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Izv. Akad. Nauk SSSR
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COMPLEX HOMOGENEOUS SPACES OF THE LIE GROUP $SO(2k+1, 2l+1)$.

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F. M. MALYŠEV

Abstract. Let G be a connected Lie group, with Lie algebra which is the real form of the second category of type D_n. This paper lists all the connected closed subgroups U of G such that there exists a complex structure on the manifold $M = G/U$ which is invariant under G, and it also describes all such structures on M.

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Introduction

Let G be a connected real Lie group and let U be a connected closed subgroup. We are interested in the question of whether the manifold $M = G/U$ has a complex structure, invariant under G, and also in describing all such structures on M.

Some light is thrown on the history of the question in [7], where this problem is solved for semisimple Lie groups of first category.

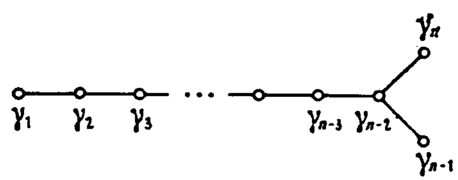
The present paper is an extension of [7], and this problem is solved here for the case when the Lie algebra of the group G is $\mathfrak{g} = \mathfrak{so}(2k+1, 2l+1)$, $k + l \geqslant 1$. This is precisely all real forms of the second category of type D_n, $n = k + l + 1$. (In this context we note that the title of this paper is not quite exact.)

As is well known, the aforementioned problem can be formulated as a problem on finding complex subalgebras $\mathfrak{h} \subset \mathfrak{g}^{\mathbf{C}}$ admitting a decomposition of the form

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \mathfrak{h}. \tag{1}$$

It is just this problem which we solve here. The Lie subalgebra corresponding to the subgroup U will be $\mathfrak{u} = \mathfrak{g} \cap \mathfrak{h}$. Obviously it is sufficient to classify the subalgebras $\mathfrak{h}$ satisfying (1) up to conjugacy with respect to automorphisms induced by automorphisms of $\mathfrak{g}$.

Let us quote first of all the classification proposed in [1] of all real forms of type D_n of the second category. Let



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be the Dynkin scheme for the simple complex Lie algebra $\mathfrak{k}^{\mathbf{C}}$ constructed from a Cartan subalgebra $\bar{\mathfrak{b}}^{\mathbf{C}}$, where $\bar{\mathfrak{b}} \subset \mathfrak{k}$ is a Cartan subalgebra in the compact form $\mathfrak{k}$ in some ordering. Let us denote by τ_0 the involutive automorphism of $\mathfrak{k}^{\mathbf{C}}$ given by

$$\tau_0(e_{\pm\gamma_i}) = e_{\pm\gamma_i}, \quad i = 1, \ldots, n-2;$$

$$\tau_0(e_{\pm\gamma_n}) = e_{\pm\gamma_{n-1}}; \quad \tau_0(e_{\pm\gamma_{n-1}}) = e_{\pm\gamma_n},$$

where e_{γ_i} are the elements of a Weil basis for $\mathfrak{k}^{\mathbf{C}}$. Then τ_0 generates an involutive automorphism on $\mathfrak{k}$, and we have $\mathfrak{k} = \mathfrak{k}_0 \dot{+} L_0$, where $\tau_0|\mathfrak{k}_0 = 1$ and $\tau_0|L_0 = -1$. Also $\mathfrak{k}_0^{\mathbf{C}}$ will be a simple complex Lie algebra of type B_{n-1} and $\mathfrak{k}_0$ will be its compact real form with Cartan subalgebra $\mathfrak{b}_0 = \ker((\tau_0 - 1)_{\bar{\mathfrak{b}}})$. One can suppose that the roots of $\mathfrak{k}_0^{\mathbf{C}}$ for the Cartan subalgebra $\mathfrak{b}_0^{\mathbf{C}}$ are $\varphi' = \varphi|_{\mathfrak{b}_0^{\mathbf{C}}}$, $\varphi \in \Sigma$. Here Σ is a system of roots for the Lie algebra $\mathfrak{k}^{\mathbf{C}}$. Let us denote the system of roots thus obtained for the Lie algebra $\mathfrak{k}_0^{\mathbf{C}}$ by Σ'. One may suppose that the simple roots in Σ' are $\gamma_1', \ldots, \gamma_{n-2}', \gamma_{n-1}' = \gamma_n$, where the last one is short. We choose a vector h_k in the space $i\mathfrak{b}_0$ according to the rule

$$\gamma_k'(h_k) = \frac{1}{2}, \quad \gamma_i'(h_k) = 0, \quad i = 1, \ldots, n-1, \quad i \neq k, \quad k = 1, \ldots, \left[\frac{n-1}{2}\right],$$

and put $h_0 = 0$ by definition. Then

$$\tau_k = e^{\operatorname{ad}(2\pi i h_k)} \tau_0$$

is an involutive automorphism of $\mathfrak{k}^{\mathbf{C}}$ inducing an involutive automorphism on $\mathfrak{k}$, and $\mathfrak{k} = \mathfrak{k}_k \dot{+} L_k$, where $\tau_k|\mathfrak{k}_k = 1$ and $\tau_k|L_k = -1$. Putting $\mathfrak{g}_k = \mathfrak{k}_k \dot{+} iL_k \cong \mathfrak{so}(2k+1, 2n-2k-$, we obtain all the real forms of the second category for $\mathfrak{k}^{\mathbf{C}}$.

If $\mathfrak{b}_1 = \ker((\tau_0 + 1)_{\bar{\mathfrak{b}}})$, then $\bar{\mathfrak{b}} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$, and $\mathfrak{b} = \mathfrak{b}_0 \oplus i\mathfrak{b}_1$ will be the maximal compact Cartan subalgebra for all the forms $\mathfrak{g}_k$.

In this paper it will be assumed that $\mathfrak{g}_k$ is imbedded in $\mathfrak{k}^{\mathbf{C}}$ by the method described.

Everywhere in future $\dot{+}$ will denote the direct sum of vector spaces and $\oplus$ the direct sum of Lie algebras. We also use the notation, associated with parabolic subalgebras, introduced in [7].

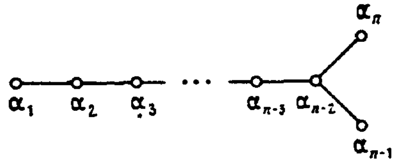
§1. Basic results

In this section we formulate the results of this paper, i.e. we list the subalgebras $\mathfrak{h} \subset \mathfrak{g}^{\mathbf{C}}$ satisfying (1), for $\mathfrak{g} = \mathfrak{g}_k$. (Throughout the whole paper we will suppose that $\mathfrak{g} = \mathfrak{g}_k$ in (1).)

For this let us introduce an ordering in Σ such that all standard parabolic subalgebras $\mathfrak{h}$ corresponding to this ordering give the decomposition (1). All such orderings are described in [2] and are listed in the following way: we choose a regular vector $x_0 \in \mathfrak{b}_0$ and suppose that $\varphi > 0$, $\varphi \in \Sigma$, if $\varphi(ix_0) > 0$. Let us examine what, in essence, this method of ordering means in our case. For this, as usual, we put $h_\varphi \in i\bar{\mathfrak{b}}$ for each $\varphi \in \Sigma$, so that $(h_\varphi, x) = \varphi(x)$ ($x \in \mathfrak{b}^{\mathbf{C}}$), where (,) is the Cartan-Killing form; similarly $h_{\varphi'} \in i\mathfrak{b}_0$ when $\varphi' \in \Sigma'$. Let us note that $h_{\varphi'}$ is the projection, parallel to $i\mathfrak{b}_1$, of h_φ onto $i\mathfrak{b}_0$; in this case we say that φ' is a projection of φ. It is obvious that φ' is a long root in Σ' if and only if $h_\varphi \in i\mathfrak{b}_0$. What we have said implies that the introduction of such an ordering means the following: we chose an arbitrary ordering of Σ' and declare the simple roots in Σ to be the long simple roots in Σ' and two roots in Σ which project onto the short simple root in Σ'.

Throughout the whole paper we consider a given fixed ordering. The choice of the subalgebras $\mathfrak{h}$ which will be described here will differ with different orderings.

Let



be the scheme of a Dynkin system of roots Σ, corresponding to the chosen ordering. We use the following notation:

$$\alpha_n=\alpha,\quad \alpha_{n-1}=\beta,\quad \alpha_{n-2}=\gamma,\quad \alpha_{n-3}=\delta,\quad \Pi_0=\{\alpha_1,\ \ldots,\ \alpha_{n-2}\},$$

$$\Pi=\Pi_0\cup\{\alpha,\beta\},$$

$$\Phi_\alpha=[\Pi_0\cup\{\alpha\}]\setminus[\Pi_0],\quad \Phi_\beta=[\Pi_0\cup\{\beta\}]\setminus[\Pi_0],$$

$$\Phi_{\alpha\beta}=[\Pi]\setminus([\Pi_0]\cup\ \Phi_\alpha\cup\Phi_\beta),$$

$$\Phi_{-\beta}=-\Phi_\beta,\quad \Phi_{-\alpha}=-\Phi_\alpha,\quad \Phi_{-\alpha,-\beta}=-\Phi_{\alpha,\beta}.$$

We denote by ι the projection, parallel to $i\mathfrak{g}_k$, of $\mathfrak{g}_k^{\mathbf{C}}$ onto $\mathfrak{g}_k$.

REMARK 1. *According to* [3] *we will have* $\iota(\mathfrak{g}_\varphi)=\iota(\mathfrak{g}_{-\varphi})$ *if* $h_\varphi\in i\mathfrak{b}_0$, *and* $\iota(\mathfrak{g}_\varphi)=\iota(\mathfrak{g}_{-\psi})$ *if* $h_\varphi\notin i\mathfrak{b}_0$ *and* $\varphi'=\psi'$.

Let us proceed to list the subalgebras $\mathfrak{h}$. All of them, according to the previous remark, will satisfy (1), since the latter is equivalent to $\iota(\mathfrak{h})=\mathfrak{g}_k$.

I. Let $\{\alpha,\beta\}\subset\Phi\subset\Pi$, or $\Phi\subset\Pi_0$. Then $\mathfrak{h}=(\mathfrak{s}_\Phi^{\mathbf{C}}\oplus\mathfrak{a})\dotplus\mathfrak{p}_\Phi^{\mathfrak{u}}$, where $\mathfrak{a}\subset\mathfrak{b}_\Phi^{\mathbf{C}}$, $\iota(\mathfrak{a})=\mathfrak{b}_\Phi$; $\mathfrak{u}=\mathfrak{s}_\Phi\oplus\mathfrak{a}\cap\mathfrak{b}_\Phi$.

II. Let $\Phi\subset\Pi_0$ and $\alpha_i\in\Pi_0\cup\{\beta\}$, $\alpha_{i-1}\notin\Phi$. Put

$$\Psi=\{\varphi\in\Phi_{-\beta}\mid\varphi\geqslant-\beta-\alpha_{n-2}-\ \ldots\ -\alpha_i\},$$

$$\Psi_1=\Phi_{\alpha,\beta}\cup\Phi_\beta\cup(\Phi_\alpha\setminus\{\alpha\})\cup([\Pi_0]\setminus[\Phi])\cup\Psi.$$

Then

$$\mathfrak{h}=(\mathfrak{s}_\Phi^{\mathbf{C}}\oplus\mathfrak{a})\dotplus\dot{\sum_{\varphi\in\Psi_1}}\mathfrak{g}_\varphi\dotplus\mathfrak{r},$$

where $\mathfrak{a}\subset\mathfrak{b}_\Phi^{\mathbf{C}}$, $\iota(\mathfrak{a})=\mathfrak{b}_\Phi$ and $\mathfrak{b}_\Phi^{\mathbf{C}}\oplus\mathfrak{a}\ni h_\varphi$ $(\varphi\in\Psi)$. Here $\mathfrak{b}_\Phi=[\{h_\varphi|\varphi\in\Phi\}]_{\mathbf{R}}$. When $i\neq n-1$ we have $\mathfrak{r}=\mathfrak{g}_\alpha$, and if $i=n-1$, then $\mathfrak{r}=\{0\}$ or $\mathfrak{g}_\alpha$. If $\mathfrak{r}=\mathfrak{g}_\alpha$, then

$$\mathfrak{u}=(\mathfrak{s}_\Phi\oplus\mathfrak{a}\cap\mathfrak{b}_\Phi)\dotplus\iota\Big(\sum_{\varphi\in\Psi}\mathfrak{g}_\varphi\Big),$$

and if $\mathfrak{r}=\{0\}$, then $\mathfrak{u}=\mathfrak{s}_\Phi\oplus\mathfrak{a}\cap\mathfrak{b}_\Phi$. We note that $\{\alpha_i,\alpha_{i+1},\ldots,\alpha_{n-2}\}\subset\Phi$.

