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## INDUCED EXTENDED CALCULUS ON THE QUANTUM PLANE

Ch. Chryssomalakos, P. Schupp, B. Zumino

**Abstract.** The non-commutative differential calculus on quantum groups can be extended by introducing, in analogy with the classical case, inner product operators and Lie derivatives. For the case of  $GL_q(n)$  we show how this extended calculus induces by coaction a similar extended calculus, covariant under  $GL_q(n)$ , on the quantum plane. In this way, inner product operators and Lie derivatives can be introduced on the plane as well. The situation with other quantum groups and quantum spaces is briefly discussed. Explicit formulas are given for the two dimensional quantum plane.

### §1. Introduction

The differential calculus on quantum groups [13, 14] involves functions on the group, differentials, differential forms and derivatives. The basic differential operator  $d$  satisfies the standard properties such as linearity,  $(d1) = 0$ ,  $d^2 = 0$ , and the undeformed Leibniz rule

$$(dfg) = (df)g + (-1)^k f(dg), \quad (1)$$

where  $k$  is the degree of the form  $f$ . As shown explicitly for  $GL_q(n)$  in [9], this calculus can be extended by the introduction of inner product operators  $i_X$  in terms of which Lie derivatives are given by the undeformed formula

$$\mathcal{L}_X = di_X + i_X d, \quad (2)$$

valid for forms of any degree.

The covariant differential calculus on the quantum plane similarly involves functions, differentials, differential forms, and derivatives. The differential operator  $d$  on the plane satisfies again the standard undeformed properties such as (1). It is natural to ask whether the calculus on the plane can also be extended by the introduction of inner product operators and Lie derivatives related to each other by a formula analogous to (2). In this paper we show that this is indeed possible and that the extended calculus on the plane can actually be considered as being induced by the extended calculus on the group  $GL_q(n)$  under which the quantum plane is covariant.

For differentiable manifolds, the introduction of the inner product operator and its relation (2) with the Lie derivative provided a kind of algebraization of the differential

calculus. We find it pleasing that the same undeformed relations (1) and (2) are valid in the more clearly algebraic context of non-commutative calculus.

The extended calculus on the plane can also be established independently, without reference to that on  $GL_q(n)$ . It represents an appropriate combination of a bosonic and a fermionic quantum calculus on the plane [5, 6, 15], the differentials being fermionic and the inner product operators fermionic derivatives with respect to the differentials. The formulas expressing operations on the quantum group in terms of operations on the plane provide then a realization of the former on the plane.

Most of this paper is concerned with the quantum group  $GL_q(n)$  and with the quantum plane covariant under its coaction. In this case, the realization mentioned above is especially simple. For instance, the basic vector fields of the quantum Lie algebra of  $GL_q(n)$  are realized by differential operators on the quantum plane, linear in the coordinates and their corresponding derivatives. For other quantum groups the realization is more complicated and requires quantum pseudodifferential operators, i.e., nonlinear functions of the coordinates and the derivatives. This is shown explicitly for  $SL_q(n)$  in §4. A general procedure for obtaining such realizations, applicable to other quantum groups, is also outlined.

In §2 we collect some well known properties of Hopf algebras, actions and coactions which we shall use later, mostly in order to establish our notation. We also review the extended differential calculus on quantum groups, with special emphasis on  $GL_q(n)$ . In §3 we use these results to work out the extended calculus on the quantum plane. In §4 we explain how, for groups other than  $GL_q(n)$ , the vector fields are realized on the corresponding quantum space by means of quantum pseudodifferential operators. Finally, in the appendix, we give explicitly all commutation relations of the extended calculus for the two-dimensional quantum plane.

## §2. Preliminaries

We collect here, in the interest of self-containment, several definitions and results that we will need later. Detailed treatments of the topics touched upon here can be found in the references.

