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GAUSSIAN MEASURE OF TRANSLATED BALLS IN A BANACH SPACE

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Let E be a real separable Banach space and let μ be a centered Gaussian measure on E. Then the function F on $(0, \infty) \times E$ defined by

$$F(s, z) = \mu\{x \in E: \|x - z\| \leqslant s\}$$

has remarkable properties. For instance, if $z \in \operatorname{supp}(\mu)$ is fixed, then it possesses everywhere right and left derivatives and the derivative $F'(\cdot, z)$ exists except possibly at countable many points. Further properties have been proved in [5, 8, 9, 10, 17, 19]. As shown in [4] (corollary 4.4) $F(s, \cdot)$ is continuous on $\operatorname{supp}(\mu)$, $s > 0$ fixed. The aim of this paper is to prove that F is Gateaux differentiable as a function on $\operatorname{supp}(\mu)$. Moreover, its derivative $d(s, z)$ is a linear and continuous functional on $\operatorname{supp}(\mu)$ which satisfies

$$\|d(s, z)\| \leqslant F'(s, z), \quad s > 0 \text{ and } z \in \operatorname{supp}(\mu).$$

This extends earlier results where F was investigated as a function on the reproducing kernel Hilbert space $\mathrm{H}(\mu)$ (cf. [11, theorem 6.2] or [7]). Observe that the topology on $\mathrm{H}(\mu)$ does not coincide with the topology generated by E ($\dim \mathrm{H}(\mu) = \infty$). So the differentiability of F on $\operatorname{supp}(\mu)$ does not follow by an easy extension procedure.

Finally we apply properties of F to the limit

$$D_p(z) = \lim_{t \to 0} \int_E t^{-1}(\|x + z + ty\|^p - \|x + z\|^p)\, d\mu(x),$$

$y, z \in \operatorname{supp}(\mu)$ and $1 \leqslant p < \infty$.

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1. Notation and basic properties of Gaussian measures. Throughout this paper μ is a centred Gaussian measure on the real separable Banach space E, i. e. $a(\mu)$ is centred Gaussian on $\mathbf{R}$ for each $a \in E'$ (topological dual space). The covariance operator R of μ is defined by

$$Ra = \int_E \langle x, a \rangle x \, d\mu(x) \quad \text{(Bochner integral)}$$

and it maps E' into E.

If we regard E' as a subspace of $L_2(E, \mu)$, then $E_2'(\mu)$ denotes the closure of E' with respect to the L_2-norm. Let Q: $E' \to E_2'(\mu)$ be the natural quotient map, i. e. $Q(a) = \langle \cdot, a \rangle$. Since Q has a dense range its dual operator Q' is injective. Moreover, $Q'f = \int_E f(x)\, x\, d\mu(x)$ for any $f \in E_2'(\mu)$ and, consequently, Q' maps $E_2'(\mu)$ into E. Clearly we have $R = Q'Q$. Let $H(\mu) \subseteq E$ be the image of $E_2'(\mu)$ with respect to Q' and write $\tilde{y}$ instead of $(Q')^{-1}(y)$, $y \in H(\mu)$. Then we introduce a scalar product on $H(\mu)$ by

$$\langle y_1, y_2 \rangle_\mu = \int_E \tilde{y}_1 \tilde{y}_2\, d\mu, \quad y_1, y_2 \in H(\mu),$$

which makes $H(\mu)$ to a Hilbert space (reproducing kernel Hilbert space of μ). We denote by $\|\cdot\|_\mu$ the corresponding Hilbert norm on $H(\mu)$. Of course, the embedding of $H(\mu)$ into E is continuous (it is even a compact operator (cf. [4])). By construction the range $R(E')$ of R is a subspace of $H(\mu)$ and $\widetilde{Ra} = Qa$, $a \in E'$, i. e. we have $\widetilde{Ra} = a$ μ-a. e. and

$$\| Ra \|_\mu^2 = \int_E |\langle x, a \rangle|^2\, d\mu(x) = \langle Ra, a \rangle.$$

Let supp (μ) be the support of μ. Then supp (μ) is a closed subspace of E ([2] or [11]) and $H(\mu) \subseteq$ supp (μ). Moreover,

$$\overline{R(E')} = \overline{H(\mu)} = \operatorname{supp}(\mu)$$

where the closure is taken in E [2, theorem IX 2.1]. For later use we mention that $x \in$ supp (μ) iff $\mu\{y \in E: \| x - y \| < \varepsilon\} > 0$, $\varepsilon > 0$.

For each $y \in E$ the translated measure μ_y is defined by $\mu_y(\cdot) = \mu(\cdot + y)$.

The next proposition plays an important role in our subsequent investigations.

