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STABILITY CRITERIA FOR THE ACTION OF A SEMISIMPLE GROUP ON A FACTORIAL MANIFOLD

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Abstract. In this work it is proved that, for the regular action of a semisimple irreducible algebraic group G on an affine space, the existence of a closed orbit of maximum dimension is equivalent to the existence of an invariant open set at any point of which the stationary subgroup is reductive. This result is established for the action of G on manifolds of a special type (the so-called factorial manifolds). There are given several other conditions equivalent to the existence of a closed orbit of maximum dimension for the action of G on an arbitrary affine manifold.

Let k be an algebraically closed field of characteristic zero, which will be viewed in the sequel as the universal domain.

Definition 1. An irreducible affine manifold X is called *factorial* if the ring $k[X]$ of regular functions defined on it is factorial and every invertible regular function is a constant.

Let G be an irreducible semisimple algebraic group acting regularly on X. This action induces k-automorphisms of the ring $k[X]$ and of the field $k(X)$ of rational functions on X. Let O_x denote the orbit of the point $x \in X$. Let m_G be the maximum dimension of the orbits of the action of G on X. It is not hard to show that each orbit is a smooth quasi-projective manifold. Points with $\dim O_x = m_G$ are called points of general position. Standard arguments connected with tangent spaces show that points of general position form an open subset in X. Let us denote it by Ω. Let $\Pi = X - \Omega$. As follows from the definition, Π is a G-invariant submanifold in X, for every point x of which $\dim O_x < m_G$.

Definition 2. The action is called *locally transitive* if $\dim X = m_G$.

Clearly, if the action is locally transitive then there exists in X a unique orbit of dimension m_G, and it is the set Ω. Since $\dim G \geq m_G$, if $\dim X > \dim G$ the action cannot be locally transitive.

Definition 3. The action is called *stable* if X contains a G-invariant open set such that every orbit lying within it is closed in X (see [7]).

The main result of this report is

Theorem 1. *In order that the regular action of an irreducible semisimple algebraic group on a factorial manifold be stable, it is necessary and sufficient that there exist in the manifold a G-invariant open set U, at any point of which the stationary subgroup is reductive.*

Since finite groups are reductive, we have

Corollary 1. *If* $m_G = \dim G$ *then the action of* G *on* X *is stable.*

Indeed, it suffices to take $U = \Omega$.

Note that if the action is stable then the intersection $U \cap \Omega$ is always nonempty. If the action is locally transitive but not transitive, then clearly there does not exist a closed orbit of maximum dimension. In view of these remarks we obtain

Corollary 2. *If the action is locally transitive but* X *is not a homogeneous space, then the stationary subgroup of a point of general position cannot be reductive.*

Note that the theorem is applicable to the case when the action is a linear presentation of G in the m-dimensional vector space k^m, since k^m is a factorial manifold. Below we will show that there exist actions on factorial manifolds other than the affine space k^m.

For the case $X = k^m$ and for a linear action there are given in [4] sufficiently broad conditions that $m_G = \dim G$. Namely, if for all simple normal subgroups G_i of G the index $l_{G_i} > 1$, then $m_G = \dim G$. Recall that the index is equal to

$$l_{G_i} = \frac{\operatorname{Tr} Y^2}{\operatorname{Tr} (\operatorname{ad} Y)^2}, \quad Y \in \mathfrak{G}_i,$$

where $\mathfrak{G}_i$ is the Lie algebra of G_i (it is known that this quotient does not depend on the choice of the element $Y \in \mathfrak{G}_i$; see [5]). In particular, for example, for irreducible presentations of simple Lie groups the condition $m_G = \dim G$ is equivalent to the fact that the dimension of the space of the presentation is greater than the dimension of the group.

Remark 1. If the manifold X is not factorial then the theorem may or may not hold. We give some examples.