**2.1. Hopf Algebras, Actions And Coactions.** We start with the definition of a *Hopf algebra* [1, 3, 11]. An associative unital algebra  $\mathcal{A}$  over a field  $k$  is called a Hopf algebra if it possesses a *coproduct*  $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ , a *counit*  $\epsilon : \mathcal{A} \rightarrow k$  and an *antipode*  $S : \mathcal{A} \rightarrow \mathcal{A}$  which obey the relations

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{3}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \tag{4}$$

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \tag{5}$$

$$(\epsilon \otimes \text{id}) \circ \Delta = \text{id}, \tag{6}$$

$$(\text{id} \otimes \epsilon) \circ \Delta = \text{id}, \tag{7}$$

$$(S \otimes \text{id}) \circ \Delta(a) = \epsilon(a)1_{\mathcal{A}}, \tag{8}$$

$$(\text{id} \otimes S) \circ \Delta(a) = \epsilon(a)1_{\mathcal{A}}, \tag{9}$$

$$\Delta(1_{\mathcal{A}}) = 1_{\mathcal{A}} \otimes 1_{\mathcal{A}}, \tag{10}$$

$$\epsilon(1_{\mathcal{A}}) = 1. \tag{11}$$

We will often use the Sweedler notation for the coproduct

$$\Delta(a) \equiv \sum_i a_i^{(1)} \otimes a_i^{(2)} \equiv a_{(1)} \otimes a_{(2)}. \tag{12}$$

Also,

$$(\Delta \otimes \text{id}) \circ \Delta(a) \equiv a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \tag{13}$$

and so on for higher powers of the coproduct.

Two Hopf algebras  $\mathcal{U}$  and  $\mathcal{A}$  are said to be *dually paired* if there exists a non-degenerate inner product  $\langle \cdot, \cdot \rangle : \mathcal{U} \otimes \mathcal{A} \rightarrow k$  which relates the two Hopf structures as follows ( $x, y \in \mathcal{U}, a, b \in \mathcal{A}$ ):

$$\langle x, ab \rangle = \langle x_{(1)}, a \rangle \langle x_{(2)}, b \rangle, \tag{14}$$

$$\langle xy, a \rangle = \langle x, a_{(1)} \rangle \langle y, a_{(2)} \rangle, \tag{15}$$

$$\langle 1_{\mathcal{U}}, a \rangle = \epsilon(a), \tag{16}$$

$$\langle x, 1_{\mathcal{A}} \rangle = \epsilon(x), \tag{17}$$

$$\langle S(x), a \rangle = \langle x, S(a) \rangle. \tag{18}$$

We will assume, as is the case in the applications we consider, that  $S^{-1}$  exists. An algebra that satisfies the Hopf algebra conditions except those involving the antipode is called a *bialgebra*. Given an algebra  $\mathcal{B}$  and a bialgebra  $\mathcal{A}$ , we say “ $\mathcal{A}$  coacts from the right on  $\mathcal{B}$ ” if there exists a *coaction*  $\Delta_{\mathcal{A}} : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$  satisfying ( $b, c \in \mathcal{B}$ ) the relations

$$\Delta_{\mathcal{A}}(bc) = \Delta_{\mathcal{A}}(b)\Delta_{\mathcal{A}}(c), \tag{19}$$

$$(\Delta_{\mathcal{A}} \otimes \text{id}) \circ \Delta_{\mathcal{A}} = (\text{id} \otimes \Delta) \circ \Delta_{\mathcal{A}}, \tag{20}$$

$$(\text{id} \otimes \epsilon) \circ \Delta_{\mathcal{A}}(b) = b, \tag{21}$$

$$\Delta_{\mathcal{A}}(1_{\mathcal{B}}) = 1_{\mathcal{B}} \otimes 1_{\mathcal{A}}. \tag{22}$$

We will use a Sweedler-like notation for coactions

$$\Delta_{\mathcal{A}}(b) \equiv \sum_i b_i^{(1)} \otimes b_i^{(2')} \equiv b^{(1)} \otimes b^{(2')}, \tag{23}$$

where, we remind the reader,  $b^{(1)} \otimes b^{(2')} \in \mathcal{B} \otimes \mathcal{A}$ .

An algebra  $\mathcal{A}$  acts from the left on an algebra  $\mathcal{B}$  if there exists a map  $\triangleright : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}, a \otimes b \mapsto a \triangleright b$ , satisfying ( $a, a' \in \mathcal{A}, b \in \mathcal{B}$ )

$$(aa') \triangleright b = a \triangleright (a' \triangleright b), \tag{24}$$

$$1_{\mathcal{A}} \triangleright b = b. \tag{25}$$

Right coactions of  $\mathcal{A}$  on  $\mathcal{B}$  give rise to actions from the left of  $\mathcal{A}^*$  (the dual of  $\mathcal{A}$ ) on  $\mathcal{B}$  according to ( $a \in \mathcal{A}^*, b \in \mathcal{B}$ )