III. Let $\alpha_i\in\Pi_0$, $\varphi_i=\alpha+\alpha_{n-2}+\cdots+\alpha_i$, $\Phi\subset\{\alpha_1,\ldots,\alpha_{i-2}\}$, $\Phi_0=\{\alpha_i,\ldots,\alpha_{n-2}\}$ and

$$\Psi=\Phi_{\alpha,\beta}\cup\Phi_\beta\cup([\Pi_0]\setminus[\Phi\cup\Phi_0])\cup\{\varphi\in\Phi_\alpha\mid\varphi>\varphi_i\}.$$

Then

$$\mathfrak{h}=(\mathfrak{s}_\Phi^{\mathbf{C}}\oplus e^{\operatorname{ad}(ie_{\varphi_i})}(\mathfrak{s}_{\Phi_0}^{\mathbf{C}}\oplus\mathfrak{a}))\dotplus\dot{\sum_{\varphi\in\Psi}}\mathfrak{g}_\varphi,$$

where $\mathfrak{a} \subset \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$. If $\mathfrak{a}' = \{x \in \mathfrak{b}^{\mathbf{C}}_{\Phi_0} \oplus \mathfrak{a} | \varphi_i(x) = 0\}$, then $\iota(\mathfrak{a}') = \mathfrak{b}_\Phi$. $0 \neq t \in \mathbf{C}$ is an arbitrary parameter;

$$\mathfrak{u} = \mathfrak{s}_\Phi \oplus \mathfrak{s}_{\Phi_0\setminus\{\alpha_i\}} \oplus \mathfrak{b}_\Phi \cap \mathfrak{a}'.$$

IV. Let $\Phi \subset \{\alpha_1, \dots, \alpha_{n-5}\}$, $\Phi_0 = \{\delta, \gamma, \alpha, \beta\}$. Then

$$\mathfrak{h} = (\mathfrak{s}^{\mathbf{C}}_\Phi \oplus \mathfrak{f} \oplus \mathfrak{a}) \dotplus \mathfrak{p}^u_{\Phi\cup\Phi_0},$$

where $\mathfrak{a} \subset \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$, $\iota(\mathfrak{a}) = \mathfrak{b}_{\Phi\cup\Phi_0}$; $\mathfrak{u} = \mathfrak{s}_\Phi \oplus \mathfrak{f} \cap \mathfrak{s}_{\Phi_0} \oplus \mathfrak{a} \cap \mathfrak{b}_{\Phi\cup\Phi_0}$. Here

IV_1: $\mathfrak{f} = \mathfrak{spin}(7, \mathbf{C}) \subset \mathfrak{s}^{\mathbf{C}}_{\Phi_0}$, and then $\mathfrak{f} \cap \mathfrak{s}_{\Phi_0}$ is a compact or noncompact form of type G_2 depending on whether $\mathfrak{s}_{\Phi_0} = \mathfrak{so}(1, 7)$ or $\mathfrak{so}(3, 5)$.

IV_2:

$$\mathrm{IV}_2: \mathfrak{s}^{\mathbf{C}}_{\Phi_0} \supset \mathfrak{f} = \mathfrak{i} \dotplus \sum^{\bullet}_{\varphi\in\{\alpha+\gamma,\alpha+\gamma+\delta,\alpha+\gamma+\beta,\alpha+\beta+\gamma+\delta\}} \mathfrak{g}_\varphi \dotplus \mathfrak{s}^{\mathbf{C}}_{\{\beta\}} \dotplus \mathfrak{s}^{\mathbf{C}}_{\{\delta\}} \dotplus \mathfrak{r}$$

$$\dotplus [te_{-\delta-\gamma} + e_{\beta+\gamma}] \dotplus [te_{-\gamma} - e_{\beta+\gamma+\delta}] \dotplus [e_{\delta+\gamma} + te_{-\beta-\gamma}] \dotplus [e_\gamma - te_{-\beta-\gamma-\delta}],$$

where $\mathfrak{i} \subset [h_{\alpha+\beta+\gamma}]_{\mathbf{C}} \oplus \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$ is a one-dimensional subspace not contained in $\mathfrak{b}_{\Phi\cup\Phi_0}$, $\mathfrak{r} = \mathfrak{g}_{\alpha+\beta+2\gamma+\delta} \dotplus \mathfrak{g}_\alpha$ or $\mathfrak{r} = [e_{\alpha+\beta+2\gamma+\delta} \dotplus te_\alpha]$, and $t \in \mathbf{C}$ is a parameter; here $\mathfrak{u}$ is the same for all t, and when $t = 0$ we have a subalgebra $\mathfrak{h}$ of type I or II; therefore $\mathfrak{u}$ need not be described. This remark will be relevant in all future cases where the one-dimensional parameter t oc

V. Let $\Phi \subset \Pi_0\setminus\{\alpha_{i-1}, \alpha_i, \alpha_{i+1}, \alpha_{n-2}\}$, $i = 1, \dots, n-3$, and $\Phi_0 = \{\alpha_i, \alpha, \beta\}$. Then

$$\mathfrak{h} = (\mathfrak{s}^{\mathbf{C}}_\Phi \oplus \mathfrak{f} \oplus \mathfrak{a}) \dotplus \mathfrak{p}^u_{\Phi\cup\Phi_0},$$

where $\mathfrak{a} \subset \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$, and $\iota(\mathfrak{a}) = \mathfrak{b}_{\Phi\cup\Phi_0}$. Here

V_1: $\mathfrak{s}^{\mathbf{C}}_{\Phi_0} \not\supseteq \mathfrak{f}$ is a semisimple algebra such that $\mathfrak{s}^{\mathbf{C}}_{\Phi_0} = \mathfrak{s}_{\Phi_0} + \mathfrak{f}$, $\mathfrak{u} = \mathfrak{s}_\Phi \oplus \mathfrak{f} \cap \mathfrak{s}_{\Phi_0} \oplus \mathfrak{b}_{\Phi\cup\Phi_0} \cap \mathfrak{a}$, and $\mathfrak{f} \cap \mathfrak{s}_{\Phi_0} \cong \mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbf{R})$ depending on whether $\mathfrak{s}_{\{\alpha_i\}} \cong \mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbf{R})$.

V_2: $\mathfrak{f} = \mathfrak{s}^{\mathbf{C}}_{\{\beta\}} \dotplus [e_{\alpha_i} + te_{\pm\alpha}] \dotplus \mathfrak{i}$, $\mathfrak{i} \subset [h_{\alpha_i} - h_{\pm\alpha}]_{\mathbf{C}} \oplus \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$ is a one-dimensional subspace not lying in $\mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$, and $t \in \mathbf{C}$ is a parameter.

V_3: $\mathfrak{f} = \mathfrak{g}_\alpha \dotplus [te_{-\beta} + e_{\alpha_i}] \dotplus [e_\beta + te_{-\alpha_i}] \dotplus [h_\beta - h_{\alpha_i}]_{\mathbf{C}} \dotplus \mathfrak{i}$, $\mathfrak{i} \subset [h_\alpha]_{\mathbf{C}} \oplus \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$ is a one-dimensional subspace not lying in $\mathfrak{b}^{\mathbf{C}}_{\Phi\cup\Phi_0}$, and $t \in \mathbf{C}$ is a parameter.

VI. Let $\Phi \subset \Pi_0\setminus\{\alpha_{n-2}\}$. Then $\mathfrak{h} = (\mathfrak{s}^{\mathbf{C}}_\Phi \oplus \mathfrak{f}) \dotplus \mathfrak{p}^u_{\Phi\cup\{\alpha,\beta\}}$. Here

VI_1: $\mathfrak{f} = \mathfrak{s}^{\mathbf{C}}_{\{\beta\}} \oplus \mathfrak{a}$, where $\mathfrak{g}_\alpha \not\subset \mathfrak{a} \subset \mathfrak{b}^{\mathbf{C}}_{\Phi\cup\{\alpha,\beta\}} \oplus \mathfrak{g}_\alpha$, $\iota(\pi(\mathfrak{a})) = \mathfrak{b}_{\Phi\cup\{\alpha,\beta\}}$, and π is the projection, parallel to $\mathfrak{g}_\alpha$, onto $\mathfrak{b}^{\mathbf{C}}_\Phi$;

$$\mathfrak{u} = \mathfrak{s}_\Phi \oplus (\mathfrak{a} \oplus \mathfrak{g}_{-\beta}) \cap (\mathfrak{b}_{\Phi\cup\Phi_0} \oplus \iota(\mathfrak{g}_\alpha)),$$

where $\Phi_0 = \{\alpha, \beta\}$.

$$\mathrm{VI}_2: \mathfrak{f} = \mathfrak{g}_\beta \dotplus \mathfrak{a},\ \mathfrak{g}_\alpha \not\subset \mathfrak{a} \subset \mathfrak{b}^{\mathbf{C}}_\Phi \dotplus \mathfrak{g}_\alpha,\ \mathfrak{a} \not\subset \mathfrak{b}^{\mathbf{C}}_\Phi,\ \iota(\cdot \cap \mathfrak{b}^{\mathbf{C}}_\Phi) = \mathfrak{b}_\Phi,\ \alpha|_{\mathfrak{a}\cap\mathfrak{b}^{\mathbf{C}}_\Phi} = 0;$$

$$\mathfrak{u} = \mathfrak{s}_\Phi \oplus \mathfrak{a} \cap \mathfrak{b}_\Phi.$$

VI_3: $\mathfrak{f} = \mathfrak{g}_\alpha \dotplus \widetilde{\mathfrak{a}}$, where $\widetilde{\mathfrak{a}} \subset \mathfrak{g}_\beta \dotplus \mathfrak{g}_{-\beta} \dotplus \mathfrak{b}^{\mathbf{C}}_\Phi$, $(\mathfrak{g}_\beta \cup \mathfrak{g}_{-\beta}) \cap \widetilde{\mathfrak{a}} = \{0\}$, and $\widetilde{\mathfrak{a}}$ projects "onto" $\mathfrak{g}_\beta$. If $\widetilde{\mathfrak{a}} \cap \mathfrak{b}^{\mathbf{C}}_\Phi = \mathfrak{a}$, then $\iota(\mathfrak{a}) = \mathfrak{b}_\Phi$ and $\beta|_{\mathfrak{a}} = 0$; $\mathfrak{u} = \mathfrak{s}_\Phi \oplus \mathfrak{a} \cap \mathfrak{b}_\Phi$.

The aim of the present paper is to prove that this list exhausts all the subalgebras $\mathfrak{h}$ in $\mathfrak{g}^{\mathbf{C}}_k$ satisfying (1).

To complete the picture let us formulate the basic result of [7], which will be frequently used.

THEOREM 1. *Let* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{c}$ *be a reductive Lie algebra with a semisimple part* $\mathfrak{g}_0$ *of the first category, and let the decomposition* (1) *hold. Then* $\mathfrak{h} = (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}) \dot{+} \mathfrak{p}_\Phi^u$, *where* $\mathfrak{a} \subset \mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{c}^{\mathbf{C}}$, $\iota(\mathfrak{a}) = \mathfrak{b}_\Phi \oplus \mathfrak{c}$ *and* $\mathfrak{b}$ *is a compact Cartan subalgebra in* $\mathfrak{g}_0$.

§2. α-subalgebras

DEFINITION 1. A subalgebra $\mathfrak{h} \subset \mathfrak{g}_k^{\mathbf{C}}$ satisfying (1) is called an *α-subalgebra* if $\mathfrak{h} \subset \mathfrak{p}_{\Pi_0 \cup \{\beta\}}$.