Let $G = SL(2, k)$. Consider the action of G on the space of binary forms of degree m. This is an irreducible presentation of G in the $(m+1)$-dimensional space k^{m+1}. Consider the set of binary forms having a not less than $[(m+2)/2]$-multiple linear factor. Let us denote it by X. Then X is an affine algebraic manifold on which G acts regularly. It is easy to see that zero lies in the closure of each orbit of a point $x \in X$. Hence, other than zero, there do not exist orbits of the action of G on X. On the other hand, it is not hard to see that if $m > 4$, then $m_G = 3$. Since $\dim G = 3$, the stationary subgroups of points of general position for the action of G on X are finite, and hence also reductive. It follows from Theorem 1 that X is not factorial.

Remark 2. If the action of G on X is stable and U is an open G-invariant subset such that any orbit lying in U is closed in X, then U does not coincide, generally speaking, with Ω. In fact, if, for example, $X = k^m$, $G = SL(2, k)$, and the action is that of an irreducible linear presentation, it follows, as noted earlier, that if $m > 5$, then X contains three-dimensional orbits whose closures contain zero.

The proof of the theorem will be carried out below. First, we will derive certain necessary and sufficient conditions for the stability of the action of G on X.

Lemma 1. *Let the algebraic group G, having no characters, act on the factorial manifold X. Then the field of invariant rational functions $I_G(X)$ coincides with the field of quotients of the algebra $I_G[X]$ of regular invariants.*

Proof. Let $f \in I_G(X)$. Since $k(X)$ is the field of quotients of the algebra $k[X]$, there exists a presentation (not unique) $f = p/q$, where $p, q \in k[X]$. If, however, p and q are relatively prime, then the presentation $f = p/q$ is uniquely defined up to multiplication of p and q by an invertible element. Now let $\sigma \in G$. It is not hard to see that if p and q are relatively prime then so are p^σ and q^σ. Since $f^\sigma = p^\sigma/q^\sigma = p/q$, it follows that $p = \epsilon(\sigma)\, p^\sigma$ and $q = \epsilon(\sigma)\, q^\sigma$, so that $\epsilon(\sigma)$ is invertible. Since the invertible elements of the ring $k[X]$ coincide, by hypothesis, with the field of constants, it follows that $\epsilon(\sigma)$ is a character on G, and hence $\epsilon(\sigma) \equiv 1$, and p and q are invariants.

Lemma 2. *Let the irreducible algebraic group G, having no characters, act on the factorial manifold X. Then the algebra of invariants $I_G[X]$ is also factorial, and all the invertible elements of this algebra are constants.*

Proof. Let $p \in I_G[X]$ and let $p = p_1 \cdots p_k$ be a factorization of p into prime factors in the ring $k[X]$. It is not hard to see that if α and γ are, respectively, a prime and an invertible element of $k[X]$, then for every $\sigma \in G$ the elements α^σ and γ^σ are also, respectively, prime and invertible. Due to the irreducibility of G the element p_i cannot be associated with the element p_j^σ for $i \neq j$ if p_i and p_j are not associated. Hence $p_i = \epsilon(\sigma)\, p_i^\sigma$, where $\epsilon(\sigma)$ is an invertible element. But then, since $\epsilon(\sigma) \in k$ and G has no characters, it follows that $\epsilon(\sigma) \equiv 1$ and the p_i are invariant for every i. The fact that every invertible element of $I_G[X]$ is a constant follows from the definition of a factorial manifold.

Since X is affine, the algebra $k[X]$ is finitely generated. The group G is semisimple; hence one can apply the results of [1], from which it follows that the algebra of invariant regular functions $I_G[X]$ is finitely generated.

Since a semisimple group has no characters, it follows from Lemma 1 that $I_G(X)$ is the field of quotients of $I_G[X]$ and the maximum number of algebraically independent invariants in $I_G[X]$ is equal to the degree of transcendence of the field $I_G(X)$ over the subfield of constants k. Let us denote this number by M.

$I_G[X]$ is finitely generated; hence there exists an affine manifold W such that the algebra $k[W]$ of regular functions on W is isomorphic to $I_G[X]$. The embedding

$$\tau^*: \mathrm{I}_G[X] \to k[X]$$

induces a regular mapping

$$\tau: X \to W$$

(in [1] it is shown that this mapping is surjective).