Proposition 1.1 (Cameron — Martin's formula). *For each $y \in H(\mu)$ the measures μ and μ_y are mutually absolutely continuous and*

$$d\mu_y = \exp(-\tilde{y} - \| y \|_\mu^2/2)\, d\mu.$$

P r o o f cf. [4, corollary 2.1].

R e m a r k s. 1) In the special case $y = Ra$, $a \in E'$, this yields

$$d\mu_{Ra} = \exp(-\langle \cdot, a \rangle - \langle Ra, a \rangle/2)\, d\mu.$$

2) Observe that $\tilde{y}(\mu)$ is Gaussian on $\mathbf{R}$, $y \in H(\mu)$. Thus the above written function is μ-integrable.

Let us state a further basic property of Gaussian measures.

Proposition 1.2. *If A and B are measurable subsets of E, then*

$$\mu_*(\lambda A + (1-\lambda) B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for all $\lambda \in (0, 1)$. Here $\mu_*(C) = \sup\{\mu(K)$: $K \subseteq C$, K — compact$\}$.

P r o o f cf. [3, corollary 2.1].

2. Existence and properties of the right derivative. As in the introduction the function F on $(0, \infty) \times E$ is defined by

$$F(s, z) = \mu\{x \in E: \| x - z \| \leqslant s\} = \mu(U_s(z))$$

where $U_s(z)$ denotes the closed ball of radius $s > 0$ with centre $z \in E$.

Proposition 2.1. *Suppose thas $z \in E$ and that $0 < \lambda < 1$. Then for all $y_1, y_2 \in E$*

$$F(s, z + \lambda y_1 + (1-\lambda) y_2) \geqslant F(s, z + y_1)^\lambda F(s, z + y_2)^{1-\lambda}.$$

P r o o f. This is an immediate consequence of proposition 1.2. Observe that

$$U_s(z+\lambda y_1+(1-\lambda)y_2)=\lambda U_s(z+y_1)+(1-\lambda)U_s(z+y_2).$$

Corollary 2.1. *If* $s>0$, $z\in\operatorname{supp}(\mu)$ *and* $y\in E$, *then the function*

$$t\to\log F(s,z+ty)$$

is a concave mapping from $\mathbf{R}$ *into* $[-\infty,0)$. *Moreover, it is finite in some neighborhood of zero.*

P r o o f. The first property follows directly from prorosition 2.1. Because of $z\in\operatorname{supp}(\mu)$ we have $F(s,z)>0$ for all $s>0$. Consequently, if $\|ty\|<s$, then $F(s,z+ty)>0$ as well and this ends the proof.

Now we are in position to apply well-known properties of concave functions to $\log F(s,z+ty)$ (cf. [6]). Doing so the following is true.

Proposition 2.2. *Suppose that* $s>0$ *and that* $z\in\operatorname{supp}(\mu)$.

a) *For each* $y\in E$ *the limit*

$$d(s,z)(y)=\lim_{t\downarrow 0}t^{-1}(F(s,z+ty)-F(s,z))$$

exists.

b) *We have*

$$-d(s,z)(-y)=\lim_{t\uparrow 0}t^{-1}(F(s,z+ty)-F(s,z))$$

and

$$d(s,z)(y)\leqslant -d(s,z)(-y).$$

c) *Moreover, this right derivative coincides with*

$$d(s,z)(y)=\lim_{t\downarrow 0}\frac{E(s,z)}{t}\log\frac{F(s,z+ty)}{F(s,z)}.$$

R e m a r k s. 1) If $z\notin\operatorname{supp}(\mu)$, then there are easy examples where $d(s,z)(y)$ does not exist. It may even happen that

$$\lim_{t\downarrow 0}F(s,z+ty)\neq F(s,z)$$

in this case. For instance, choose μ as in the example below, i. e. μ is the standard Gaussian measure on $\mathbf{R}$ regarded as measure in $\mathbf{R}^2$. Setting $\|x\|_\infty=\max(|\xi_1|,|\xi_2|)$, $x=(\xi_1,\xi_2)$, it is easy to see that $F(1,(0,1))>0$ while $F(1,(0,1+t))=0$ for all $t>0$.

2) As we mentioned in the introduction the existence of $d(s,z)(y)$ is well-known in the case $y\in H(\mu)$. Let us repeat the arguments for the proof of the existence in this case. If $y\in H(\mu)$, then in view of proposition 1.1 we have

$$F(s,z+ty)=\mu_{ty}(U_s(z))=\int_{U_s(z)}\exp(-t\tilde{y}(x)-(t^2/2)\|y\|_\mu^2)\,d\mu(x)$$

and

$$t^{-1}(F(s,z+ty)-F(s,z))=\int_{U_s(z)}t^{-1}(\exp(-t\tilde{y}-(t^2/2)\|y\|_\mu^2)-1)\,d\mu.$$

Using the mean value theorem and the fact that $\tilde{y}(\mu)$ is Gaussian we may apply Lebesgue's theorem and obtain

$$d(s,z)(y)=-\int_{U_s(z)}\tilde{y}(x)\,d\mu(x).$$

So we have proved the following proposition.