Let us examine the construction of α-subalgebras. We have

$$\mathfrak{p}_{\Pi_0 \cup \{\beta\}} = \mathfrak{b}^{\mathbf{C}} \dot{+} \sum_{\substack{\varphi \in \Sigma \\ \varphi > 0}}^{\cdot} \mathfrak{g}_\varphi \dot{+} \sum_{\varphi \in -[\Pi_0 \cup \{\beta\}]}^{\cdot} \mathfrak{g}_\varphi.$$

According to Remark 1,

$$(\mathfrak{g}_k \cap \mathfrak{p}_{\Pi_0 \cup \{\beta\}})^{\mathbf{C}} = \mathfrak{b}^{\mathbf{C}} \dot{+} \sum_{\varphi \in \pm[\Pi_0]}^{\cdot} \mathfrak{g}_\varphi \dot{+} \sum_{\varphi \in \Phi_\alpha \cup \Phi_{-\beta}}^{\cdot} \mathfrak{g}_\varphi.$$

Put

$$\sum_{\varphi \in \Phi_\alpha \cup \Phi_{-\beta}}^{\cdot} \mathfrak{g}_\varphi = V^{\mathbf{C}},$$

where $V \subset \mathfrak{g}_k$. It is clear that $V^{\mathbf{C}}$ is an Abelian subalgebra. Now $\mathfrak{b}^{\mathbf{C}} \dot{+} \Sigma^{\cdot}_{\varphi \in \pm[\Pi_0]} \mathfrak{g}_\varphi$ is a reductive Lie algebra with center $\mathfrak{z}^{\mathbf{C}} = \{x \in \mathfrak{b}^{\mathbf{C}} | \varphi(x) = 0\ (\varphi \in \Pi_0)\}$, $\mathfrak{z} \subset \mathfrak{g}_k$. Therefore $\mathfrak{b}^{\mathbf{C}} \dot{+} \Sigma^{\cdot}_{\varphi \in \pm[\Pi_0]} \mathfrak{g}_\varphi = (\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}}$, and $\mathfrak{s} \subset \mathfrak{g}_k$ is a simple Lie algebra of the first category since $\mathfrak{b}_0 = \mathfrak{b}_{\Pi_0} = [ih_{\alpha_1}, \ldots, ih_{\alpha_{n-2}}] \subset \mathfrak{b}_0$ is a compact Cartan subalgebra in $\mathfrak{s}$. Let us put

$$\mathfrak{p} = \sum_{\varphi \in \Phi_\beta \cup \Phi_{\alpha,\beta}}^{\cdot} \mathfrak{g}_\varphi.$$

Suppose we have an α-subalgebra $\mathfrak{h}$. According to Lemma 1 of [7],

$$\mathfrak{h} = \mathfrak{h} \cap ((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V)^{\mathbf{C}} \dot{+} \mathfrak{n} \dot{+} \mathfrak{h} \cap \mathfrak{p}, \tag{2}$$

where $\mathfrak{n}$ is a subspace of $\mathfrak{h}$ and

$$((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V)^{\mathbf{C}} = ((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V) + \mathfrak{h}_1, \tag{3}$$

where $\mathfrak{n}_1 = \mathfrak{h} \cap ((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V)^{\mathbf{C}}$. In addition, $\mathfrak{n} \dot{+} \mathfrak{h} \cap \mathfrak{p}$ projects "onto" $\mathfrak{p}$ in the decomposition $((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V)^{\mathbf{C}} \dot{+} \mathfrak{p}$.

Let us denote by π and π_1 the projections of $((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V)^{\mathbf{C}}$ onto $(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}}$ and $V^{\mathbf{C}}$ respectively in the decomposition $(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}} \dot{+} V^{\mathbf{C}}$. $V^{\mathbf{C}}$ is an Abelian ideal in $((\mathfrak{s} \oplus \mathfrak{z}) \dot{+} V)^{\mathbf{C}}$, and therefore π is a homomorphism.

According to (3)

$$(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}} = (\mathfrak{s} \oplus \mathfrak{z}) + \pi(\mathfrak{h}_1). \tag{4}$$

Theorem 1 asserts in this case that

$$\pi(\mathfrak{h}_1) = (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}) \dot{+} \mathfrak{p}_\Phi^u, \tag{5}$$

where Φ is a subsystem of simple roots of the algebra $\mathfrak{s}^{\mathbf{C}}$ for the Cartan subalgebra $'\mathfrak{b}^{\mathbf{C}}$, and $'\mathfrak{b}$ is a compact Cartan subalgebra in $\mathfrak{s}$.

REMARK 2. *One may suppose* $'\mathfrak{b} = \mathfrak{b}_0$.

If this is not so, then we consider, instead of the subalgebra $\mathfrak{h}$, the subalgebra $'\mathfrak{h}$ conjugate to $\mathfrak{h}$ under an inner automorphism of the algebra $\mathfrak{b}$ which takes $'\mathfrak{b}$ to $\mathfrak{b}_0$.

REMARK 3. *One can also suppose that the ordering of the system of roots* $\pm[\Pi_0]$ *on which* (5) *depends* (*it is this system of roots, since* $'\mathfrak{b} = \mathfrak{b}_0$) *is chosen so that its system of simple roots is precisely* Π_0.

Suppose this were not so, and let the system of simple roots for $\pm[\Pi_0]$ with respect to the new ordering be Π_1. Consider the adjoint representations of the Lie algebra $\mathfrak{s}^{\mathbf{C}}$ in the spaces $\dot{\Sigma}_{\varphi\in\Phi_\alpha}\mathfrak{g}_\varphi$ and $\dot{\Sigma}_{\varphi\in\Phi_\beta}\mathfrak{g}_\varphi$; they are invariant and irreducible. Let $\bar{\alpha}|_{\mathfrak{b}_0}$ ($\bar{\alpha} \in \Phi_\alpha$) and $\bar{\beta}|_{\mathfrak{b}_0}$ ($\bar{\beta} \in \Phi_\beta$) be the smallest weights with respect to Π_1 for these spaces. Then $\Pi_1 \cup \{\bar{\alpha}, \bar{\beta}\}$ will be a system of simple roots in Σ, and moreover $\mathfrak{p}_{\Pi_0\cup\{\beta\}} = \mathfrak{p}_{\Pi_1\cup\{\bar{\beta}\}}$, $\Phi_\alpha = \Phi_{\bar{\alpha}}$, $\Phi_{\pm\beta} = \Phi_{\pm\bar{\beta}}$, $\Phi_{\alpha,\beta} = \Phi_{\bar{\alpha},\bar{\beta}}$ and the equations (2)–(5) remain true. Therefore what was required in Remark 3 will be satisfied for the ordering of Σ thus obtained.

We put $V_0 = V^{\mathbf{C}} \cap \mathfrak{h}_1$. Since $V^{\mathbf{C}}$ is an Abelian ideal in $(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}} \dotplus V^{\mathbf{C}}$, then V_0 is $\pi(\mathfrak{h}_1)$-invariant with respect to the adjoint representation. Clearly the adjoint representation of $\mathfrak{a} \oplus \mathfrak{s}_\Phi^{\mathbf{C}}$ is completely reducible on $V^{\mathbf{C}}$. Let $V^{\mathbf{C}} = V_0 \dotplus V_1$ be an $\mathfrak{a} \oplus \mathfrak{s}_\Phi^{\mathbf{C}}$-invariant decomposition. Let us put $\hat{\mathfrak{h}}_1 = \{x \in \mathfrak{h}_1 | \pi_1(x) \in V_1\}$, $\hat{\pi} = \pi|\hat{\mathfrak{h}}_1$, $\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} = \hat{\pi}^{-1}(\mathfrak{s}_\Phi^{\mathbf{C}})$ and $\hat{\mathfrak{a}} = \hat{\pi}^{-1}(\mathfrak{a})$.

Let us introduce a few definitions.

DEFINITION 2. Let $\varphi, \psi \in R = \Phi_{\alpha,\beta} \cup \pm\Phi_\beta \cup \Phi_\alpha \cup \pm[\Pi_0] \cup \{0\}$. The equation

$$\varphi = \psi \tag{6}$$

will be called an *equation of type* (6) if

$$\iota(\{x \in \mathfrak{b}^{\mathbf{C}} | \varphi(x) = \psi(x)\}) = \mathfrak{b}.$$

ASSERTION 1. *An equation* $\varphi = \psi$, $\varphi, \psi \in R$, *is of type* (6) *if and only if* $h_\varphi - h_\psi \notin i\mathfrak{b}_0 \cup i\mathfrak{b}_1$.

The proof follows simply from the fact that $i\mathfrak{b}_0$ and $i\mathfrak{b}_1$ are perpendicular.

We shall say that a subspace $X \subset \mathfrak{b}^{\mathbf{C}}$ satisfies the equation $\varphi = \psi$, $\varphi, \psi \in \Sigma \cup \{0\}$, if $(\varphi - \psi)|_X \equiv 0$. Clearly if $\iota(X) = \mathfrak{b}$, $\varphi, \psi \in R$ and X satisfies the equation $\varphi = \psi$, then the latter is of type (6).

DEFINITION 3. An α-subalgebra $\mathfrak{h}$ is said to be *regular* if $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ does not satisfy any equation of type (6), and to be *singular* otherwise.

We recall that

$$\mathfrak{b}_\Phi = [\{ih_\varphi | \varphi \in \Phi\}]_{\mathbf{R}}, \quad \mathfrak{a} \subset \mathfrak{t}_\Phi^{\mathbf{C}} = \{x \in \mathfrak{b}^{\mathbf{C}} | \varphi(x) = 0 \ (\varphi \in \Phi)\}, \quad \iota(\mathfrak{a}) = \mathfrak{b}_\Phi.$$

DEFINITION 4. A subalgebra $\mathfrak{h} \subset \mathfrak{g}_k^{\mathbf{C}}$ satisfying (1) will be called an (α, β)*-subalgebra* if $\mathfrak{h} \subset \mathfrak{p}_{\Pi_0}$.

DEFINITION 5. An α-subalgebra is called a *strict* α*-subalgebra* if it is not an (α, β)-subalgebra.

§3. Classification of regular strict α-subalgebras

In this section we suppose that $\mathfrak{h}$ is a regular strict α-subalgebra.

Let us first prove a lemma about a more general situation.

LEMMA 1. *In the notation of the preceding section one can find $\Omega_0, \Omega_1 \subset \Sigma$ such that $\Omega_0 \cup \Omega_1 = \Phi_\alpha \cup \Phi_{-\beta}$ and $V_i = \dot{\Sigma}_{\varphi\in\Omega_i}\mathfrak{g}_\varphi$, $i = 0, 1$. And if $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ does not satisfy any equations of the form $\psi = 0$, $\psi \in \Omega_1$, then one can find a vector $v \in V_1$ such that*

$$\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} \oplus \hat{\mathfrak{a}} = e^{\operatorname{ad} v}(\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}). \tag{7}$$

The proof of the first part of the lemma follows from the fact that $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ cannot satisfy an equation $\varphi = \psi$, $\varphi, \psi \in \Phi_\alpha \cup \Phi_{-\beta}$, in view of Assertion 1.

We proceed to the proof of the second assertion. By Mal'cev's theorem one can find an element $v_1 \in V_1$ such that $e^{\operatorname{ad} v_1}\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} = \mathfrak{s}_\Phi^{\mathbf{C}}$. Let us consider the subalgebra

$$e^{\operatorname{ad} v_1}(\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} \oplus \hat{\mathfrak{a}}).$$

Clearly

$$[\mathfrak{s}_\Phi^{\mathbf{C}}, \pi_1(e^{\operatorname{ad} v_1}\hat{\mathfrak{a}})] = \{0\}. \tag{8}$$

Let $V_1 = V_{10} \dotplus V_{11}$ be a $\mathfrak{s}_\Phi^{\mathbf{C}}$-invariant decomposition, and moreover let $\mathfrak{s}_\Phi^{\mathbf{C}}$ act trivially on V_{10} and nontrivially on V_{11}. From (8) we have

$$\pi_1(e^{\operatorname{ad} v_1}\hat{\mathfrak{a}}) \subset V_{10} = \dot{\sum_{\varphi\in\Omega_{10}}} \mathfrak{g}_\varphi$$

for a certain $\Omega_{10} \subset \Omega_1$. Let us consider an element $X_0 \in \mathfrak{a}$ such that $\operatorname{ad} X_0$ is nondegenerate on V_{10}. Such an element exists because in view of the assumption of the lemma the set

$$\mathfrak{a} \setminus \Big(\bigcup_{\varphi\in\Omega_{10}} \{x \in \mathfrak{a} \mid \varphi(x) = 0\} \Big)$$

is nonempty. Further one can find an element $v_0 \in V_{10}$ such that $X_0 + v_0 \in e^{\operatorname{ad} v_1}\hat{\mathfrak{a}}$. Let $v_2 = (\operatorname{ad} X_0)^{-1}v_0 \in V_{10}$. Then $e^{\operatorname{ad} v_2}(e^{\operatorname{ad} v_1}\hat{\mathfrak{a}}) = \mathfrak{a}$. In fact let $X + w \in e^{\operatorname{ad} v_1}\hat{\mathfrak{a}}$, $X \in \mathfrak{a}$ and $w \in V_{10}$; then

$$e^{\operatorname{ad} v_2}(X + w) = X - \operatorname{ad} X((\operatorname{ad} X_0)^{-1}v_0) + w = X,$$

since $0 = [X + w, X_0 + v_0] = (\operatorname{ad} X)v_0 - (\operatorname{ad} X_0)w$. According to (8), putting $v = -v_1 - v_2$ we obtain (7). The lemma is proved.