From the above it follows that the field $k(W)$ of rational functions on W is isomorphic to the field $\mathrm{I}_G(X)$, where the isomorphism is given by the mapping

$$\tau^*: k(W) \to k(X).$$

Definition 4. The manifold W is called the *orbit space* for the action of G on X.

Clearly $\dim W = M$. Since $k[W]$ is isomorphic to the ring $\mathrm{I}_G[X]$, whereas this ring is factorial, has no zero divisors and every invertible element is a constant, we have that W is factorial.

This allows one to construct factorial manifolds other than affine space. Namely, if H is a semisimple irreducible normal subgroup of a semisimple irreducible algebraic group G, then, taking, for example, the linear presentation of G in k^m and constructing the orbit space for the action of H on k^m, we obtain a factorial manifold, in general different from affine space, on which the group G/H acts.

One can give also another example. Let G be an irreducible algebraic group acting on a factorial manifold, where its radical is unipotent and irreducible, and the algebra of regular functions invariant relative to the action of the radical is finitely generated. Then if R is the radical of G, and W is the orbit space of the action of R on the factorial manifold, it follows that W is a factorial manifold on which the semisimple algebraic group G/R acts.

Definition 5. If the algebraic group G acts regularly on the manifold Y, then by the *factor-manifold* we will mean the pair $(Y/G, \pi)$, where Y/G is the manifold and π is the regular mapping of Y onto Y/G, and:

1) $\pi(x) = \pi(y)$ if and only if $O_x = O_y$;

2) the natural embedding of the field of functions $\pi^*: k(Y/G) \to k(Y)$ induces an isomorphism of $k(Y/G)$ and $\mathrm{I}_G(Y)$.

Since τ is a regular mapping, the fibers of this mapping, i.e. the sets of the form $\tau^{-1}(x)$, where $x \in W$, are G-invariant submanifolds of X. Generally speaking, the fibers do not coincide with the orbits, and hence W is not, generally speaking, a factor-manifold of X under the action of G. It is easy to show that W is a factor-manifold if and only if all orbits have the same dimension.

However, we do have

Theorem 2. *Let G be an algebraic group acting on the manifold X. Let there exist a manifold W and a rational mapping $\tau: X \to W$ such that the embedding $\tau^*: k(W) \to k(X)$ induces an isomorphism of the fields $k(W)$ and $\mathrm{I}_G(X)$. Then* *there exist a proper submanifold* $W_1 \subset W$ *and a G-invariant submanifold* $X_1 \subset X$ *such that* $(W - W_1, \tau_1)$ *is a factor-manifold of* $X - X_1$ *under the action of* G *(here* τ_1 *is the restriction of* τ *to* $X - X_1$*).*

Proof. See [2].

Let us recall yet another known result which we will need.

Theorem 3 (Theorem on the fibers of the mapping). *Let there be given a regular mapping* $\tau: Y \longrightarrow Z$ *of a quasi-projective manifold* Y *onto a quasi-projective manifold* Z*. Then there exists in* Z *a nonempty open set* V *such that for any points* $x, y \in V$ *and* $z \in Z - V$ *we have*

$$\dim \tau^{-1}(x) = \dim \tau^{-1}(y) = s < \dim \tau^{-1}(z) \text{ and } \dim Y = \dim Z + s.$$

From Theorems 2 and 3 it follows that $M = \dim X - m_G$, and there exist in W an open set W_1 and in X a G-invariant submanifold X_1 such that, for any point x of this set W_1, $\tau^{-1}(x) - X_1$ is an orbit of maximum dimension. However, the question of whether this orbit is closed in X remains open. It follows that the action of G on X is locally transitive if and only if the field of invariants $I_G(X)$ coincides with the field of constants k.

Theorem 4. *The following conditions are equivalent:*

1. *The action of* G *on* X *is stable.*
2. *There exists a closed orbit of maximum dimension.*
3. $\dim W > \dim \tau(\Pi)$.

Proof. Consider the set $\tau(\Pi)$ and its closure $\overline{\tau(\Pi)} \subset W$. Suppose that $\dim \tau(\Pi) < \dim W$. Then $\tau(\Pi)$ is a proper submanifold of the orbit space W, and the set $W - \tau(\Pi)$ is dense in W (a set is called dense if it contains a nonempty open subset of X).