Proposition 2.3. *For any* $y \in \mathrm{H}(\mu)$

$$d(s, z)(y) = -\int_{U_s(z)} \tilde{y}(x)\, d\mu(x).$$

In particular,

$$d(s, z)(Ra) = -\int_{U_s(z)} \langle x, a\rangle\, d\mu(x), \quad a \in E'.$$

Corollary 2.2. $d(s, z)$ *is a linear and continuous functional on* $\mathrm{H}(\mu)$ *and*

$$\| d(s, z) \|_{\mathrm{H}(\mu)} \leqslant F(s, z)^{1/2}.$$

R e m a r k s. 1) Observe that we did not assume $z \in \operatorname{supp}(\mu)$ in the formulation of proposition 2.3. This is possible because we only took translations in direction of elements in $\mathrm{H}(\mu)$. As mentioned above in general $d(s, z)(y)$ does not exist for arbitrary $y, z \in E$.

2) corollary 2.2. does not imply the continuity of $d(s, z)$ on $[\mathrm{H}(\mu), \|\cdot\|_E]$. Indeed, if $\dim \mathrm{H}(\mu) = \infty$, then the inverse of the embedding from $\mathrm{H}(\mu)$ into E is not continuous. Recall that this embedding is a compact operator.

3) In general the right derivative $d(s, z)$ is not linear on the whole space E. To see this we give an easy example.

E x a m p l e. Let μ be the standard Gaussian measure on $\mathbf{R}$ regarded as measure in $\mathbf{R}^2$, i. e. we have

$$\operatorname{supp}(\mu) = \{(\xi, 0): \xi \in \mathbf{R}\} \subseteq \mathbf{R}^2.$$

Endow $\mathbf{R}^2$ with the norm $\|\cdot\|_1$ defined by $\|x\|_1 = |\xi_1| + |\xi_2|$, $x = (\xi_1, \xi_2)$. If $z = (1, 0) \in \operatorname{supp}(\mu)$ and $y = (0, 1)$, then we obtain

$$F(s, z + ty) = (2\pi)^{-1/2} \int_{1-s+|t|}^{1+s-|t|} \exp(-u^2/2)\, du$$

and

$$d(s, z)(y) = -(2\pi)^{-1/2} \exp((1 + s^2)/2)(e^s + e^{-s}) < 0.$$

On the other hand,

$$d(s, z)(-y) = d(s, z)(y)$$

which proves that $d(s, z)$ is not linear on $\mathbf{R}^2$.

R e m a r k. Observe that we have chosen an element $y \notin \operatorname{supp}(\mu)$ in order to disprove the linearity of $d(s, z)$. This choice was in fact necessary as we shall see below (theorem 3.1).

3. The Gateaux differentiability of F on supp (μ). The aim of this section is to show that $d(s, z)$ is linear and continuous on $\operatorname{supp}(\mu)$ which means that $F(s, \cdot)$ is a Gateaux differentiable function on $\operatorname{supp}(\mu)$. In a first step we prove that $-d(s, z)$ is sublinear.

Proposition 3.1 *We have*

$$d(s, z)(y_1) + d(s, z)(y_2) \leqslant d(s, z)(y_1 + y_2)$$

for all $s > 0$, $z \in \operatorname{supp}(\mu)$ *and* $y_1, y_2 \in E$.

P r o o f. We start with proposition 2.2:

$$d(s, z)(y_1 + y_2) = \lim_{t \downarrow 0} \frac{F(s, z)}{t} \log \frac{F(s, z + ty_1 + ty_2)}{F(s, z)}.$$

proposition 2.1 lets us conclude

$$\log \frac{F(s, z+ty_1+ty_2)}{F(s, z)} \geqslant \frac{1}{2}\left[\log \frac{F(s, z+2ty_1)}{F(s, z)} + \log \frac{F(s, z+2ty_2)}{F(s, z)}\right].$$

Multiplying both sides of this inequality by $F(s, z)/t$ ($F(s, z) > 0$ as well as $t > 0$) the assertion follows from proposition 2.2 by letting $t > 0$ tend to zero.

Since $d(s, z)$ is positive homogeneous, i. e.

$$d(s, z)(\alpha, y) = \alpha d(s, z)(y), \quad \alpha > 0.$$

we obtain the following rather helpful corollary.

Corollary 3.1. *If $s > 0$ and $z \in \operatorname{supp}(\mu)$, then $d(s, z)$ is continuous on E iff the supremum*

$$\sup \{| d(s, z)(y) |: y \in E, \| y \| \leqslant 1\}$$

is finite.