Suppose we have a regular strict α-subalgebra $\mathfrak{h}$. Obviously Lemma 1 is valid. Consider $\bar{\mathfrak{h}} = e^{\operatorname{ad}(-v)}\mathfrak{h} \supset \mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$. In view of the regularity, we obtain, by considering the adjoint representation of $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ in $\mathfrak{p}_{\Pi_0\cup\{\beta\}}$,

$$\bar{\mathfrak{h}} = \bar{\mathfrak{h}} \cap \mathfrak{p} \dotplus \bar{\mathfrak{h}} \cap V^{\mathbf{C}} \dotplus \mathfrak{h} \cap (\mathfrak{s}^{\mathbf{C}} \oplus \mathfrak{z}^{\mathbf{C}}). \tag{9}$$

Hence, according to (5), (7) and the regularity condition, we conclude that

$$\bar{\mathfrak{h}} \cap (\mathfrak{s}^{\mathbf{C}} \oplus \mathfrak{z}^{\mathbf{C}}) = (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}) \dotplus \mathfrak{p}_\Phi^u.$$

Therefore (9) can be rewritten in the form

$$\bar{\mathfrak{h}} = \bar{\mathfrak{h}} \cap \mathfrak{p} \dotplus \bar{\mathfrak{h}} \cap V^{\mathbf{C}} \dotplus \mathfrak{p}_\Phi^u \dotplus (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}). \tag{10}$$

Since, in the decomposition (2), $\mathfrak{n} \dotplus \mathfrak{h} \cap \mathfrak{p}$ is projected "onto" $\mathfrak{p}$ with respect to the decomposition $((\mathfrak{z} \oplus \mathfrak{s}) + V)^{\mathbf{C}} \dotplus \mathfrak{p}$, and since $(\mathfrak{s}^{\mathbf{C}} \oplus \mathfrak{z}^{\mathbf{C}}) \dotplus V^{\mathbf{C}}$ is invariant under $e^{\operatorname{ad}(-v)}$, it follows that $e^{\operatorname{ad}(-v)}(\mathfrak{n} \dotplus \mathfrak{h} \cap \mathfrak{p})$ will, as before, be projected "onto" $\mathfrak{p}$, i.e. $\bar{\mathfrak{h}} \cap \mathfrak{p} = \mathfrak{p}$. Further, from (7) we obtain

$$\bar{\mathfrak{h}} \cap V^{\mathbf{C}} = V_0 = \sum_{\varphi \in \Omega_0}{}^{\cdot} \mathfrak{g}_\varphi.$$

From (10), according to the remarks we have just made,

$$\bar{\mathfrak{h}} = \mathfrak{p} \dotplus \sum_{\varphi \in \Omega_0}{}^{\cdot} \mathfrak{g}_\varphi \dotplus \mathfrak{p}_\Phi^u \dotplus (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}) = \sum_{\varphi \in R_0}{}^{\cdot} \mathfrak{g}_\varphi \dotplus \mathfrak{b}_\Phi^{\mathbf{C}} \dotplus \mathfrak{a},$$

where $R_0 = \Phi_{\alpha,\beta} \cup \Phi_\beta \cup \Omega_0 \cup [\Pi_0] \cup (-[\Phi])$. Put $\Omega_{-\beta} = \Phi_{-\beta} \cap \Omega_0$. We prove that $\Omega_{-\beta} \neq \varnothing$ and $\mathfrak{h}$ is a type II subalgebra listed in §1. In fact, if $\Omega_{-\beta} \neq \varnothing$, then, since R_0 is a closed subset of the roots and $[\Pi_0] \subset R_0$, we will have $-\beta \in R_0$. From this, since $\Phi_{\alpha,\beta} \subset R_0$ we obtain $\Phi_\alpha \backslash \{\alpha\} \subset R_0$. Consequently, according to Remark 1, $V^{\mathbf{C}} = V + V_0$. Therefore $v = v_1 + v_0$, where $v_1 \in V$ and $v_0 \in V_0$. Since $v_0 \in \bar{\mathfrak{h}}$, we have $e^{\operatorname{ad} v_0} \bar{\mathfrak{h}} = \bar{\mathfrak{h}}$. Therefore

$$\mathfrak{h} = e^{\operatorname{ad} v} \bar{\mathfrak{h}} = e^{\operatorname{ad} v_1} \bar{\mathfrak{h}}.$$

From the latter it is clear that, instead of $\mathfrak{h}$, we can consider $\bar{\mathfrak{h}}$, which is obviously of type II.

We prove that $\Omega_{-\beta} \neq \varnothing$ by contradiction. In fact, according to Remark 1, $v = v_1 + v_2$, where $v_1 \in V$ and $v_2 \in \Sigma^{\cdot}_{\varphi \in \Phi_\alpha} \mathfrak{g}_\varphi$. Therefore, instead of $\mathfrak{h}$, we can consider $e^{\operatorname{ad}(-v_1)}\mathfrak{h} = e^{\operatorname{ad} v_2} \bar{\mathfrak{h}}$; but the latter is contained in $\mathfrak{p}_{\Pi_0}$ if $\Omega_{-\beta} = \varnothing$, which contradicts the fact that we have a strict α-subalgebra.

Thus we have proved

Proposition 1. *Any regular strict α-subalgebra is of type* II.

§4. Singular α-subalgebras

We shall say that a singular α-subalgebra is of the *first type* if we have (7), and of the *second type* otherwise.

Suppose we have a singular α-subalgebra of the second type. Then we can find $\varphi_0 \in \Phi_\alpha \cup \Phi_{-\beta}$ (but, according to Assertion 2 (see the end of this section), only one) such that $\varphi_0 \equiv 0$ on $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$; otherwise Lemma 1 would hold, i.e. (7). Let us show that in this case we can find an element $v \in V_1$ such that

$$\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} \oplus \hat{\mathfrak{a}} = e^{\operatorname{ad} v} (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \tilde{\mathfrak{a}}); \tag{11}$$

$\mathfrak{g}_{\varphi_0} \subset \tilde{\mathfrak{a}} \subset \mathfrak{a} \oplus \mathfrak{g}_{\varphi_0}$ is a subspace projecting "onto" $\mathfrak{a}$ but not coinciding with $\mathfrak{a}$. Let us prove this. According to Mal'cev's theorem we may suppose $\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} = \mathfrak{s}_\Phi^{\mathbf{C}}$. Let $V_1 = V_{10} + V_{11}$ be an $\mathfrak{s}_\Phi^{\mathbf{C}}$-invariant decomposition, as in the proof of Lemma 1. It is clear that $\pi_1(\hat{\mathfrak{a}}) \subset V_{10}$. If $e_{\varphi_0} \in$ [illegible] $+ V_{11}$, then, as in the proof of Lemma 1, we would have (7) and consequently $e_{\varphi_0} \in$ [illegible] Let $V_{10} = \mathfrak{g}_{\varphi_0} \dotplus V_{01}$ be a $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$-invariant decomposition; then

$$\mathfrak{a} \dotplus V_{10} = (\mathfrak{a} \dotplus V_{01}) \oplus \mathfrak{g}_{\varphi_0}.$$

Let ρ be the projection, parallel to $\mathfrak{g}_{\varphi_0}$, of $\mathfrak{a} \dotplus V_{10}$ onto $\mathfrak{a} \dotplus V_{01}$. As in the proof of Lemma 1, there exists an element $v \in V_{01}$ such that

$$e^{\operatorname{ad}(-v)}(\rho(\hat{\mathfrak{a}})) = \mathfrak{a} = \rho(e^{\operatorname{ad}(-v)} \hat{\mathfrak{a}}).$$

For the v just obtained (11) is also fulfilled with $\widetilde{\mathfrak{a}} = e^{\mathrm{ad}(-v)}\hat{\mathfrak{a}}$.

THEOREM 2. *Let $\mathfrak{h}$ be a singular α-subalgebra. Then there exists an α-subalgebra $\overline{\mathfrak{h}}$, conjugate to $\mathfrak{h}$ under an automorphism generated by an inner automorphism of $\mathfrak{g}_{k}$, such that for this $\overline{\mathfrak{h}}$ we may suppose $v = 0$ in* (7) *or* (11).

PROOF. If we put $\mathfrak{h}_2 = ((\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{b}_\Phi^{\mathbf{C}}) + V^{\mathbf{C}}) \cap \mathfrak{h}_1$, then from (3)–(5) and Lemma 1 of [7] we have

$$\mathfrak{h}_2 = (\hat{\mathfrak{s}}_\Phi^{\mathbf{C}} \oplus \hat{\mathfrak{a}}) \dotplus V_0$$

and

$$(\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{b}_\Phi^{\mathbf{C}}) \dotplus V^{\mathbf{C}} = ((\mathfrak{s}_\Phi \oplus \mathfrak{b}_\Phi) \dotplus V) + \mathfrak{h}_2,$$

i.e. according to (7) or (11)

$$((\mathfrak{s}_\Phi \oplus \mathfrak{b}_\Phi) \dotplus V)^{\mathbf{C}} = ((\mathfrak{s}_\Phi \oplus \mathfrak{b}_\Phi) \dotplus V) + e^{\mathrm{ad}\, v}((\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}') \dotplus V_0), \tag{12}$$

where $\mathfrak{a}' = \mathfrak{a}$ or $\widetilde{\mathfrak{a}}$ respectively. Put $\overline{\mathfrak{h}} = e^{\mathrm{ad}(-v)}\mathfrak{h}$. We have $v \in V_1 = \dot{\Sigma}_{\varphi\in\Omega_1}\mathfrak{g}_\varphi$. Let $v = \Sigma_{\varphi\in\Omega_1}v_\varphi$, $v_\varphi \in \mathfrak{g}_\varphi$. We may suppose that $v_\varphi = 0$, if there is an element $\varphi \in \Omega_0$ such that $\varphi' = -\psi'$. In fact, in this case, according to Remark 1 we have $v_\varphi = v_{\varphi 1} + v_{\varphi 2}$, where $v_{\varphi 1} \in V$, $v_{\varphi 2} \in V_0$ and $e^{\mathrm{ad}(-v_\varphi)}\mathfrak{h} = e^{\mathrm{ad}(-v_{\varphi 1})}\mathfrak{h}$, and, instead of $\mathfrak{h}$, we can consider $e^{\mathrm{ad}(-v_{\varphi 1})}\mathfrak{h}$.

Further we may suppose that $v_\varphi = 0$ if $\varphi = \varphi_0$ or $\varphi' = -\varphi_0'$. In fact, if $\varphi = \varphi_0$, then

$$e^{\mathrm{ad}\, v_\varphi}|_{\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}'} = 1,$$

and if $\varphi' = -\varphi_0'$, we can write $v_\varphi = v_{\varphi 1} + v_{\varphi 2}$, where $v_{\varphi 1} \in V$ and $v_{\varphi 2} \in \mathfrak{g}_{\varphi_0}$.

Finally we may suppose that $v_\varphi = 0$ if $\varphi \in \Phi_{-\beta}$. In fact, let $\varphi \in \Phi_{-\beta}$; then $v_\varphi = v_{\varphi 1} + v_{\varphi 2}$, where $v_{\varphi 1} \in V$, $v_{\varphi 2} \in \mathfrak{g}_{\varphi_1}$, $\varphi' = -\varphi_1'$ and $\varphi_1 \in \Phi_\alpha$. Obviously we can consider v_φ instead of $v_{\varphi 2}$.

According to these last remarks, by (12) and Lemma 1 of [7] we must have $\mathfrak{b}^{\mathbf{C}} = \mathfrak{b} + \mathfrak{m}$, where $\mathfrak{m} = \{x \in \mathfrak{b}_\Phi^{\mathbf{C}} \ni \mathfrak{a}|\varphi_1(x) = 0\}$ for a certain $\varphi_1 \in \Omega_1 \cap \Phi_\alpha$ such that $v_{\varphi_1} \neq 0$. But according to Assertion 2 there is no such φ_1. Consequently we may suppose that $v_{\varphi_1} = 0$ for all $\varphi_1 \in \Omega_1 \cap \Phi_\alpha$. The theorem is proved.

The remaining part of the section will be devoted to studying equations of type (6).

DEFINITION 6. Two equations $\varphi_1 = \psi_1$ and $\varphi_2 = \psi_2$ of type (6) are said to be *equivalent* if the corresponding vectors $h_{\varphi_1} - h_{\psi_1}$ and $h_{\varphi_2} - h_{\psi_2}$ are collinear.

This is an equivalence relation.

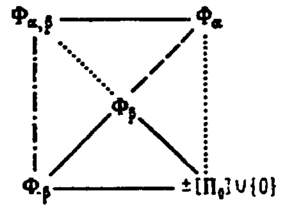
Let us formulate an assertion which we have already used repeatedly and which we shall use later on.

ASSERTION 2. *Let $X \subset \mathfrak{b}^{\mathbf{C}}$, $\iota(X) = \mathfrak{b}$, satisfy an equation $\varphi = \psi$ of type* (6). *Then X satisfies another equation $\varphi_1 = \psi_1$ of type* (6) *if and only if the equations $\varphi = \psi$ and $\varphi_1 = \psi_1$ are equivalent.*

The proof of the sufficiency is obvious, and the necessity follows from the fact that the space $[h_\varphi - h_\psi, h_{\varphi_1} - h_{\psi_1}]_{\mathbf{R}}$ either contains $i\mathfrak{b}_1$ or intersects $i\mathfrak{b}_0$ if the equations are not equivalent.

Our aim is to classify equations of type (6) up to equivalence.