$$a \triangleright b \equiv b^{(1)} \langle a, b^{(2')} \rangle. \tag{26}$$

Given two dually paired Hopf algebras  $\mathcal{U}$  and  $\mathcal{A}$ , one can construct a new algebra, their *semidirect product*  $\mathcal{A} \rtimes \mathcal{U}$ , in the following way: as a vector space,  $\mathcal{A} \rtimes \mathcal{U}$  is the tensor product of  $\mathcal{A}$  and  $\mathcal{U}$ . The product in  $\mathcal{A} \rtimes \mathcal{U}$  is defined as follows ( $a, b \in \mathcal{A}, x, y \in \mathcal{U}$ ):

$$\begin{aligned} (a \otimes x)(b \otimes y) &= a(x_{(1)} \triangleright b) \otimes x_{(2)}y \\ &\equiv ab_{(1)} \langle x_{(1)}, b_{(2)} \rangle \otimes x_{(2)}y. \end{aligned} \tag{27}$$

**2.2. Extended Calculus On Quantum Groups.** The concepts presented above are fully employed in the description of Quantum Groups [2, 13]. We examine here the case of  $\text{Fun}_q(GL(n))$  [7, 9]. Let  $\mathcal{A}$  be the algebra generated by the unit  $1_{\mathcal{A}}$  and the elements  $A_{ij}$  of an  $n \times n$  matrix  $A$  modulo the relations

$$\hat{R}_{12}A_1A_2 = A_1A_2\hat{R}_{12}. \tag{28}$$

The  $n^2 \times n^2$  matrix  $\hat{R}$  is a solution of the quantum Yang-Baxter equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{29}$$

and satisfies the characteristic equation

$$\hat{R}^2 - \lambda\hat{R} - 1 = 0, \quad \lambda \equiv q - q^{-1}. \tag{30}$$

The explicit form of  $\hat{R}$  is given by  $\hat{R}_{ij,kl} = R_{ji,kl}$  where  $R$  is the  $GL_q(n)$   $R$ -matrix of [7], given by

$$R = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \lambda \sum_{i > j} e_{ij} \otimes e_{ji}, \tag{31}$$

where  $i, j = 1, \dots, n$  and  $e_{ij}$  is the  $n \times n$  matrix with single nonzero element (equal to 1) at  $(i, j)$ . The above algebra can be endowed with a Hopf structure via the definitions

$$\Delta(A_{ij}) = A_{ik} \otimes A_{kj}, \tag{32}$$

$$\epsilon(A_{ij}) = \delta_{ij}, \tag{33}$$

$$S(A_{ij}) = (A^{-1})_{ij}. \tag{34}$$

We now construct  $\mathcal{U}$  so that  $\mathcal{A}$  and  $\mathcal{U}$  are dually paired Hopf algebras.  $\mathcal{U}$  is generated by the unit  $1_{\mathcal{U}}$  and the elements  $X_{ij}$  of the matrix  $X$  whose inner product with the generators of  $\mathcal{A}$  is given by the formulas

$$\langle X_1, A_2 \rangle = \lambda^{-1}(1 - \hat{R}^2)_{12}, \tag{35}$$

$$\langle X, 1_{\mathcal{A}} \rangle = 0 \tag{36}$$

and satisfy the relation

$$\hat{R}_{12}X_2\hat{R}_{12}X_2 - X_2\hat{R}_{12}X_2\hat{R}_{12} = \lambda^{-1}(\hat{R}_{12}^2X_2 - X_2\hat{R}_{12}^2). \tag{37}$$

Notice that for  $GL_q(n)$ , the right hand side of (35) simplifies to  $-\hat{R}_{12}$  due to the characteristic equation (30)—we leave it as is though since this form is valid for other quantum groups as well.