Let us now estimate the absolute value of $d(s, z)(y)$. To do so we need properties of the function $s \to F(s, z)$ where now $z \in \operatorname{supp}(\mu)$ is fixed.

Proposition 3.2. *Let $z \in \operatorname{supp}(\mu)$ be a fixed element.*

a) *$F(\cdot, z)$ is continuous on $(0, \infty)$ and admits right and left derivatives everywhere on $(0, \infty)$.*

b) *The derivative $F'(\cdot, z)$ exists except possibly at countable many points where $F'(\cdot, z)$ has a jump downwards.*

c) *For any measurable function $f: (0, \infty) \to \mathbf{R}$ we have*

$$\int_E f(\| x - z \|)\, d\mu(x) = \int_0^\infty f(u) F'(u, z)\, du$$

provided that one of the integrals exists.

P r o o f. There properties of $F(\cdot, z)$ ase well-known (cf. [10]). They follow from the concavity of the function $\log F(\cdot, z)$ which is a direct consequence of proposition 1.2.

R e m a r k s. 1) As recently shown by A. Ehrhard [8] the function $\Phi^{-1} \circ F(\cdot, z)$ is concave. Here

$$\Phi(u) = (2\pi)^{-1/2} \int_{-\infty}^{u} e^{-v^2/2}\, dv.$$

This implies $\sup_{s > s_0} F'(s, z) < \infty$ for each $s_0 > 0$. But as shown in [13, 15, 16, 22], it depends on geometric properties of E whether or not this remains true in the case $s_0 = 0$. Bounds for $F'(s, 0)$ have been given in [19] and in [9]. Moreover, Talagrand [17] recently proved that $F'(\cdot, 0)$ is even continuous. We do not know whether this is also valid for arbitrary $z \in \operatorname{supp}(\mu)$, $z \neq 0$.

2) If $F'(\cdot, z)$ has a jump at $s > 0$, then we denote by $F'(s, z)$ the left derivative (which is larges than the right one).

Proposition 3.3. *For any $s > 0$ and any $z \in \operatorname{supp}(\mu)$ we have*

$$\sup \{| d(s, z)(y) |: y \in E, \| y \| \leqslant 1\} \leqslant F'(s, z).$$

In particular, $d(s, z)$ is continuous on E.

P r o o f. We estimate the expression

$$| t^{-1}(F(s, z+ty) - F(s, z)) |, \quad 0 < t < s. \tag{1}$$

To do so we treat the tuo cases $F(s, z+ty) \geqslant F(s, z)$ and $F(s, z) \geqslant F(s, z+ty)$ separately. In the former case (1) can be estimated by

$$t^{-1}(F(s+t\|y\|, z) - F(s, z)) \leqslant t^{-1}(F(s+t, z) - F(s, z))$$

provided that $\|y\| \leqslant 1$. Indeed, we have

$$U_s(z+ty) \leqslant U_{s+t\|y\|}(z).$$

In the latter case (1) is less than

$$t^{-1}(F(s, z) - F(s-t, z))$$

which easily follows from

$$U_{s-t}(z) \subseteq U_{s-t\|y\|}(z) \subseteq U_s(z+ty), \quad \|y\| \leqslant 1.$$

Combining these two estimates we arrive at

$$|t^{-1}(F(s, z+ty) - F(s, z))| \leqslant$$
$$\leqslant \max\{t^{-1}(F(s+t, z) - F(s, z)), t^{-1}(F(s, z) - F(s-t, z))\}$$

whenever $\|y\| \leqslant 1$ and $0 < t < s$. But this proves the desired estimate by taking the limit $t \downarrow 0$. Recall that we defined $F'(\cdot, z)$ as left derivative.

Propositions 3.3, 2.3 and corollary 2.2 give us the following surprising estimates.

Corollary 3.2. *For any* $y \in \mathrm{H}(\mu)$ *and any* $z \in \mathrm{supp}(\mu)$ *we have*

$$\left|\int_{U_s(z)} \tilde{y}(x)\, d\mu(x)\right| \leqslant F'(s, z)\|y\|.$$

Consequently,

$$\left|\int_{U_s(z)} \langle x, a\rangle\, d\mu_1(x)\right| \leqslant F'(s, z)\|Ra\|$$

for all $a \in E'$.

Now we are going to formulate and to prove the main result of this paper.

Theorem 3.1. *Suppose that* $s > 0$ *and that* $z \in \mathrm{supp}(\mu)$. *Then* $d(s, z)$ *is a linear and continuous functional on* supp (μ). *Its norm* (in supp (μ)′) *can be estimated as follows*:

$$\|d(s, z)\| \leqslant F'(s, z).$$

P r o o f. H (μ) is a dense subspace of supp (μ) and $d(s, z)$ is linear on H (μ) (corollary 2.2). By continuity (proposition 3.3) it is linear on supp (μ) as well. The estimate of the norm is a direct consequence of proposition 3.3.