For $x \in W - \tau(\Pi)$, the fiber $\tau^{-1}(x)$ is a G-invariant submanifold lying in Ω. Let $y \in \tau^{-1}(x)$. Let us show that $O_y = \tau^{-1}(x)$. In fact, it is clear that O_y, $\overline{O}_y$ and $\Gamma = \overline{O}_y - O_y$ lie in $\tau^{-1}(x)$. Since $\dim \Gamma < \dim O_y$, for every $z \in \Gamma$ we have $\dim O_z < \dim O_y$, which is impossible since by construction y and z are points of general position. Thus $\Gamma = \emptyset$ and $\overline{O}_y = O_y$. On the other hand, if $z \in \tau^{-1}(x) - O_y$, then the orbits O_z and O_y give us two closed G-invariant nonintersecting sets. But such sets are separated by the algebra of invariants, and therefore cannot lie in one single fiber (see [1]).

Thus, if $\dim \pi(\Pi) < M$, then for every $x \in W - \tau(\Pi)$ the fiber $\tau^{-1}(x)$ is an orbit which is thus closed in X, and so $\tau^{-1}(W - \overline{\tau(\Pi)})$ is a G-invariant open set containing only orbits closed in X. Hence the action is stable.

Conversely, if there exists at least one closed orbit of maximum dimension then $\dim \tau(\Pi) < M$.

In fact, if Π is empty then there is nothing to show. If Π is nonempty and O_x is a closed orbit of maximum dimension, then clearly $O_x \cap \Pi \neq \emptyset$, and we obtai- two nonintersecting closed G-invariant sets. As has been proved in [1], there exists a G-invariant regular function f which is equal to zero on Π and equal to one on O_x. Since f is nonconstant, the submanifold in the space of orbits W defined by the equation $f = 0$ is proper. It remains only to note that $\tau(\Pi)$ lies in this submanifold.

Finally, let us derive yet another purely algebraic criterion for the existence of a closed orbit of maximum dimension, and hence for the stability of the action.

Theorem 5. *The following properties are equivalent to the stability of the action:*

1. *Every nonzero G-invariant ideal in the algebra $k[X]$ contains a nonzero G-invariant function.*

2. *The ideal $I(\Pi)$ of all regular functions on X equal to zero on Π contains a nonzero G-invariant function.*

Proof. Let I be some G-invariant nonzero ideal in $k[X]$. Let Y be the submanifold in X determined by this ideal. Suppose that there exists a closed orbit of maximum dimension. Then, as proved above, there exists an open set U containing only orbits which are closed in X. There is a point $x \in U - Y$. Due to the G-invariance of U and Y we have $O_x \cap Y = \emptyset$. But O_x and Y are closed G-invariant nonintersecting submanifolds in X, so that there is a regular function $f \in I_G[X]$ which is equal to one on O_x and zero on Y. Hence f lies in the ideal $I(Y)$ of all regular functions on X equal to zero on Y. By a theorem of Hilbert on roots, $f^m \in I$ for some integer $m > 0$. It remains only to note that f^m is also invariant.

Conversely, if hypothesis 1 of Theorem 5 holds, consider the ideal $I(\Pi)$ of all regular functions equal to zero on Π. It is G-invariant (here again it clearly suffices to consider the case when Π is nonempty). Let $f \in I(\Pi) \cap I_G[X]$, $f \not\equiv 0$. Assuming as earlier, that f is a regular function on W, we see that $\tau(\Pi)$ lies in a proper submanifold of the orbit space W defined by the equation $f = 0$. Thus $\dim W > \dim \tau(\Pi)$, and hence the action is stable.

To prove Theorem 1 we use criterion 3 of Theorem 4.

In the sequel we will need the following two lemmas.

Lemma 3. *Let Y be a quasi-projective normal irreducible manifold, $Y \subset \mathbf{P}^m$. Let $\Gamma = \overline{Y} - Y$, where $\overline{Y}$ is the closure of Y in $\mathbf{P}^m$. Let Y be biregularly isomorphic to the affine manifold $Z \subset k^m$. Then Γ is an unmixed manifold whose codimension in $\overline{Y}$ is equal to one (a manifold is called unmixed if all its irreducible components have identical dimension).*

This assertion is a corollary of results of Kleiman [8].

