}(y)|)), \quad y \in C_j, \tag{5.10}$$

and

$$f(x+iy) = X_j(L_j^{-1}(x) + iL_j^{-1}(y)) \int_{\mathcal{R}^n} g_j(t) e^{2\pi i\langle L_j^{-1}(x)+iL_j^{-1}(y),t\rangle}\, dt, \quad x+iy \in \mathcal{R}^n + iC_j. \tag{5.11}$$

We have proved the following result.

Theorem 5.3. *Let C be a polygonal cone in $\mathcal{R}^n$. Let $f(z)$ be holomorphic in T^C and satisfy* (3.1) *for* $2 < r < \infty$. *Then there exist quadrant cones C_j, $j = 1, \ldots, m$; nonsingular linear transformations L_j mapping $C_{\bar{0}}$ one to one and onto C_j; functions g_j having* $\operatorname{supp}(g_j) \subseteq \bar{C}_{\bar{0}}$ *almost everywhere and satisfying* (5.10); *and polynomials X_j such that* (5.11) *holds where* $C = \cup_{j=1}^{m} C_j$.

A similar result to Theorem 5.3 can be proved for C being a regular cone.

Can an ultradistributional boundary value be obtained for $f(z)$ in Theorem 5.3 and for the corresponding result for C being a regular cone?

6. Future research

In the future we shall attempt to answer the questions posed in section 5 concerning ultradistributional boundary values for the functions $f(z)$ considered there. We further desire to obtain Cauchy and Poisson integral representations for the functions $f(z)$ in sections 4 and 5 where possible and to develop the theory of these functions to the greatest extent possible.

Recall the quadrants C_μ defined in section 5. Tillmann [16] has characterized the analytic functions in tubes $\mathcal{R}^n + iC_\mu$ which have distributional boundary values in the Schwartz space $\mathcal{D}'_{L^r}$ [14, p. 199–200] and in so doing has shown that the Cauchy kernel function corresponding to the tubes $\mathcal{R}^n + iC_\mu$ is an element of the test spaces $\mathcal{D}_{L^s}$, $1 < s < \infty$. In [6, Lemma 2.1] we have shown that in the quadrant setting the Cauchy kernel is also in $\dot{B} \cap \mathcal{D}_{L^\infty}$. (See [14, p. 199] for the definitions of $\dot{B}$ and $\mathcal{D}_{L^s}$.) We know that the general Cauchy kernel defined in (2.2) corresponding to arbitrary regular cones C is an element of $\dot{B} \cap \mathcal{D}_{L^q}$ for all q, $1 \leq p \leq 2$, $1/p + 1/q = 1$ [11, Lemma 3.1, p. 601] and by [8, Theorem 4.1] is in the ultradifferentiable spaces $\mathcal{D}(*, L^s)$, $2 \leq s \leq \infty$, where $*$ is either (M_p) or $\{M_p\}$. We ask if $K(z - t)$ as defined in (2.2) is an element of $\mathcal{D}_{L^s}$, $1 < s < 2$, or $\mathcal{D}(*, L^s)$, $1 < s < 2$, as a function of $t \in \mathcal{R}^n$ for $z \in T^C$ for arbitrary regular cones C. If this were the case, much analysis concerning the Cauchy integral of distributions and ultradistributions could be extended to the cases $\mathcal{D}'_{L^s}$ and $\mathcal{D}'(*, L^s)$, $1 < s < 2$.

Поступило в мае 1993 г.

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