Let us state a reformulation of theorem 3.1.

Theorem 3.2. *For any* $s > 0$ *the function* $F(s, \cdot)$ *is Gateaux differentiable on* supp (μ). *Its derivative at the point* z *is* $d(s, z)$ *and* $\|d(s, z)\| \leqslant F'(s, z)$.

Our next objective is to find some formula for the calculation of $d(s, z)$. For this reason we extend $d(s, z)$ as functional on supp (μ) via Hahn—Banach theorem to an element $\tilde{d}(s, z) \in E'$. Then $R(\tilde{d}(s, z))$ (recall that R denotes the covariance operator of μ) makes sense and, surprisingly, this element of E can be calculated in a rather easy way.

Proposition 3.4. *Suppose that* $s > 0$ *and that* $z \in \mathrm{supp}(\mu)$.

a) *Under these assumptions we have*

$$R(\tilde{d}(s, z)) = -\int_{U_s(z)} x\, d\mu(x).$$

b) *Let* **P** *be the orthogonal projection from* $L_2(E, \mu)$ *into* $E_2'(\mu)$. *Then*

$$d(s, z) = \tilde{d}(s, z) = -\mathbf{P}(\chi_{U_s(z)}) \quad \mu\text{-a. e.}.$$

Here χ_A *is defined as the characteristic function of* $A \subseteq E$.

P r o o f. a) Taking $a \in E'$ we obtain

$$\langle R(\tilde{d}(s, z)), a\rangle = \langle Ra, \tilde{d}(s, z)\rangle = d(s, z)(Ra)$$

which by corollary 2.2 coincides with

$$-\int_{U_s(z)} \langle x, a\rangle\, d\mu(x) = \left\langle -\int_{U_s(z)} x\, d\mu(x), a\right\rangle.$$

This being true for all $a \in E'$ proves assertion a). To prove b) we regard $\tilde{d}(s, z)$ as an element of $L_2(E, \mu)$. Then $\tilde{d}(s, z) = d(s, z)$ ($d(s, z)$ is μ-a.e. defined on E) and $d(s, z) \in E_2'(\mu)$. Thus it suffices to show that $\chi_{U_s(z)} + d(s, z)$ belongs to the orthogonal complement of $E_2'(\mu)$. Because of

$$R(\tilde{d}(s, z)) = \int_E d(s, z)(x)\, x\, d\mu(x)$$

a) implies

$$\int_E [d(s, z)(x) + \chi_{U_s(z)}(x)]\, x\, d\mu(x) = 0,$$

i. e. $\chi_{U_s(z)} + d(s, z)$ is orthogonal to every element in $Q(E') \subseteq E_2'(\mu)$. But by construction $Q(E')$ is dense in $E_2'(\mu)$ which completes the proof.

Corollary 3.3. a) *We have* $F(s, z)^{-1}R(\tilde{d}(s, z)) \in U_s(-z)$.

b) $d(s, z) \neq 0$ *whenever* $0 < s < \|z\|$.

P r o o f. a) is an easy consequence of proposition 3.4. Indeed, the barycentre over a closed convex set belongs to this set too. To prove b) we may assume $E = \operatorname{supp}(\mu)$. By a) we have $F(s, z)^{-1}R(d(s, z)) \in U_s(-z)$ and $0 \notin U_s(-z)$ in view of the relation between s and $\|z\|$. Thus $R(d(s, z)) \neq 0$ which clearly yields $d(s, z) \neq 0$ as asserted.

R e m a r k s. 1) Property a) of the preceding corollary easily gives

$$\lim_{s \downarrow 0} (F(s, z))^{-1} R(\tilde{d}(s, z)) = -z \quad (\text{in } E).$$

2) Unfortunately we do not know whether or not $d(s, z) \neq 0$ in the remaining case $0 < \|z\| \leqslant s$. Observe that $d(s, 0) = 0$.

4. Applications to derivatives of integrals. Recall that the norm of E is Gateaux differentiable at $x \in E$ provided that

$$d_x(y) = \lim_{t \to 0} t^{-1}(\|x + ty\| - \|x\|)$$

exists for all $y \in E$. Then $d_x \in E'$ with $\|d_x\| = 1$.

For our subsequent investigations we need the following infinite dimensional version of Rademacher's theorem:

Proposition 4.1. *There exists a measurable subset* $B \subseteq E$ *possessing the following properties:*

(i) *We have*

$$\nu_z(B) = \nu(B + z) = 1$$

for each centred Gaussian measure ν *on* E *with* supp $(\nu) = E$ *and for each* $z \in E$.