The coproduct of  $X_{ij}$  is of the form

$$\Delta(X_{ij}) = X_{ij} \otimes 1_{\mathcal{U}} + \mathcal{O}_{ij,kl} \otimes X_{kl} \tag{38}$$

(with  $\mathcal{O}_{ij,kl} \in \mathcal{U}$  typically non-linear in the  $X_{ij}$ 's) and gives rise to the following  $X - \mathcal{A}$  commutation relations in  $\mathcal{A} \bowtie \mathcal{U}$ :

$$X_1A_2 = A_2\hat{R}_{12}X_2\hat{R}_{12} + \lambda^{-1}A_2(1 - \hat{R}^2)_{12}. \tag{39}$$

One can now introduce an (undeformed) exterior derivative  $d$  which maps  $k$ -forms to  $k + 1$ -forms (functions being 0-forms) and satisfies the relations

$$d^2 = 0, \tag{40}$$

$$\begin{aligned} (df)g &= (df)g + (-1)^k f(dg) \\ &\equiv df g + (-1)^k f dg, \end{aligned} \tag{41}$$

where  $g$  is any form and  $f$  is a  $k$ -form.<sup>1</sup> The matrix  $\Omega$  of Cartan-Maurer forms is given by [9, 17]

$$\Omega \equiv A^{-1}dA = -dA^{-1}A \tag{42}$$

and satisfies

$$\Omega_1 A_2 = A_2 \hat{R}_{12}^{-1} \Omega_2 \hat{R}_{12}^{-1}, \tag{43}$$

$$\Omega_1 dA_2 = -dA_2 \hat{R}_{12}^{-1} \Omega_2 \hat{R}_{12}. \tag{44}$$

Inner derivations  $i_X$  and Lie derivatives  $\mathcal{L}_X$  complete the construction. They obey, among other, the relations

$$i_{1\mu} = 0, \tag{45}$$

$$\begin{aligned} i_{X_1} A_2 &= ((X_1)_{(1)} \triangleright A_2) i_{(X_1)_{(2)}} \\ &= A_2 \hat{R}_{12} i_{X_2} \hat{R}_{12}, \end{aligned} \tag{46}$$

$$i_{X_1} dA_2 = X_1 \triangleright A_2 - d((X_1)_{(1)} \triangleright A_2) i_{(X_1)_{(2)}}, \tag{47}$$

$$\mathcal{L}_X \equiv i_X d + di_X, \tag{48}$$

$$\mathcal{L}_X d = d\mathcal{L}_X. \tag{49}$$

Using  $\Omega$  instead of  $dA$  and substituting explicit expressions for the inner products that occur in (47) we get

$$\hat{R}_{12} i_{X_2} \hat{R}_{12} \Omega_2 = -\Omega_2 \hat{R}_{12} i_{X_2} \hat{R}_{12} + \lambda^{-1}(1 - \hat{R}^2)_{12}. \tag{50}$$

We undertake now the task of letting the structure described above induce a corresponding one on the quantum plane—this is the subject of the next section.

### §3. Extended Calculus On The Quantum Plane

We work in the following with the (noncommutative) algebra  $\mathcal{P}$  of functions on the quantum plane, generated by the unit  $1_{\mathcal{P}}$  and the coordinates  $x_j, j = 1, \dots, n$ , which satisfy the relations

$$x_2 x_1 = q^{-1} \hat{R}_{12} x_2 x_1 \tag{51}$$

(notice that the above can be obtained from the “standard” quantum plane of [5, 12] by letting  $q \rightarrow q^{-1}$  and remembering that  $\hat{R}_{12}(q^{-1}) = \hat{R}_{21}^{-1}(q)$ ). The algebra  $\mathcal{P}$  admits the right coaction  $\Delta_{\mathcal{A}} : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{A}$  (with  $\mathcal{A} \equiv \text{Fun}_q(GL(n))$ ) defined on the generators  $x_k$  by the formula

$$\Delta_{\mathcal{A}}(x_k) = x_l \otimes (A^{-1})_{kl} \tag{52}$$

( $A$  is a  $GL_q(n)$  matrix) and extended to the whole of  $\mathcal{P}$  multiplicatively - we take of course  $\Delta_{\mathcal{A}}(1_{\mathcal{P}}) = 1_{\mathcal{P}} \otimes 1_{\mathcal{A}}$ . Our next step will be to give commutation relations between elements of  $\mathcal{A} \rtimes \mathcal{U}$  and  $\mathcal{P}$ . To accomplish this, notice that (52) allows us to embed  $\mathcal{P}$  in  $\mathcal{P} \otimes \mathcal{A} \rtimes \mathcal{U}$  (since  $\mathcal{A} \rtimes \mathcal{U}$  contains  $\mathcal{A}$  as a subalgebra) via  $x_i \mapsto x_i^{(1)} \otimes x_i^{(2')}$ . Then, the trivial embedding  $\mathcal{A} \rtimes \mathcal{U} \rightarrow 1_{\mathcal{P}} \otimes \mathcal{A} \rtimes \mathcal{U}$  allows us to compute, for example, commutation relations between elements of  $\mathcal{U}$  and  $\mathcal{P}$  ( $\chi \in \mathcal{U}$ ):

$$\begin{aligned} \chi x_i &\mapsto (1 \otimes \chi) \Delta_{\mathcal{A}}(x_i) \\ &= x_i^{(1)} \otimes (x_i^{(2')})_{(1)} \left\langle \chi_{(1)}, (x_i^{(2')})_{(2)} \right\rangle \chi_{(2)} \end{aligned}$$