ASSERTION 3. *The equation $\varphi = \psi$, $\varphi, \psi \in R$, is of type* (6) *if and only if φ and ψ belong to different sets in the diagram*



(13)

and these sets are joined by some line. Moreover, if for two equations these lines are distinct, then the equations are not equivalent.

The proof follows immediately from Assertion 1.

In future we shall interpret the roots Σ as sequences

$$(0\ldots0\underset{i}{\pm1}0\ldots0\underset{j}{\pm1}0\ldots0)$$

of length n, where $1 \leqslant i < j \leqslant n$, and suppose the simple roots to be

$$(1-10\ldots0)=\alpha_1,\ \ (01-10\ldots0)=\alpha_2,\ \ldots,$$
$$(0\ldots01-1)=\alpha=\alpha_n,\ \ (0\ldots011)=\beta=\alpha_{n-1}.$$

The problem of classifying equations of type (6) up to equivalence is solved by using Assertion 3. For the sake of simplicity we immediately quote the answer as Table 1. Here we use the notation

$$(0\ldots0\underset{i}{\pm1}0\ldots0\underset{j}{\pm1}0\ldots0)=\pm i\pm j$$

for $1 \leqslant i, j \leqslant n-1$, and suppose that ±1 always stands in the nth place, i.e. for example $(0\ldots0\pm10\ldots-1)$ is denoted by $i-1$, the sign $+$ is omitted and the equality sign in the equation is also omitted. The boxed equations in Table 1 are not of type (6), but they will be useful to us in the future.

REMARK 4. *Looking through Table* 1, *we note that if $\varphi = \psi_1$ and $\varphi = \psi_2$ are two equivalent equations of type* (6), *then $\psi_1 = \psi_2$. The one exception is in the equivalence class* 15, *where it is possible to have*

$$0=(0\ldots0\underset{i}{1}0\ldots01)=(0\ldots0\underset{i}{-1}0\ldots0-1).$$

§5. Classification of singular strict α-subalgebras of the first type

In this and in the following section, by a singular strict α-subalgebra we understand a singular α-subalgebra which is not an (α, β)-subalgebra or a subalgebra of type II.

Suppose we have a singular strict α-subalgebra $\mathfrak{h}$ of the first type. According to Theorem 2 we may suppose that $\mathfrak{h}_2 = (\mathfrak{s}^{\mathbf{C}}_{\Phi} \oplus \mathfrak{a}) \dotplus V_0$. Also, according to Assertion 2, $\mathfrak{d}^{\mathbf{C}}_{\Phi} \oplus \mathfrak{a} \subset \mathfrak{h}$ satisfies some equivalence class of equations of type (6). We shall examine each case separately. Moreover we shall suppose that the spaces $\mathfrak{p}_{\Pi_0\cup\{\beta\}}$ and $\mathfrak{h}$ are decomposed into weight subspaces with respect to the adjoint representation of $\mathfrak{d}^{\mathbf{C}}_{\Phi} \oplus \dot{\mathfrak{a}}$. These decompositions are completely defined by the corresponding equivalence class and according to Remark 4 are

Type	Equivalence class
1	$i{-}1j1$
2	$ij{-}j{-}1$ $\boxed{-i{-}jj1}$
3	$i{-}1{-}ij$ $\boxed{-i1i{-}j}$
$1 \leqslant i, j \leqslant n-1;\ i \neq j$	
4	$ij{-}k{-}1,\ ik{-}j{-}1,\ jk{-}i{-}1$ $\boxed{-i{-}jk1,\ -i{-}kj1,\ -j{-}ki1}$
$1 \leqslant i, j, k \leqslant n-1;\ i \neq j \neq k;\ i \neq k$	
5	$-i{-}1j1,\ -j{-}1i1$
6	$-iji1,\ -i{-}1i{-}j$
7	$i{-}jj1,\ -j{-}1{-}ij$
$1 \leqslant i < j \leqslant n-1$	
8	$ijk1,\ i{-}1{-}jk,\ j{-}1{-}ik$ $\boxed{-i1j{-}k,\ -j1i{-}k}$
9	$ikj1,\ i{-}1j{-}k,\ k{-}1{-}ij$ $\boxed{-i1{-}jk,\ -k1i{-}j}$
10	$jki1,\ k{-}1i{-}j,\ j{-}1i{-}k$
11	$i{-}1jk,\ i{-}jk1,\ i{-}kj1,\ -k{-}1{-}ij,\ -j{-}1{-}ik$
12	$j{-}1ik,\ -ijk1,\ j{-}ki1\ -i{-}1{-}jk,\ -k{-}1i{-}j$
13	$k{-}1ij,\ -ikj1,\ -jki1,\ -j{-}1i{-}k,\ -i{-}1j{-}k$
$1 \leqslant i < j < k \leqslant n-1$	
14	$i{-}10,\ ijj1,\ j{-}1{-}ij$; $i+1 \leqslant j \leqslant n-1$; $jij1,\ j{-}1j{-}i$; $1 \leqslant j \leqslant i-1$
15	$j{-}1ji,\ 1 \leqslant j \leqslant n-1;\ i \neq j$; $-ijj1,\ -j{-}1i{-}j,\ i+1 \leqslant j \leqslant n-1$; $j{-}ij1,\ -j{-}1{-}ji,\ 1 \leqslant j \leqslant i-1$; $0i1,\ 0{-}i{-}1,\ i1{-}i{-}1$
$1 \leqslant i \leqslant n-1$	

simple to construct. Let $\varphi = \psi$ be an equation in the corresponding equivalence class; we shall say that the vector $c_1 e_\varphi + c_2 e_\psi \in \mathfrak{h}$, $0 \neq c_1, c_2 \in \mathbf{C}$, is *diagonal* if $e_\varphi \notin \mathfrak{h}$. We shall say $\varphi \in R_0$ if $\mathfrak{g}_\varphi \subset \mathfrak{h}$. Further we shall suppose that we have drawn the diagram

$$\begin{array}{ccc} \Phi_{\alpha,\beta} & \Phi_\beta & \Phi_\alpha \\ & & \Phi_{-\beta} \\ \Phi^u & -\Phi^u & \pm[\Phi]\{0\} \end{array}$$

where $\Phi^u = [\Pi_0] \setminus [\Phi]$, and we shall denote the possible diagonals by lines on this diagram, i.e. we join the roots φ and ψ if the equation $\varphi = \psi$ is contained in the equivalence class under consideration. From (2), (5), Remark 1 and the condition $\iota(\mathfrak{h}) = \mathfrak{g}_k$ it follows that such likely-looking diagonals as $\varphi = \psi$, where $\varphi \in \Phi_\beta$, $\psi \in \Phi_{\alpha,\beta} \cup [\Pi_0]$ or $\varphi \in \Phi_\alpha \cup \Phi_{-\beta}$, $\psi \in -[\Pi_0]$, and also those with $\varphi \in \pm[\Phi]$, automatically cannot occur as diagonals, and therefore they cannot be drawn. It is clear that, if there are no diagonals, then $\mathfrak{h}$ is of type II. Further, we shall always bear Remark 1 in mind, and we shall use Lemma 1 of [7] without mentioning it specifically.

First let us prove that in cases 1–9, 11 and 14 there can be no singular strict α-subalgebras of the first type. In fact, in cases 1, 3, 7, 8, 9, 11 and 14 we have $\Omega_{-\beta} = \Omega_0 \cap \Phi_{-\beta} \neq \varnothing$, and therefore $-\beta \in R_0$; further one can find a vector of the form $e_\beta + X \in \mathfrak{h}$ such that $[e_\beta + X, e_{-\beta}] = \mathfrak{h}_\beta \in \mathfrak{d}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ which does not satisfy our equation in all cases, apart from 9 when $k = n-1$ and 14 when $i = n-1$. But in these cases it is easy to see that $\Phi_\alpha \setminus \{\alpha\} \subset R_0$ and therefore we have a decomposition of type II. In cases 2 and 4, if $\mathfrak{h}$ is not of type II, we have $c_1 e_\varphi + c_2 e_{-\psi} \in \mathfrak{h}$, where $0 \neq c_1, c_2 \in \mathbf{C}$, $\varphi \in \Phi_{\alpha,\beta}$ and $\psi \in \Phi_\beta$, whence

$$[e_\psi,\ c_1 e_\varphi + c_2 e_{-\psi}] = c_2 h_\psi \in \mathfrak{d}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$$

and the corresponding equations 2 and 4 are not satisfied. It is obvious that in case 5 the diagonals are only possible when $i = n-2$ and $j = n-1$. In this case if $e_\beta + te_{-\beta-\gamma} \in \mathfrak{h}$, $t \in \mathbf{C}$, then

$$[e_\gamma,\ e_\beta+te_{-\beta-\gamma}]=e_{\beta+\gamma}-te_{-\beta}\in\mathfrak{h},$$

and then

$$[e_\beta+te_{-\beta-\gamma},\ e_{\beta+\gamma}-te_{-\beta}]=-t(h_{\beta+\gamma}+h_\beta)\in\mathfrak{h},$$

which satisfies the equation $\beta=-\beta-\gamma$ only when $t=0$. Case 6 is completely analogous to case 5.

10. Obviously $-\beta\in R_0$, and therefore $\Phi_\alpha\backslash\{\alpha\}\subset R_0$. Consequently diagonals are possible only when $k=n-1$, $j=i+1$ and $\Phi\subset\Pi_0\backslash\{\alpha_{i-1},\alpha_i,\alpha_{i+1},\alpha_{n-2}\}$. Thus we have a subalgebra V_2 with $+\alpha$.

12. It is clear that $\Omega_{-\beta}=\varnothing$, and diagonals are possible only when $k=n-1$, $j=i+1$ and $\Phi\subset\Pi_0\backslash\{\alpha_{i-1},\alpha_i,\alpha_{i+1},\alpha_{n-2}\}$. There cannot be any diagonals between $\Phi_{\alpha,\beta}$ and Φ_α. Further, if $e_\beta+te_{-\alpha_i}$, $se_{-\beta}+e_{\alpha_i}\in\mathfrak{h}$, then

$$[e_\beta+te_{-\alpha_i},\ se_{-\beta}+e_{\alpha_i}]=sh_\beta-th_{\alpha_i}\in\mathfrak{h},$$

which satisfies 12 only if $s=t$, i.e. we have a subalgebra of type V_3.

13. It is clear that diagonals are possible only when $k=n-1$, $j=n-2$, $i=n-3$ and $\Phi\subset\{\alpha_1,\dots,\alpha_{n-5},\delta\}$. Further if $e_\gamma+te_{-\beta-\gamma-\delta}\in\mathfrak{h}$, then $[e_\delta,e_\gamma+te_{-\beta-\gamma-\delta}]=e_{\gamma+\delta}-te_{-\beta-\gamma}\in\mathfrak{h}$. Analogously if $e_{\beta+\gamma}+se_{-\gamma-\delta}\in\mathfrak{h}$, then $e_{\beta+\gamma+\delta}-se_{-\gamma}\in\mathfrak{h}$. The vector

$$[e_{\gamma+\delta}-te_{-\beta-\gamma},\ e_{\beta+\gamma}+se_{-\gamma-\delta}]=sh_{\gamma+\delta}+th_{\beta+\gamma}\in\mathfrak{h}$$

must satisfy 13, and therefore $s=-t$. We also have

$$[e_{\beta+\gamma}+se_{-\gamma-\delta},\ e_\gamma-se_{-\beta-\gamma-\delta}]=-2se_{-\delta}\in\mathfrak{h},$$
$$[e_{\gamma+\delta}+se_{-\beta-\gamma},\ e_\gamma-se_{-\beta-\gamma-\delta}]=-2se_{-\beta}\in\mathfrak{h}.$$

Thus in this case we have a subalgebra of type IV_2.

15. We note that $\Omega_{-\beta}=\varnothing$ and therefore $\Phi_\alpha\in R_0$. And if $i=n-1$, then we have precisely a subalgebra of type VI_3.

Let us examine the case $i\neq n-1$ in detail. Put $\psi_j=\beta+\alpha_{n-2}+\cdots+\alpha_j$, $i\leqslant j\leqslant n-1$, and $\varphi_j=\alpha_i+\alpha_{i+1}+\cdots+\alpha_{j-1}$, $i<j\leqslant n-1$. Then diagonals are possible only when $\Phi\subset\Pi_0\backslash\{\alpha_{i-1},\alpha_i\}$. The subalgebra $\mathfrak{h}$ will have the form

$$\mathfrak{h}=(\mathfrak{s}^{\mathbf{C}}_\Phi\oplus\mathfrak{a})\dotplus\sum_{\varphi\in\Psi}{}^{\cdot}\,\mathfrak{g}_\varphi\dotplus\sum_{j=i+1}^{n-1}{}^{\cdot}\,([e_{\psi_j}+te_{-\varphi_j}]\dotplus[se_{-\psi_j}+e_{\varphi_j}])$$
$$\dotplus[e_{\psi_i}+tse_{-\psi_i}+h_0],$$

where $h_0=sh_{\psi_j}-th_{\varphi_j}$ for any $j=i+1,\dots,n-1$,

$$\mathfrak{a}\subset\mathfrak{z}^{\mathbf{C}}_\Phi,\ \Psi=\Phi_{\alpha,\beta}\cup\Phi_\alpha\cup(\Phi_\beta\backslash\{\psi_i,\psi_{i+1},\dots,\psi_{n-1}\})$$
$$\cup([\Pi_0]\backslash([\Phi]\cup\{\varphi_{i+1},\dots,\varphi_{n-1}\})),$$

and $(t,s)\in\mathbf{C}^2$ is a parameter. As in the proof of Theorem 2, we can, instead of $\mathfrak{h}$, consider the subalgebra $e^{\operatorname{ad}(-se_{-\psi_i})}\mathfrak{h}$; and this, as is easily seen, will be an (α,β)-subalgebra.