(ii) *If* $x \in B$, *then* d_x *exists.*

Proof cf. [1] or [14].

Remark. In particular, for any fixed Gaussian measure μ on E with supp $(\mu) = E$ and any $z \in E$ the Gateaux derivative d_{x+z} exists for μ-almost all $x \in E$. Indeed,

$$\mu_*\{x \in E: \quad d_{x+z} \text{ exists}\} \geqslant \mu\{x \in E: \quad x + z \in B\} = \mu_{-z}(B) = 1.$$

Proposition 4.2. *Suppose that* $1 \leqslant p < \infty$. *Then for all* $y, z \in \in$ supp (μ) *the limit*

$$D_p(z)(y) = \lim_{t \to 0} \int_E t^{-1}(\| x + z + ty \|^p - \| x + z \|^p)\, d\mu(x)$$

exists. Moreover, $D_p(z) \in$ supp $(\mu)'$ *and*

$$D_p(z)(y) = p \int_E \| x + z \|^{p-1} \langle y, d_{x+z} \rangle\, d\mu(x).$$

Proof. Since $y, z \in$ supp (μ) we may assume supp $(\mu) = E$. Consequently, in view of the preceding remark the derivative d_{x+z} is a μ-a. e. well-defined element of E'. Because of

$$| t^{-1}(\| x + z + ty \|^p - \| x + z \|^p) | \leqslant p(\| x + z \| + | t | \| y \|)^{p-1} \| y \|$$

the existence and representation of $D_p(z)$ follow from Lebesgue's theorem. Of course, $D_p(z)$ is linear on supp (μ) and it is continuous because of

$$| D_p(z)(y) | \leqslant p \int_E \| x + z \|^{p-1} d\mu(x) \| y \|.$$

This completes the proof.

Remark. We have $\| D_p(z) \| \leqslant p \int_E \| x + z \|^{p-1} d\mu(x)$.

Our next aim is to calculate $D_p(z)$ as an integral over $d(\cdot, z)$. More precisely, the following is valid:

Theorem 4.1. *Suppose that* $1 \leqslant p < \infty$ *and that* $z \in$ supp (μ). *Then*

$$D_p(z) = -p \int_0^\infty s^{p-1} d(s, z)\, ds.$$

Proof. Let us assume supp $(\mu) = E$. The right hand integral is well-defined in the sense of Bochner (in E'). Indeed, $\| d(s, z) \| \leqslant F'(s, z)$ (theorem 3.1) and, hence, by proposition 3.2

$$\int_0^\infty s^{p-1} \| d(s, z) \| ds \leqslant \int_0^\infty s^{p-1} F'(s, z)\, ds = \int_E \| x - z \|^{p-1} d\mu(x) < \infty.$$

Consequently, it suffices to prove that $D_p(z)$ and the functional defined by the integral are equal on the dense subspace $\mathrm{H}(\mu) \subseteq E$. Choosing $y \in \in \mathrm{H}(\mu)$ we have

$$D_p(z)(y) = \lim_{t \to 0} t^{-1}\left[\int_E \| x + z \|^p d\mu_{-ty}(x) - \int_E \| x + z \|^p d\mu(x)\right] =$$

$$= \lim_{t \to 0} \int_E \| x + z \|^p t^{-1}[\exp(t\tilde{y} - (t^2/2)\| y \|_\mu^2) - 1]\, d\mu = \int_E \| x + z \|^p \tilde{y}(x)\, d\mu(x)$$

where we used the same arguments as in the proof of proposition 2.3. On the other hand, the symmetry of μ yields

$$\int_E \tilde{y}(x)\, d\mu(x) = 0,$$

i. e. we have

$$d(s, z)(y) = -\int_{U_s(z)} \tilde{y}(x)\, d\mu(x) = \int_{\|x-z\|>s} \tilde{y}(x)\, d\mu(x) = -\int_{\|x+z\|>s} \tilde{y}(x)\, d\mu(x)$$

and

$$p\int_0^\infty s^{p-1} d(s, z)(y)\, ds = -p\int_0^\infty s^{p-1} \int_{\|x+z\|>s} \tilde{y}(x)\, d\mu(x)\, ds.$$

Finally, by changing the order of integration in the lastwritten integral it coincides with $-\int_E \| x + z \|^p \tilde{y}(x)\, d\mu(x)$ which ends the proof.

As above the following representations of the Hahn—Banach extension $\tilde{D}_p(z) \in E'$ are valid

Corollary 4.4. a) $R(\tilde{D}_p(z)) = \int_E \| x + z \|^p x\, d\mu(x)$.

b) $\tilde{D}_p(z) = D_p(z) = \mathbf{P}(\|\cdot + z\|^p)$ μ-a. e.