---

<sup>1</sup>( $df$ ) is the action of  $d$  on  $f$  (the differential of  $f$ ), denoted by  $df$  (to be read as a single symbol).

$$\begin{aligned}
 &= (x_i^{(1)})^{(1)} \otimes (x_i^{(1)})^{(2')} \langle \chi_{(1)}, x_i^{(2')} \rangle \chi_{(2)} \\
 &= \Delta_{\mathcal{A}}(x_i^{(1)}) \langle \chi_{(1)}, x_i^{(2')} \rangle (1 \otimes \chi_{(2)}) \\
 \Rightarrow \chi x_i &= x_i^{(1)} \langle \chi_{(1)}, x_i^{(2')} \rangle \chi_{(2)}.
 \end{aligned} \tag{53}$$

We will now specify  $\chi$  to be one of the bicovariant generators  $X_{kl}$  introduced in the previous section. Using (38) for their coproduct and remembering that  $i_{1l} = 0$ , we find

$$i_{X_{kl}} x_i = x_i^{(1)} \langle \mathcal{O}_{kr,ls}, x_i^{(2')} \rangle i_{X_{sr}}. \tag{54}$$

To facilitate upcoming computations, we introduce now a simplified notation as follows. We write for the coaction of  $x_i$ ,

$$x_i = (A^{-1})_{ij} \bar{x}_j, \tag{55}$$

where  $\bar{x}_j$  are some “fixed” coordinates on the plane and the elements of  $A$  are taken to commute with them. To compute explicitly the relations implied by (54) we start from the  $i - A$  commutation relations of (47) to get

$$\begin{aligned}
 i_1 A_2 &= A_2 \hat{R}_{12} i_2 \hat{R}_{12} \\
 \Rightarrow A_2^{-1} i_1 &= \hat{R}_{12} i_2 \hat{R}_{12} A_2^{-1}.
 \end{aligned} \tag{56}$$

Multiplying now from the right by  $\bar{x}_2$  (which commutes with  $i_1$  as well) we find

$$x_2 i_1 = \hat{R}_{12} i_2 \hat{R}_{12} x_2. \tag{57}$$

In the same spirit, we introduce the differentials  $dx_i$  via

$$dx_i = d((A^{-1})_{ij}) \bar{x}_j = -\Omega_{ij} x_j. \tag{58}$$

With the help of (43), this implies

$$(dx_2) x_1 = q \hat{R}_{12} x_2 (dx_1), \tag{59}$$

while (44) similarly induces

$$dx_2 dx_1 = -q \hat{R}_{12} dx_2 dx_1. \tag{60}$$

On the other hand, using the general relation (47) we get ( $i_{kl} \equiv i_{X_{kl}}$ )

$$i_{kl} dx_i = x_j \langle X_{kl}, A_{ij}^{-1} \rangle - dx_j \langle \mathcal{O}_{kr,ls}, (A^{-1})_{ij} \rangle i_{rs}. \tag{61}$$

Considering the first term in (61) we would like now to realize the inner derivations  $i_{kl}$  of the quantum group in terms of inner derivations  $i_k$  on the plane via

$$\begin{aligned}
 i_{kl} &\sim x_m \langle X_{kl}, A_{nm}^{-1} \rangle i_n \\
 &\equiv Q_{kn,ml} x_m i_n.
 \end{aligned} \tag{62}$$

Substitution in (61) gives

$$Q_{kr,sl} x_s i_r dx_i = Q_{ki,jl} x_j - Q_{rm,ns} \langle \mathcal{O}_{kl,rs}, (A^{-1})_{ij} \rangle dx_j x_n i_m. \tag{63}$$