Summarizing this section, we can formulate

Proposition 2. *Every singular strict α-subalgebra of the first type is one of the subalgebras of type* IV_2, V_2 *with* $+\alpha$, V_3 *or* VI_3.

§6. The classification of singular strict α-subalgebras of the second type

Suppose we have a singular strict α-subalgebra $\mathfrak{h}$ of the second type, and suppose that $\mathfrak{b}_\Phi^C \oplus \mathfrak{a}$ satisfies the equation $\varphi_0 = 0$, $\varphi_0 \in \Phi_\alpha \cup \Phi_{-\beta}$, of type (6). Then

ASSERTION 4. *If $\psi \in R$ and ψ is neither the right- nor the left-hand side of an equation equivalent to $\varphi_0 = 0$, then $\varphi_0 + \psi$ is not a root. If $\psi_1 = \psi_2$ is equivalent to $\varphi_0 = 0$ but distinct from $\varphi_0 = -\varphi_0$, then one may suppose that $\psi_1 = \psi_2 + \varphi_0$, and $\psi_1 + \varphi_0$ is not a root.*

The proof is obvious.

ASSERTION 5. *One can find a vector X_0 in the space $\tilde{\mathfrak{a}}$ (for the definition of $\tilde{\mathfrak{a}}$ see* (11)) *such that* ad X_0, *in terms of the basis $\{e_\varphi, h_{\alpha_i} | \varphi \in R\backslash\{0\}, i = 1, \dots, n\}$, has the form*

$$\begin{array}{c} e_{\varphi_1}\, e_{\varphi_2} \cdots e_{\varphi_{2m-1}}\, e_{\varphi_{2m}}\, e_{\varphi_{2m+1}} \cdots e_{\varphi_r} \quad e_{\varphi_0}\ h_{\alpha_1} \cdots h_{\alpha_n}\ e_{-\varphi_0} \\ \left(\begin{array}{c|c} \begin{matrix} \begin{matrix}\lambda_1 & 1\\ 0 & \lambda_1\end{matrix} & & & & \\ & \ddots & & 0 & \\ & & \boxed{\begin{matrix}\lambda_m & 1\\ 0 & \lambda_m\end{matrix}} & & \\ 0 & & & \lambda_{m+1} & \\ & & & & \ddots \\ & & & & & \lambda_r \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & & & \\ & 0 & & * \\ & & \ddots & \\ 0 & & & 0 \end{matrix} \end{array}\right) \end{array} \tag{14}$$

The last column can only occur in the case 15, all the $\lambda_i \in \mathbf{C}$ are distinct and nonzero, and $\{\varphi_1 = \varphi_2, \dots, \varphi_{2m-1} = \varphi_{2m}, \varphi_0 = 0 \ (= -\varphi_0)\} = \Lambda$ is an equivalence class of equations of type (6). Here $\{0 = -\varphi_0\} \subset \Lambda$ if Λ is of type 15.

The proof follows from Assertion 4 and the fact that the set

$$\mathfrak{a} \backslash ((\tilde{\mathfrak{a}} \cap \mathfrak{a}) \cup (\bigcup_{\substack{\varphi, \psi \in R \\ \{\varphi = \psi\} \not\subset \Lambda}} \{x \in \mathfrak{a} \mid \varphi(x) = \psi(x)\})),$$

is nonempty according to Assertion 2.

Let us note that the invariant subspaces for transformations of the type (14) are the direct sums of subspaces corresponding to the blocks with distinct eigenvalues, and the eigenspace corresponding, for example, to the block

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

is $\mathfrak{g}_{\varphi_1}$.

The class Λ is either of type 14 or of type 15. Let us take these possibilities separately. Moreover, we shall bear in mind the remarks made at the beginning of the preceding section.

14. Since $\Omega_{-\beta} \neq \varnothing$, we obviously have $i = n - 1$ and $\Omega_{-\beta} = \{-\beta\}$. According to Assertion 5 and the above remark, we have

$$\Phi_{\alpha,\beta} \cup \Phi_\beta \cup [\Pi_0] \cup (\Phi_\alpha \setminus \{\alpha\}) \subset R_0.$$

Furthermore $\Phi \subset \Pi_0 \setminus \{\alpha_{n-2}\}$, since otherwise we would have $\alpha \in R_0$, and this would contradict the definition of a singular α-subalgebra of the second type. Thus we have a subalgebra of type VI_1.

15. Again $i = n - 1$, for otherwise we would have $h_\beta \in \mathfrak{h}$, which does not satisfy 15. Obviously we can find $e_\beta + ce_{-\beta} + h_0 \in \mathfrak{h}$, where $c \in \mathbf{C}$, $h_0 \in \mathfrak{b}^{\mathbf{C}}$ and $e_{-\beta} + h_1 \in \tilde{\mathfrak{a}} \subset \mathfrak{h}$, $h_1 \in \mathfrak{a}$. Then

$$[e_{-\beta}+h_1,\quad e_\beta+ce_{-\beta}+h_0] = -h_\beta+\beta(h_0)e_{-\beta}\in\tilde{\mathfrak{a}},$$

which is impossible since $-h_\beta$ does not satisfy 15. Thus in this case there are no subalgebras.

§7. The classification of (α, β)-subalgebras

Suppose we have an (α, β)-subalgebra $\mathfrak{h}$. By definition

$$\mathfrak{h} \subset \mathfrak{p}_{\Pi_0} = \mathfrak{b}^{\mathbf{C}} + \sum_{\varphi \in \pm[\Pi]_0} \mathfrak{g}_\varphi \dotplus \mathfrak{q},$$

where

$$\mathfrak{q} = \sum_{\varphi \in \Phi_{\alpha,\beta} \cup \Phi_\alpha \cup \Phi_\beta} \mathfrak{g}_\varphi.$$

According to Lemma 1 of [7],

$$\mathfrak{h} = \mathfrak{h} \cap (\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}} \dotplus \mathfrak{n} \dotplus \mathfrak{h} \cap \mathfrak{q},$$

where $\mathfrak{n}$ is a subspace of $\mathfrak{h}$ and

$$(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}} = (\mathfrak{s} \oplus \mathfrak{z}) \dotplus \mathfrak{h}_2,$$

where $\mathfrak{h}_2 = \mathfrak{h} \cap (\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}}$, and the space $\mathfrak{n} + \mathfrak{h} \cap \mathfrak{q}$ projects "onto" $\mathfrak{q}$ in the decomposition $(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}} \dotplus \mathfrak{q}$. The aim of the present section is basically to find $\mathfrak{n}$. Obviously $\mathfrak{h}_2 = (\mathfrak{s}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}) \dotplus \mathfrak{p}_\Phi^u$, where $\Phi \subset \Pi_0$ (see Remarks 2 and 3 of §2).

If $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ does not satisfy any equation of type (6), it is obvious that $\mathfrak{n} = \{0\}$, i.e. $\mathfrak{h}$ is a subalgebra of type I. Let $\mathfrak{b}_\Phi^{\mathbf{C}} \oplus \mathfrak{a}$ satisfy some equation of type (6). Again we shall bear in mind the remark made at the beginning of §5. Looking through Table 1, we note that $\mathfrak{n} = \{0\}$ automatically in all cases apart from 6, 8, 12, 13, 14 and 15. In cases 6, 8, 12 and 13 it is easy to show that $\varphi \in [\Pi_0]$, so that h_φ does not satisfy the corresponding equation, and so in these cases $\mathfrak{n} = \{0\}$.

It only remains to consider cases 14 and 15. It is clear that we can confine ourselves to one of them. Suppose we have 14. We suppose that $\mathfrak{h}$ is not a subalgebra of type I. Obviously when $i = n - 1$ we have a decomposition of type VI_2.

Let $i \neq n - 1$. Put $\{\alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-2}\} = \Phi_0$ and $\varphi_i = \alpha + \alpha_{n-2} + \cdots + \alpha_i$. Clearly in this case

$$\Phi_0 \setminus \{\alpha_i\} \subset \Phi \subset \{\alpha_1, \ldots, \alpha_{i-2}\} \cup (\Phi_0 \setminus \{\alpha_i\}).$$

If we consider the projection of $\mathfrak{n} \dotplus \mathfrak{s}^{\mathbf{C}}_{\Phi_0 \setminus \{\alpha_i\}}$ onto $(\mathfrak{s} \oplus \mathfrak{z})^{\mathbf{C}}$, we obtain $\mathfrak{s}^{\mathbf{C}}_{\Phi_0}$. Then by Mal'cev's theorem we have

$$\mathfrak{n} \dot{+} \mathfrak{s}^{\mathbf{C}}_{\Phi_0 \setminus \{\alpha_i\}} = e^{\operatorname{ad} v}(\mathfrak{s}^{\mathbf{C}}_{\Phi_0}), \quad 0 \neq v \in \sum_{\substack{\varphi \in \Phi_\alpha \\ \varphi \leqslant \varphi_i}}^{\cdot} \mathfrak{g}_\varphi.$$

Since we have case 14, from Lemma 1 of [7] and Assertion 2 we conclude that $v \in \mathfrak{g}_{\varphi_i}$, i.e. in this case we have a decomposition of type III.

Combining the results of Propositions 1 and 2 and the results of this and the preceding sections, we obtain

PROPOSITION 3. *Any α-subalgebra is of one of the following types*: I *with* $\Phi \subset \Pi_0$, II, III, IV_2, V_2 *with* $+\alpha$, V_3, VI_1, VI_2 *or* VI_3.

§8. Decomposition (1) with $\mathfrak{g} = \mathfrak{so}(1, 7)$, $\mathfrak{so}(3, 5)$, $\mathfrak{su}(2) \oplus \mathfrak{su}(2, \mathbf{C})$, or $\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{C})$

First let us study the decomposition

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} \dot{+} \mathfrak{h} \tag{15}$$

in the case when $\mathfrak{g}$ is a real semisimple Lie algebra and $\mathfrak{h}$ is a maximal complex semisimple subalgebra in $\mathfrak{g}^{\mathbf{C}}$.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two isomorphic complex simple Lie algebras and $T\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is an isomorphism, then we put

$$(\mathfrak{g}_1, \mathfrak{g}_2) = \{x \in \mathfrak{g}_1 \oplus \mathfrak{g}_2 \mid x = x_1 + Tx_1 \, (x_1 \in \mathfrak{g}_1)\},$$

and denote by $\mathfrak{g}_{1,2}$ the simple real form of the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

LEMMA 2. *The irreducible decomposition* (15) *is one of the following*:

1°. $\mathfrak{so}(8, \mathbf{C}) = \mathfrak{so}(1, 7) + \mathfrak{spin}(7, \mathbf{C}), \mathfrak{so}(8, \mathbf{C}) = \mathfrak{so}(3, 5) + \mathfrak{spin}(7, \mathbf{C})$;

2°. $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g}_{1,2} + (\mathfrak{g}_1 \oplus \mathfrak{g}_2')$, *where* $\mathfrak{g}_2'$ *is a maximal semisimple subalgebra in* $\mathfrak{g}_2$;

3°. $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = (\mathfrak{g}_0 \oplus \mathfrak{g}_{2,3}) - ((\mathfrak{g}_1, \mathfrak{g}_2) \oplus \mathfrak{g}_3)$, *where* $\mathfrak{g}_0^{\mathbf{C}} = \mathfrak{g}_1$;

4°. $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4 = (\mathfrak{g}_{1,2} \oplus \mathfrak{g}_{3,4}) + (\mathfrak{g}_1 \oplus (\mathfrak{g}_2, \mathfrak{g}_3) \oplus \mathfrak{g}_4)$.

The maximal semisimple subalgebras in the complex semisimple Lie algebra $\bigoplus_{i=1}^k \mathfrak{g}_i$, where the $\mathfrak{g}_i$ are simple ideals, are the algebras [4]:

a)
$$\mathfrak{g}_j' \oplus \bigoplus_{\substack{i=1 \\ i \neq j}}^{k} \mathfrak{g}_i,$$

where $\mathfrak{g}_j'$ is a maximal semisimple subalgebra in $\mathfrak{g}_j$;

b)
$$(\mathfrak{g}_j, \mathfrak{g}_l) \oplus \bigoplus_{\substack{i=1 \\ i \neq j,\, i \neq l}}^{k} \mathfrak{g}_i.$$

In view of this the proof of the lemma follows immediately from the results of [5].