5. Examples and concluding remarks. 1) In the case $E = \mathbf{R}^n$ the results of this paper are more or less known. Here we have

$$R(\mathbf{R}^n) = \mathrm{H}(\mu) = \mathrm{supp}(\mu)$$

and

$$d(s, z) = -R^{-1}\Big(\int_{U_s(z)} x\, d\mu(x)\Big).$$

Writting the covariance operatos R in the form

$$Ra = \sum_{j=1}^m \lambda_j \langle a, e_j\rangle e_j$$

with $\lambda_j > 0$ and $\{e_1, \ldots, e_m\} \subseteq \mathbf{R}^n$ orthonormal, it follows

$$d(s, z) = -\sum_{j=1}^m \lambda_j^{-1}\Big(\int_{U_s(z)} \langle x, e_j\rangle\, d\mu(x)\Big) e_j.$$

2) A similar representation of $d(s, z)$ is valid in a separable Hilbert space H. Here

$$d(s, z) = -\sum_{j=1}^\infty \lambda_j^{-1}\Big(\int_{U_s(z)} \langle x, e_j\rangle\, d\mu(x)\Big) e_j$$

where $\lambda_j > 0$ with $\sum_{j=1}^\infty \lambda_j < \infty$ and $\{e_j\} \subseteq H$ orthonormal represent the covariance operator R (cf. [18]).

3) Since in a Hilbert space H_x the derivative d_x coincides with $x/\| x \|$, $x \neq 0$, we obtain the following representation of $D_p(z)$ in this case:

$$D_p(z) = p\int_H \| x + z \|^{p-2}(x + z)\, d\mu(x).$$

Concluding remarks. 1) The original motivation for the investigation of $d(s, z)$ and $D_p(z)$ was the following problem. Let μ_1, μ_2 be two centred Gaussian measures on E. For which numbers $p > 0$ does

$$\int_E \|x + z\|^p d\mu_1(x) = \int_E \|x + z\|^p d\mu_2(x), \quad z \in E, \tag{2}$$

always imply $\mu_1 = \mu_2$?

Let $D_p^1(z)$ and $D_p^2(z)$ be the derivatives with respect to μ_1 and μ_2, respectively (cf. section 4). Then (2) clearly implies $D_p^1(z) = D_p^2(z)$ for all $z \in E$. Observe that we may assume without losing generality

$$\operatorname{supp}(\mu_1) = \operatorname{supp}(\mu_2) = E.$$

To see this replace μ_1 and μ_2 by $\mu_1 * \mu$ and $\mu_2 * \mu$ where μ is centred Gaussian with $\operatorname{supp}(\mu) = E$. Then we have $\mu_1 = \mu_2$ iff $\mu_1 * \mu = \mu_2 * \mu$ and, moreover, these two new measures satisfy (2) as well. Thus the above mentioned problem leads to the following question: for which numbers $p > 0$ does $D_p(z)$ describe μ completely? In the meantime (cf. [12]) the former problem has been solved in Hilbert spaces and some L_q-spaces by different methods. But these results dispel the belief that the set

$$\{p > 0; \quad (2) \text{ always implies } \mu_1 = \mu_2\}$$

is independent of the underlying Banach space E.

2) Several questions about the subject of this paper remain open. Let us only mention some of them.

(i) How are the elements $d(s, z)$ or $D_p(z)$ distributed in $\operatorname{supp}(\mu)'$? For instance, for which $s > 0$ and $z \in \operatorname{supp}(\mu)$ do we have $d(s, z) \neq 0$ or $D_p(z) \neq 0$, respectively? It follows from theorem 3.4 and remark (1) after proposition 3.2 that

$$\sup_{s > s_0} \|d(s, z)\| < \infty, \quad s_0 > 0.$$

Does this remain true for $s_0 = 0$ in arbitrary Banach spaces? Is it possible to describe those $y \in \operatorname{supp}(\mu)$ for which $d(s, z)(y) = 0$ or $D_p(z)(y) = 0$? In different words, for which elements y the measure $\mu(U_s(z + ty))$ is «almost» equal to $\mu(U_s(z))$?

(ii) In order to disprove the existence or the linearity of $d(s, z)$ we used non-smooth norms ($\|\cdot\|_\infty$ and $\|\cdot\|_1$, respectively). Do existence and (or) linearity of $d(s, z)$ ($\operatorname{supp}(\mu) \neq E$) depend on smoothness properties of the norm? Observe, that the Euclidean norm in $\mathbf{R}^2$ does not work in both of the examples.

(iii) How do $d(s, z)$ and $D_p(z)$ depend on s and z? It is easy to see that $d(s, -z) = -d(s, z)$ as well as $D_p(-z) = -D_p(z)$. Moreover, the mappings $s \to d(s, z)$ and $z \to d(s, z)$ are continuous from $(0, \infty)$ or $\operatorname{supp}(\mu)$ into $E_2'(\mu)$, respectively (cf. proposition 3.4). Are they continuous as mappings into $\operatorname{supp}(\mu)'$?