We now attempt to extract  $i_k - dx_l$  commutation relations from this equation. To succeed in this, it will be necessary to resort to explicit numerical computation that will parallel the above. We start from (50) and use (58) to get

$$i_2 \hat{R}_{12} (dx_2) = x_2 - \hat{R}_{12}^{-1} (dx_2) i_1 \tag{64}$$

(this is the explicit form of (61)). It is time now for some  $R$ -matrix trickery. Written out explicitly in terms of indices, (64) gives

$$\mathbf{i}_{kk'} R_{k'i, jk''} dx_{k''} + \hat{R}_{ik, i'k'}^{-1} dx_{k'} \mathbf{i}_{i'j} = x_k \delta_{ij}. \quad (65)$$

However, the  $R$ -matrix satisfies the relations

$$\begin{aligned} R_{12}^{t_1} &= D_1 \tilde{R}_{12} D_1^{-1} \quad \text{or} \\ R_{k'i, jk''} &= D_j D_{k'}^{-1} \tilde{R}_{ji, k'k''}, \end{aligned} \quad (66)$$

where

$$D_{ij} \equiv D_i \delta_{ij} = \text{diag}(q^{-2n+1}, q^{-2n+3}, \dots, q^{-1}) \quad (67)$$

and  $\tilde{R} \equiv ((R^{-1})^{t_1})^{-1}$  (with  $t_1$  denoting transposition in the first space). Using this in (65) we get

$$\begin{aligned} D_j D_{k'}^{-1} \tilde{R}_{ji, k'k''} \mathbf{i}_{kk'} dx_{k''} &= x_k \delta_{ij} - \hat{R}_{ik, i'k'}^{-1} dx_{k'} \mathbf{i}_{i'j} \\ \Rightarrow D_p^{-1} \mathbf{i}_{kp} dx_q &= -D_j^{-1} (R^{-1})_{pq, ji}^{t_1} (R^{-1})_{ik, i'k'}^{t_1} dx_{k'} \mathbf{i}_{i'j} \\ &\quad + D_i^{-1} (R^{-1})_{pq, ii}^{t_1} x_k. \end{aligned} \quad (68)$$

A second property of  $R$  comes now to our help,

$$D_i^{-1} R_{iq, pi}^{-1} = \delta_{qp} \quad (69)$$

so that (68) becomes

$$D_p^{-1} \mathbf{i}_{kp} dx_q = -D_j^{-1} (R^{-1})_{pq, ji}^{t_1} (\hat{R}^{-1})_{ik, i'k'} dx_{k'} \mathbf{i}_{i'j} + \delta_{qp} x_k. \quad (70)$$

Before realizing  $\mathbf{i}_{kl}$  according to (62) we extract the inner product that appears in (62) from the  $X - A$  commutation relations (39). Using the characteristic equation for  $\hat{R}$ , it becomes

$$\langle X_{kl}, A_{nm}^{-1} \rangle = \delta_{km} D_{ln}, \quad (71)$$

so that the realization of  $\mathbf{i}_{kl}$  on the plane is given by

$$\mathbf{i}_{kl} \sim D_{ls} x_k \mathbf{i}_s. \quad (72)$$

Substituting this in (70) we find

$$\begin{aligned} x_k \mathbf{i}_p dx_q &= -D_j^{-1} (R^{-1})_{pq, ji}^{t_1} (\hat{R}^{-1})_{ik, i'k'} dx_{k'} x_{i'} D_j \mathbf{i}_j \\ &\quad + x_k \delta_{qp} \end{aligned} \quad (73)$$

or, with the help of (59),

$$x_k (\mathbf{i}_p dx_q + q \hat{R}_{jq, ip}^{-1} dx_i \mathbf{i}_j - \delta_{qp}) = 0. \quad (74)$$

The factor in parentheses supplies us with  $\mathbf{i}_p - dx_q$  commutation relations

$$\mathbf{i}_p dx_q = \delta_{pq} - q \hat{R}_{jq, ip}^{-1} dx_i \mathbf{i}_j. \quad (75)$$

A similar computation, starting from (47), gives

$$\mathbf{i}_p x_q = q^{-1} \hat{R}_{jq, ip}^{-1} x_i \mathbf{i}_j. \quad (76)$$

We wish now to include partial derivatives  $\partial_r$  with respect to the coordinates  $x_r$  as well as Lie derivatives  $\mathcal{L}_s$ . The former we introduce by realizing the vector fields  $X_{kl}$  via

$$\begin{aligned} X_{kl} &\sim \langle X_{kl}, A_{rs}^{-1} \rangle x_s \partial_r \\ &= D_{lr} x_k \partial_r. \end{aligned} \quad (77)$$