According to Lemma 2, for the $\mathfrak{g}$'s listed in the title of this section there exists a decomposition $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \mathfrak{h}_0$, where $\mathfrak{h}_0$ is a maximal semisimple subalgebra in $\mathfrak{g}^{\mathbf{C}}$.

LEMMA 3. *Suppose we have a decomposition* (1) *for* $\mathfrak{g} = \mathfrak{so}(1, 7), \mathfrak{so}(3, 5), \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbf{C})$ *or* $\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{C})$, *and* $\mathfrak{h} \not\subseteq \mathfrak{h}_0$. *Then the subalgebra* $\mathfrak{h}$ *is an* α*-subalgebra.*

PROOF. Let $\mathfrak{g} = \mathfrak{so}(1, 7)$ or $\mathfrak{so}(3, 5)$. Then $\mathfrak{h}_0 = \mathfrak{spin}(7, \mathbf{C})$, and $\mathfrak{h}_0 \cap \mathfrak{g} = \mathfrak{w}$ is a compact or noncompact form of the type G_2. $\mathfrak{so}(1, 7)$ and $\mathfrak{so}(3, 5)$ are obtained from the construction described in the Introduction when $n = 4$ and $k = 0$ or 1 respectively. We can suppose [5] that the Cartan subalgebra for $\mathfrak{h}_0$ is

$$\overline{\overline{\mathfrak{b}}} = \{x \in \mathfrak{b}^{\mathbf{C}} \mid \delta(x) = \beta(x)\},$$

and for $\mathfrak{w}^{\mathbf{C}}$ it is

$$\overline{\overline{\overline{\mathfrak{b}}}} = \{x \in \mathfrak{b}^{\mathbf{C}} \mid \delta(x) = \beta(x) = \alpha(x)\}.$$

The roots for $\mathfrak{h}_0$ will be $\varphi|_{\widetilde{\mathfrak{b}}} = \widetilde{\varphi} \in \widetilde{\Sigma}$, and those for $\mathfrak{w}^{\mathbf{C}}$ will be $\varphi|_{\overline{\overline{\mathfrak{b}}}} = \overline{\overline{\varphi}} \in \overline{\overline{\Sigma}}$, $\varphi \in \Sigma$. Now $\mathfrak{g}_{\widetilde{\varphi}} \subset \mathfrak{g}_\varphi \dotplus \mathfrak{g}_\psi$ if $\widetilde{\varphi} = \widetilde{\psi}$, and $\mathfrak{g}_{\overline{\overline{\varphi}}} \subset \mathfrak{g}_\varphi \dotplus \mathfrak{g}_\psi \dotplus \mathfrak{g}_\chi$ if $\overline{\overline{\varphi}} = \overline{\overline{\psi}} = \overline{\overline{\chi}}$. Let $\mathfrak{h}_0 = \mathfrak{w}^{\mathbf{C}} \dotplus \mathfrak{o}$ be a $\mathfrak{w}^{\mathbf{C}}$-invariant decomposition; obviously $\mathfrak{o} \cap \mathfrak{g} = \{0\}$. Suppose we have a decomposition (1) and $\mathfrak{h} \not\subset \mathfrak{h}_0$. According to Lemma 1 of [7], $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{w}^{\mathbf{C}} + \mathfrak{n} + \mathfrak{h} \cap \mathfrak{o}$, and moreover $\mathfrak{w}^{\mathbf{C}} = \mathfrak{w} + \mathfrak{h} \cap \mathfrak{w}^{\mathbf{C}}$. We conclude from Theorem 1 that

$$\mathfrak{h} \cap \mathfrak{w}^{\mathbf{C}} = (\mathfrak{s}^{\mathbf{C}}_{\Phi} \oplus \mathfrak{a}) \dotplus \mathfrak{p}^u_{\Phi},$$

where Φ is a subsystem of simple roots of $\overline{\overline{\Sigma}}$. Obviously Remarks 2 and 3 are also valid here, and we can suppose that $\Phi \not\subset \{\overline{\overline{\gamma}}, \overline{\overline{\delta}}\}$. Further we can conclude from Assertion 2 that $\mathfrak{h}$ is regular in $\mathfrak{h}_0$. But then it is easy to see that $\{\widetilde{\varphi} \in \widetilde{\Sigma} \mid \mathfrak{g}_{\widetilde{\varphi}} \subset \mathfrak{h}\}$ is a parabolic system of roots. Since the space $\mathfrak{n} \dotplus \mathfrak{h} \cap \mathfrak{o}$ must project "onto" $\mathfrak{o}$, of a necessity either $\mathfrak{g}_{-\widetilde{\alpha}-\widetilde{\gamma}}$ or $\mathfrak{g}_{-\widetilde{\delta}-\widetilde{\gamma}}$ must belong to $\mathfrak{h}$, and therefore $\mathfrak{g}_{-\widetilde{\gamma}} \subset \mathfrak{h}$. Similarly either $\mathfrak{g}_{-\widetilde{\delta}-\widetilde{\gamma}-\widetilde{\alpha}}$ or $\mathfrak{g}_{-\widetilde{\delta}-\widetilde{\gamma}-\widetilde{\beta}}$ belongs to $\mathfrak{h}$, and therefore $\mathfrak{g}_{-\widetilde{\delta}} \subset \mathfrak{h}$. Consequently $\mathfrak{h} \subset \mathfrak{p}_{\{\widetilde{\delta},\widetilde{\gamma}\}} \subset \mathfrak{p}_{\{\delta,\gamma,\beta\}}$.

Let us prove the lemma for $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbf{C})$. Let $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \mathfrak{h}$, where $\mathfrak{h} \not\subset \mathfrak{h}_0$ and $\mathfrak{g}^{\mathbf{C}} = \mathfrak{g} + \mathfrak{h}_0$ is a decomposition of type 3° in Lemma 2. There is one simple term of $\mathfrak{h}_0$ for which $\mathfrak{h}$ does not project "onto" this term, from which we conclude that $\mathfrak{h}$ does not project "onto" $\mathfrak{sl}(2, \mathbf{C})^{\mathbf{C}}$ in the decomposition $\mathfrak{g}^{\mathbf{C}} = \mathfrak{su}(2)^{\mathbf{C}} \oplus \mathfrak{sl}(2, \mathbf{C})^{\mathbf{C}}$. This is also valid for $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{C})$. The lemma is proved.

§9. Basic theorem

In this section we shall prove

THEOREM 3. *Any subalgebra* $\mathfrak{h} \subset \mathfrak{g}^{\mathbf{C}}_k$ *satisfying* (1) *is one of the subalgebras* I–VI *of* §1.

PROOF. Proposition 3 asserts that any α-subalgebra is one of the subalgebras of type I–VI. We shall take this into consideration in the proof.

Suppose we have a decomposition (1). Then we can find a maximal subalgebra $\widetilde{\mathfrak{h}}_0 \supset \mathfrak{h}$ in $\mathfrak{g}^{\mathbf{C}}_k$. According to [6], $\widetilde{\mathfrak{h}}_0$ is either semisimple or parabolic. It is obvious that $\mathfrak{g}^{\mathbf{C}}_k = \mathfrak{g}_k + \widetilde{\mathfrak{h}}_0$. If $\widetilde{\mathfrak{h}}_0$ is semisimple, then this decomposition, according to Lemma 2, occurs only in the case $\mathfrak{g}^{\mathbf{C}}_k = \mathfrak{so}(8, \mathbf{C})$. In this case, if $\mathfrak{h} = \widetilde{\mathfrak{h}}_0$, then $\mathfrak{h}$ is a subalgebra of type IV_1, and, if $\mathfrak{h} \neq \widetilde{\mathfrak{h}}_0$, the theorem is valid by Lemma 3. Consequently $\widetilde{\mathfrak{h}}_0$ can be considered to be a parabolic subalgebra and not necessarily maximal. According to [2],

$$\widetilde{\mathfrak{h}}_0 = (\mathfrak{s}^{\mathbf{C}}_{\Phi_0} \oplus \mathfrak{b}^{\mathbf{C}}_{\Phi_0}) \dotplus \mathfrak{p}^u_{\Phi_0},$$

where $\Phi_0 \not\subset \Pi$ for some ordering. (We may suppose that we have chosen this ordering in §1.) If $\{\alpha, \beta\} \not\subset \Phi_0$, then the theorem is true (it is obvious that the β-subalgebras which are defined in the same way as the α-subalgebras are conjugate to α-subalgebras under automorphisms generated by the $\mathfrak{g}_k$).

In the contrary case when $\{\alpha, \beta\} \subset \Phi_0$ we have

$$\widetilde{\mathfrak{h}}_0 \cap \mathfrak{g}_k = \mathfrak{s}_{\Phi_0} \oplus \mathfrak{b}_{\Phi_0}.$$

According to Lemma 1 of [7] we may suppose that

$$\mathfrak{h} = \mathfrak{h}_1 \dotplus \mathfrak{n}_0 \dotplus \mathfrak{h} \cap \mathfrak{p}^u_{\Phi_0}, \tag{16}$$

where $\mathfrak{h}_1 = \mathfrak{h} \cap (\mathfrak{s}^C_{\Phi_0} \oplus \mathfrak{b}^C_{\Phi_0})$ and $\mathfrak{s}^C_{\Phi_0} \oplus \mathfrak{b}^C_{\Phi_0} = (\mathfrak{s}_{\Phi_0} \oplus \mathfrak{b}_{\Phi_0}) + \mathfrak{h}_1$.

Let us denote by κ_0 the projection, parallel to $\mathfrak{b}^C_{\Phi_0}$, of $\mathfrak{s}^C_{\Phi_0} \oplus \mathfrak{b}^C_{\Phi_0}$ onto $\mathfrak{s}^C_{\Phi_0}$. If $\kappa_0(\mathfrak{h}_1) = \mathfrak{s}^C_{\Phi_0}$, then $\mathfrak{s}^C_{\Phi_0} \subset \mathfrak{h}_1$, but then $i\mathfrak{b}_1 \subset \mathfrak{h}_1$, since $\{\alpha, \beta\} \subset \Phi_0$; therefore, according to Assertion 1, $\mathfrak{n}_0 = \{0\}$ and $\mathfrak{h}$ is a subalgebra of type I. If $\kappa_0(\mathfrak{h}_1) \neq \mathfrak{s}^C_{\Phi_0}$, then there exists a maximal subalgebra $\widetilde{\mathfrak{h}}_1 \supset \kappa_0(\mathfrak{h}_1)$ in $\mathfrak{s}^C_{\Phi_0}$. Obviously

$$\mathfrak{s}^C_{\Phi_0} = \mathfrak{s}_{\Phi_0} + \widetilde{\mathfrak{h}}_1, \tag{17}$$

i.e. $\widetilde{\mathfrak{h}}_1$ is again either semisimple or parabolic.

Let $\widetilde{\mathfrak{h}}_1$ be semisimple. According to Lemma 2 one of the irreducible components of the decomposition (17) must be the decomposition 1° or 3°, and all the other components are trivial. Let the decomposition 1° or 3° in Lemma 2 have the form $\mathfrak{c}^C_0 = \mathfrak{c}_0 + \mathfrak{h}_0$. Then, in addition,

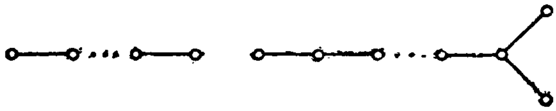
$$\mathfrak{s}_{\Phi_0} \oplus \mathfrak{b}_{\Phi_0} = \mathfrak{c} \oplus \mathfrak{c}_0, \quad \mathfrak{h}_0 \cap \mathfrak{c}_0 = \mathfrak{c}_1.$$

Obviously we have $\mathfrak{c}^C \oplus \mathfrak{c}^C_1 = (\mathfrak{c} \oplus \mathfrak{c}_1) + (\mathfrak{c}^C \oplus \mathfrak{c}^C_1) \cap \mathfrak{h}_1$ and $\mathfrak{c} \oplus \mathfrak{c}_1$ is of the first category, and therefore we can apply Theorem 1 in this last decomposition, i.e.