(iv) It is very likely that there are other classes of measures for which the measure of balls behaves quite smoothly under translations. Natural candidates are the classes of α-stable symmetric measures, $\alpha < 2$. As recently proved (cf. [20, ch. 6.10] or [21]) each α-stable symmetric measure ν can be written as follows:

$$\nu(B) = \int_\Omega \mu_\omega(B)\, d\mathbf{P}(\omega)$$

where μ_ω, $\omega \in \Omega$, are centred Gaussian measures on E. Consequently, we have

$$t^{-1}[\nu(U_s(z+ty))-\nu(U_s(z))] = \int_\Omega t^{-1}[F_\omega(s, z+ty)-F_\omega(s,z)]\,d\mathbf{P}(\omega)$$

and by the 0—1 law for Gaussian measures we obtain

$$\operatorname{supp}(\mu_\omega) = \operatorname{supp}(\nu) \ \mathbf{P}\text{-a. e.}$$

Hence, for $\mathbf{P}$-almost all ω the functions under the integral converge to $d_\omega(s, z)$, $z \in \operatorname{supp}(\nu)$. But we do not know how to estimate the function under the integral in order to apply Lebesgue's theorem. So it is very likely that the left hand quotient tends to $\int_\Omega d_\omega(s,z)\,d\mathbf{P}(\omega)$ as $t \downarrow 0$.

REFERENCES

1. *Aronszajn N.* Differentiability of Lipschitzian mappings between Banach spaces.— Studia Math., 1976, v. 57, p. 147—190.
2. *Badrikian A., Chevet S.* Mesures cylindriques, espaces de Wiener et fonctions aléatoires Gaussiennes.— Lect. Notes Math., B. 379.
3. *Borell C.* Convex measures on locally convex spaces.— Ark. Math., 1974, v. 12, p. 239—252.
4. *Borell C.* Gaussian Radon measures on locally convex spaces.— Math. Scand., 1976, v. 38, p. 265—284.
5. *Borell C.* A note on Gauss measures which agree on small balls.— Ann. Inst. H. Poincáré, ser. B, 1977, v. 13, p. 231—238.
6. *Бурбаки Н.* Функции действительного переменного. М.: Наука, 1965.
7. *Далецкий Ю. Л., Фомин С. В.* Меры и дифференциальные уравнения в бесконечномерных пространствах. М.: Наука, 1983.
8. *Ehrhard A.* Symétrisation dans l'espace de Gauss.— Math. Scand., 1983, v. 53, p. 281—301.
9. *Ehrhard A.* Sur la densite du maximum d'un processus Gaussien. These. Strasbourg, 1985, p. 87—104.
10. *Hoffmann-Jørgensen J., Shepp L. A., Dudley R. M.* On the lower tail of Gaussian seminorms.— Ann. Probab., 1979, p. 319—342.
11. *Го Х. Х.* Гауссовские меры в банаховых пространствах. М.: Мир, 1979.
12. *Linde W.* Shifted moments of Gaussian measures in Hilbert spaces. — To appear.
13. *Паулаускас В. И.* О скорости сходимости в центральной предельной теореме в некоторых банаховых пространствах.— Теория вероятн. и ее примен., 1976, т. XXI, в. 4, с. 775—791.
14. *Phelps R. R.* Gaussian null sets and differentiability of Lipschitz map on Banach spaces.— Pacific J. Math., 1978, v. 77, p. 523—531.
15. *Rhee W.* On the distribution of the norm for a Gaussian measure.— Ann. Inst. H. Poincaré, ser. B, 1984, v. 20, p. 277—286.
16. *Rhee W., Talagrand M.* Uniform convexity and the distribution of the norm for a Gaussian measure.— Probab. Th. Rel. Fields, 1986, v. 71, p. 59—67.
17. *Talagrand M.* Sur l'integrabilité des vecteurs gaussiens.— Z. Wahrscheinlichkeitstheor. verw. Geb., 1984, B. 68, S. 1—8.
18. *Вахания Н. Н., Тариеладзе В. И., Чобанян С. А.* Вероятностные распределения в банаховых пространствах. М.: Наука, 1985.
19. *Цирельсон Б. С.* Плотность распределения максимума гауссовского процесса.— Теория вероятн. и ее примен., 1975, т. XX, с. 865—873.
20. *Linde W.* Probability in Banach spaces — Stable and infinitely divisible distributions. Chichester etc.: Wiley, 1986.
21. *Marcus M. B., Pisier G.* Characterizations of almost surely continuous p-stable random Fourier series and strongly stationary processes.— Acta Math., 1984, v. 152, p. 245—301.

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