Lemma 4. *The ideal of all regular functions vanishing on an arbitrary unmixed submanifold $Y \subset X$ of codimension one in the factorial manifold X is principal.*

The proof is the same, verbatim, as for a hyperplane in affine space.

Now let the stationary subgroup of $x \in X$ be reductive. It is known that the orbit O_x is biregularly isomorphic to some affine manifold $Y \subset k^m$ precisely in this case (see [6] and [3]). Noting that an orbit is a smooth quasi-projective manifold, and hence normal, and applying Lemma 3, we obtain

Proposition 1. *If the stationary subgroup of the point $x \in X$ is reductive, then either the orbit O_x is closed in X, or the boundary $\Gamma = \overline{O}_x - O_x$ of the orbit of this $x \in X$ is an unmixed manifold and* $\dim \Gamma = \dim O_x - 1$.

Now we can prove Theorem 1.

Proof of Theorem 1. The necessity follows immediately from the result just mentioned, which is proved in [6] and [3].

Let us show the sufficiency. Let U be a G-invariant open set, at each point of which the stationary subgroup is reductive. Then $V = U \cap \Omega$ is also a G-invariant open set, at each point of which the stationary subgroup is reductive. Consider the set $\Delta = X - V$. This is a G-invariant proper submanifold in X, and clearly $\Pi \subset \Delta$.

The manifold Δ, generally speaking, is reducible. Let $\Delta_1, \cdots, \Delta_k$ be its irreducible components, let $\Delta = \bigcup_{i=1}^k \Delta_i$ (we of course assume that Δ is nonempty, since otherwise there is nothing to show, because then Π is empty and $X = \Omega$). Let us show that $\dim \tau(\Delta_i) < M$ for every i. It will follow that $\dim \tau(\Pi) < M$, i. e. the action is stable due to Theorem 4. It suffices to carry out the arguments for Δ_1.

The points in Δ_1 are of two types:

a) those lying in the closure of some orbit O_y, where y is some point of the set V;

b) those not contained in any set of the form $\Gamma = \overline{O}_y - O_y$, where y is any point of the set V.

In turn, the points x of type a) are also of two types:

a1) if y is a point of the set V and $x \in \Gamma = \overline{O}_y - O_y$, where x is a point of type a), then none of the irreducible components of Γ containing x lies in Δ_1;

a2) there is a point $y \in V$ such that $x \in \Gamma = \overline{O}_y - O_y$, where x is a point of type a) and one of the irreducible components of Γ containing x lies in Δ_1.

Let A denote the set of points of type a1), and B the ones of type a2), and C the set $\Delta_1 - (\overline{A} \cup \overline{B})$, where $\overline{A}$ and $\overline{B}$ are the closures of A and B, respectively. Clearly C consists of points of type b). Note also that all these sets are G-invariant.

The set $\overline{A}$ is a proper submanifold in Δ_1. In fact, let $x \in A$. But $x \in \Gamma = \overline{O}_y - O_y$ for a certain point of general position y, and an irreducible component of Γ containing x does not lie in Δ_1 by definition. However, since Γ lies in $\Pi \subset \Delta$ because $\dim \Gamma < \dim \overline{O}_y = m_G$, it follows that an irreducible component of Γ containing x lies in $\bigcup_{i=2}^k \Delta_i$. But then $x \in \bigcup_{i=2}^k \Delta_i$ and $x \in \Delta_1 \cap (\bigcup_{i=2}^k \Delta_i)$. Thus $A \subset \Delta_1 \cap (\bigcup_{i=2}^k \Delta_i)$, so that $\overline{A} \subset \Delta_1 \cap (\bigcup_{i=2}^k \Delta_i)$, and it remains only to note that $\Delta_1 \cap (\bigcup_{i=2}^k \Delta_i)$ is a proper submanifold in Δ_1.