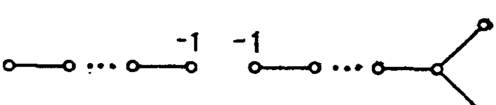
$$(\mathfrak{c}^C \oplus \mathfrak{c}^C_1) \cap \mathfrak{h}_1 = (\mathfrak{s}^C_{\Phi} \ominus \mathfrak{a}) \dotplus \mathfrak{r}^u_{\Phi}.$$

We can suppose that $\mathfrak{h}_1$ projects "onto" $\mathfrak{h}_0$ in the decomposition $\mathfrak{s}^C_{\Phi_0} \oplus \mathfrak{b}^C_{\Phi_0} = \mathfrak{c}^C \oplus \mathfrak{c}^C_0$, for otherwise, according to Lemma 3, the subalgebra $\kappa_0(\mathfrak{h}_1)$ would be an α-subalgebra for $\mathfrak{s}_{\Phi_0}$, and this case will be considered explicitly. Therefore according to Assertion 2 we can conclude that $\mathfrak{h}_0 \subset \mathfrak{h}_1$ and $\mathfrak{n}_0 = \{0\}$. Thus in this case we can say that $\mathfrak{h}$ is a subalgebra of type IV_1 or V_1.

Let us assume now that $\widetilde{\mathfrak{h}}_1$ is a parabolic subalgebra. Then again $\widetilde{\mathfrak{h}}_1 = (\mathfrak{s}^C_{\Phi_1} \oplus \mathfrak{b}^C_{\Phi_1}) \dotplus \mathfrak{p}^u_{\Phi_1}$, $\Phi_1 \not\subseteq \Phi_0$. Here we are assuming that Remarks 2 and 3 are valid. Let us prove Remark 3 for the case $\Phi_0 = \Pi\backslash\{\varphi_0\}$, $\varphi_0 \in \Pi_0$. In the general case it can be proved by induction. Suppose we have chosen an ordering in $\pm[\Phi_0]$ which is different from the induced ordering and has a system of simple roots Φ'_0:



Consider the adjoint representation of $\mathfrak{s}^C_{\Phi}$ in $\mathfrak{p}^u_{\Phi_0}$, and let $\mathfrak{g}_{\psi_0} \subset \mathfrak{p}^u_{\Phi_0}$ be the weight subspace for the weight



Taking $\Phi_0' \cup \{\psi_0\}$ to be the simple roots in Σ, we obtain a new ordering of Σ with respect to which the above construction of the proof of the theorem does not change and Remark 3 will be valid.

If $\{\alpha, \beta\} \subset \Phi_1$, then again we can write

$$\mathfrak{h}_1 = \mathfrak{h}_2 \dotplus \mathfrak{n}_1 \dotplus \mathfrak{h}_1 \cap \mathfrak{p}^u_{\Phi_2}, \tag{18}$$

where $\mathfrak{h}_2 = \mathfrak{h}_1 \cap (\mathfrak{s}^{\mathbf{C}}_{\Phi_1} \oplus \mathfrak{b}^{\mathbf{C}}_{\Phi_1} \oplus \mathfrak{b}^{\mathbf{C}}_{\tilde{\Phi}_0})$. We extend this process until we obtain one of the conditions a) $\kappa_{l-1}(\mathfrak{h}_l) = \mathfrak{s}^{\mathbf{C}}_{\Phi_{l-1}}$, b) $\tilde{\mathfrak{h}}_l$ is semisimple, or c) $\{\alpha, \beta\} \subset \Phi_l$. It is obvious that this process terminates. As we have already shown, if the process terminates with conditions a) or b), then we shall have subalgebras I, IV_1 or V_1.

Let us consider now the case when the process terminates with condition c). We can suppose that $\alpha \notin \Phi_l$. Combining (16), (18) and so on up to $\mathfrak{h}_{l-1} = \mathfrak{h}_l \dotplus \mathfrak{n}_{l-1} \dotplus \mathfrak{h}_{l-1} \cap \mathfrak{p}^u_{\Phi_{l-1}}$, we find that

$$\mathfrak{h} = \mathfrak{p}_{\Pi_0 \cup \{\beta\}} \cap \mathfrak{h} \dotplus \mathfrak{n}, \tag{19}$$

where

$$\mathfrak{n} \cap \sum_{\varphi \in \Phi_\alpha \cup \Phi_{\alpha,\beta}}^{\cdot} \mathfrak{g}_{-\varphi} = \{0\}.$$

In addition the projection, parallel to $\Sigma^{\cdot}_{\varphi \in \Phi_\alpha \cup \Phi_{\alpha,\beta}} \mathfrak{g}_{-\varphi}$, of $\mathfrak{h}$ onto $\mathfrak{p}_{\Pi_0 \cup \{\beta\}}$ will be an α-subalgebra.

We shall classify the subalgebras $\mathfrak{h}$ of type (19) in the same way as we classified the α-subalgebras. For this we shall consider the equations

$$\varphi = \psi, \tag{20}$$

where $\varphi \in \Phi_{-\alpha} \cup \Phi_{-\alpha,-\beta}$ and $\psi \in [\Pi]$. We shall classify these equations up to an equivalence; moreover, equivalent equations of type (6) will be included in the corresponding equivalence classes. For the sake of simplicity we shall quote the answer immediately. The equivalence classes of equation (20) will be the classes 2, 3, 4, 8 and 9 of Table 1 together with the boxed equations, and also classes 2′, 4′, 5′ and 15′; here class N' is obtained from the class of type N by reversing the sign belonging to the 1 in every equation in the equivalence class of type N in Table 1.

Now let us consider each of these cases separately, bearing in mind the remark made at the beginning of §5.

Let us prove that in all cases, apart from 9, $\mathfrak{h}$ will be an α-subalgebra or a β-subalgebra.

In cases 3, 8, 5′ and 15′, when $i \neq n-1$, we have $\Omega_{-\beta} = \varnothing$, since otherwise we would have $e_{-\beta} \in \mathfrak{h}$, whence $[e_\beta, e_{-\beta}] = h_\beta \in \mathfrak{h}$ and does not satisfy the corresponding equations. Therefore in these cases $\mathfrak{h}$ is a β-subalgebra.

In cases 2, when $j \neq n-1$, and 4, when $k \neq n-1$, there can be no diagonals between $\Phi_{\alpha,\beta}$ and $\Phi_{-\beta}$, since otherwise $e_{-\beta} \in \mathfrak{h}$, whence $h_\beta \in \mathfrak{h}$ and does not satisfy these equations. There are no diagonals between Φ_β and $\Phi_{-\alpha,-\beta}$, since otherwise an element $\psi \in \Phi_{\alpha,\beta}$ could be found such that $h_\psi \in \mathfrak{h}$ and does not satisfy our equations. Consequently in these cases $\mathfrak{h}$ is a β-subalgebra. Similarly in case 2′ when $i \neq n-1$ and $j \neq n-1$ we have $\Omega_{-\beta} = \varnothing$, and therefore $\alpha \in R_0$, whence $\Phi_\alpha \subset R_0$, i.e. again $\mathfrak{h}$ is a β-subalgebra.

We note that in case 2′ when $i > j$ we also have $\Omega_{-\beta} = \emptyset$. And in case 2 when $j = n - 1$, case 4 when $k = n - 1$, and case 15′ when $i = n - 1$, we have an α-subalgebra; and in case 2′ when $i > j = n - 1$, we have a β-subalgebra, because there are no diagonals between Φ_β and $\Phi_{-\alpha,-\beta}$ in 2 and 4, between $-\alpha$, α and 0 in 15′, and between Φ_α and $\Phi_{-\alpha,-\beta}$ in 2′, since $\kappa_{l-1}(\mathfrak{h}_l)$ is an α-subalgebra for $\mathfrak{s}_{\Phi_{l-1}}$.

In case 2′ when $i < j$, we may suppose that $j = n - 1$ and $i = n - 2$. If $i \neq n - 2$, we would have

$$(\underset{i}{1}\quad \underset{j}{1}) = (\underset{i}{1} - \underset{i+1}{1}) + (\underset{i+1}{1}\quad \underset{j}{1}) \in R_0,$$

whence we would conclude that there are no diagonals between $\Phi_{-\alpha,-\alpha}$ and Φ_α, i.e. $\mathfrak{h}$ would be an α-subalgebra. But if $i = n - 2$, then the vectors $c_1 e_{\alpha+\gamma} + c_2 e_{-\alpha-\beta-\gamma}$, $c_2 \neq 0$, and $k_1 e_{-\alpha-\gamma} + k_2 e_{\alpha+\beta+\gamma}$, $k_1 \neq 0$, do not belong to $\mathfrak{n}_{l-1}$, since in general we have

REMARK 5. *If* $c_1 e_\varphi + c_2 e_{-\psi} \in \mathfrak{n}_s$, $\varphi \in [\Pi]$, $\psi \in \Phi_\alpha \cup \Phi_{\alpha,\beta}$ *and* $s = 0, \ldots, l - 1$, *then* $\psi \in [\Phi_s]$.

Let us consider case 4′. If $k \neq n - 1$, then again $\Omega_{-\beta} = \emptyset$. Therefore $e_\alpha \in \mathfrak{h}$, which means $\Phi_\alpha \subset R_0$. If $c_1 e_\varphi + c_2 e_{-\psi} \in \mathfrak{h}$, $c_2 \neq 0$, $\varphi \in \Phi_{\alpha,\beta}$, $\psi \in \Phi_\alpha$, then we would have $[e_\psi, c_1 e_\varphi + c_2 e_{-\psi}] = c_2 h_\psi \in \mathfrak{b}$ not satisfying 4′. Let $k = n - 1$. Then $j = n - 2$. In the contrary case $-\beta - \gamma \notin R_0$; therefore $\gamma + \alpha \in R_0$, from which in turn it follows that

$$(\underset{p}{1}\quad -1) \in R_0, \quad p = i, j.$$

From this last relation, since $h_{(\underset{p}{1}-1)}$, $p = i, j$, does not satisfy 4′, we have

$$(\underset{p}{1}\quad \underset{k}{1}) \in R_0, \quad p = i, j,$$

but then

$$(\underset{i}{1}\quad \underset{j}{1}) = (\underset{i}{1}\quad -\underset{k}{1}) + (\underset{j}{1}\quad \underset{k}{1}) \in R_0;$$

and therefore

$$c_1 e_{(\underset{i}{-1}\quad \underset{j}{-1})} + c_2 e_{(\underset{k}{1}\quad -1)}, \quad c_1 \neq 0,$$

cannot be contained in $\mathfrak{h}$. In this way we would have an α-decomposition if $j \neq n - 2$. Further there must necessarily exist $c_1, c_2 \in \mathbf{C}$ such that

$$c_1 e_{(\underset{j}{1}\quad -1)} + c_2 e_{(\underset{i}{-1}\quad \underset{k}{-1})} \in \mathfrak{h}.$$

If $\Omega_{-\beta} = \emptyset$, this is obvious, but if $\Omega_{-\beta} \neq \emptyset$ it will follow from the fact that the projection, parallel to $\Sigma'_{\varphi \in \Phi_\alpha \cup \Phi_{\alpha,\beta}} \mathfrak{g}_{-\varphi}$, of $\mathfrak{h}$ onto $\mathfrak{p}_{\Pi_0 \cup \{\beta\}}$ is a subalgebra. Clearly

$$(-\underset{j}{1}\quad -\underset{k}{1}) = (-\underset{j}{1}\quad \underset{n-2}{1}) - \gamma - \alpha - \beta.$$

Therefore, according to Remark 5, $c_1 e_{\gamma+\alpha} + c_2 e_{(\underset{i}{-1}\ \underset{k}{-1})}$, $c_2 \neq 0$, cannot belong to $\mathfrak{h}$. Consequently $\gamma + \alpha \in R_0$, and that means $(\underset{i}{1}\ -1) \in R_0$ also. Hence in turn we have

$$(\underset{i}{1}\quad \underset{k}{1}) \in R_0, \quad (\underset{j}{1}\quad \underset{k}{1}) \in R_0,$$

and from this last condition we obtain that

$$(\underset{i}{1} \quad \underset{j}{1}) \in R_0.$$

So $\mathfrak{h}$ is an α-subalgebra in this case.

Let us consider finally case 9. When $k \neq n-1$, as usual, $\Omega_{-\beta} = \varnothing$ and $\mathfrak{h}$ is a β-subalgebra. If $k = n-1$, then $\Omega_{-\beta} = \{-\beta\}$, for otherwise $\mathfrak{h}$ would be a β-subalgebra. Furthermore $j = i+1$ and

$$\{\varphi \in \Pi_0 \mid -\varphi \in R_0\} \subset \Pi_0 \setminus \{\alpha_{i-1}, \alpha_i, \alpha_{i+1}, \alpha_{n-2}\}.$$

But then $\mathfrak{h}$ is of type V_2 with $-\alpha$. The theorem is proved.

According to the theorem just proved and the remark made in describing the subalgebra $\mathfrak{h}$ of type IV_2 in §1, we have

COROLLARY. *If G_k is a connected Lie group with Lie algebra $\mathfrak{g}_k$ and U is a closed connected subgroup of G_k, then there exists an invariant complex structure on the manifold G_k/U if and only if U is conjugate to a subgroup with a subalgebra $\mathfrak{u}$ of type* I, II, IV_1, V_1, VI_1 *or* VI_2.

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