We assert further that if $\dim \tau(\Delta_1) = \dim W$, i. e. $\overline{\tau(\Delta_1)} = W$, then $\overline{B} = \Delta_1$. Suppose that this is not so, and that $\overline{B}$ is a proper submanifold in Δ_1. Then C is a nonempty G-invariant open set in Δ_1. It is clear that $\tau(C)$ is dense in W. Since C

contains points only of type b), the set $\bigcup_{y \in V} (C \cap \overline{O}_y)$ is empty. Using Theorem 2 and also the fact that $\tau(\Delta_1)$ is dense in W, we can say that there exists an open set Θ in the orbit space W which has the following properties:

1. The fibers $\tau^{-1}(x)$, where $x \in \Theta$, are the closures of orbits of maximum dimension lying in V.

2. If $\tau_1 \colon \Delta_1 \to W$ is the restriction of τ to Δ_1, then the fibers of τ_1 lying over points in Θ all have minimum dimension.

Since $\tau_1^{-1}(\Theta)$ is an open set, the intersection $\tau_1^{-1}(\Theta) \cap C$ is nonempty and there exists a point y lying in this intersection. Let $\tau(y) = x$. Since Θ possesses property 1, $\tau^{-1}(x) = \overline{O}_z$, where z is a point in V. Since $y \in \Delta_1$ and $\Delta_1 \cap O_z = \emptyset$, it follows that $y \in \Gamma = \overline{O}_z - O_z$ and we obtain a contradiction with the fact that the set $\bigcup_{y \in V} (C \cap \overline{O}_y)$ is empty.

And thus, if $\overline{\tau(\Delta_1)} = W$, then $\Delta_1 = \overline{B}$. From this we now have a contradiction, which completes the proof of the theorem.

Since B is dense in Δ_1, the set $\tau_1^{-1}(\Theta) \cap B$ is nonempty. Moreover, by a theorem on the fibers of a mapping, the dimension of the fiber $\tau^{-1}(x)$, for $x \in \Theta$, is equal to

$$\dim \Delta_1 - \dim W = \dim \Delta_1 - (\dim X - m_G).$$

Let $y \in \tau_1^{-1}(\Theta) \cap B$ and $z = \tau(y)$. Then $\tau^{-1}(x) = O_u$, where u is a point of general position in X lying in V, and $y \in \Gamma = \overline{O}_u - O_u$. By the choice of the point y, one of the irreducible components of the manifold Γ containing it lies in Δ_1; more exactly, in the fiber $\tau_1^{-1}(z)$.

Since Γ is an unmixed manifold of dimension $m_G - 1$, the minimum dimension of a fiber of the mapping τ_1 is not less than $m_G - 1$. In other words,

$$\dim \Delta_1 - (\dim X - m_G) \geq m_G - 1 \text{ and } \dim \Delta_1 \geq \dim X - 1.$$

Since, clearly, $\dim X > \dim \Delta_1$, we have $\dim \Delta_1 = \dim X - 1$. The last equality means that, by Lemma 4, the ideal of all regular functions on X equal to zero on Δ_1 is principal: $I(\Delta_1) = (f)$. Since Δ_1 is G-invariant because G is irreducible, the ideal (f) is also G-invariant. Hence for every $\sigma \in G$ we have

$$f^\sigma = \epsilon(\sigma) f \text{ and } f = \delta(\sigma) f^\sigma,$$

and it follows that $\epsilon(\sigma) \cdot \delta(\sigma) = 1$, i. e. $\epsilon(\sigma)$ is an invertible element of the algebra $k[X]$. Since X is a factorial manifold, $\epsilon(\sigma) \in k$. But G acts on $k[X]$ as a group of k-automorphisms. Hence $\epsilon(\sigma)$ is a character on G. Due to the semisimplicity of G we have $\epsilon(\sigma) \equiv 1$, and f is thus a nonconstant invariant.

Since Δ_1 is defined by the equation $f = 0$, the set $\tau(\Delta_1)$, and thus also $\overline{\tau(\Delta_1)}$, lies in the proper submanifold of the orbit space W determined by the equation $f = 0$. We have obtained a contradiction to the fact that $\overline{\tau(\Delta_1)} = W$. Theorem 1 is proved.

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