Starting from

$$D_p^{-1} X_{kp} x_q = x_k \delta_{qp} + D_j^{-1} (R^{-1})_{jq,pi} (\hat{R}^{-1})_{ik,i'k'} x_{k'} X_{i'j} \tag{78}$$

(easily derivable from the  $X - A$  commutation relations), we obtain

$$\partial_p x_q = \delta_{qp} + q^{-1} \hat{R}_{jq,ip}^{-1} x_i \partial_j \tag{79}$$

and similarly, making use of the characteristic equation for  $\hat{R}$ ,

$$\partial_p dx_q = q \hat{R}_{jq,ip} dx_i \partial_j. \tag{80}$$

On the other hand, (79) and the  $X - X$  commutation relations give

$$\partial_k \partial_l = q^{-1} \hat{R}_{pq,kl} \partial_p \partial_q. \tag{81}$$

An exterior derivative  $d$  on the plane is given by

$$d \equiv dx_i \partial_i; \tag{82}$$

it realizes the action of  $d$  on the plane implied by (58) so that it satisfies relations, analogous to (41),

$$d^2 = 0, \tag{83}$$

$$d\alpha = d\alpha + (-1)^k \alpha d, \tag{84}$$

where  $\alpha$  is a  $k$ -form on the plane; in particular,  $dx_i = dx_i + x_i d$ .

We may now complete the extended calculus structure on the quantum plane by introducing the Lie derivatives  $\mathcal{L}_s$ , in the canonical way, through

$$\mathcal{L}_s \equiv \mathbf{i}_s d + d \mathbf{i}_s. \tag{85}$$

These enter naturally in the realization of  $\mathcal{L}_{kl}$  ( $\equiv \mathcal{L}_{X_{kl}}$ ),

$$\begin{aligned} \mathcal{L}_{kl} &= \mathbf{i}_{kl} d + d \mathbf{i}_{kl} \\ &\sim x_k D_{ls} \mathbf{i}_s d + dx_k D_{ls} \mathbf{i}_s \\ &= x_k D_{ls} \mathbf{i}_s d + dx_k D_{ls} \mathbf{i}_s + x_k D_{ls} d \mathbf{i}_s \\ \Rightarrow \mathcal{L}_{kl} &\sim x_k D_{ls} \mathcal{L}_s + dx_k D_{ls} \mathbf{i}_s. \end{aligned} \tag{86}$$

Their commutation relations with the coordinates  $x_i$  are now easily computed, namely,

$$\begin{aligned} \mathcal{L}_p x_q &= (d \mathbf{i}_p + \mathbf{i}_p d) x_q \\ &= dq^{-1} (\hat{R}^{-1})_{jq,ip} x_i \mathbf{i}_j + \mathbf{i}_p dx_q + \mathbf{i}_p x_q d \\ &= q^{-1} (\hat{R}^{-1})_{jq,ip} (dx_i + x_i d) \mathbf{i}_j + \delta_{pq} - q (\hat{R}^{-1})_{jq,ip} dx_i \mathbf{i}_j \\ &\quad + q^{-1} (\hat{R}^{-1})_{jq,ip} x_i \mathbf{i}_j d \\ \Rightarrow \mathcal{L}_p x_q &= \delta_{pq} + q^{-1} (\hat{R}^{-1})_{jq,ip} x_i \mathcal{L}_j - \lambda (\hat{R}^{-1})_{jq,ip} dx_i \mathbf{i}_j. \end{aligned} \tag{87}$$

Notice the appearance of a term, absent in the classical case, which is bilinear in the differentials and the inner derivations. Proceeding similarly, we find

$$\mathcal{L}_p dx_q = q (\hat{R}^{-1})_{jq,ip} dx_i \mathcal{L}_j. \tag{88}$$

The following, easily verifiable, identities are often useful in computations:

$$d\mathcal{L}_p = \mathcal{L}_p d, \tag{89}$$

$$d\partial_p = q^{-2} \partial_p d, \tag{90}$$

$$\partial_p = q^2 d\mathbf{i}_p + \mathbf{i}_p d. \tag{91}$$

To complete the description of the above scheme, we give below the remaining commutation relations among the generators of the extended calculus:

$$\mathcal{L}_p \partial_q = q^{-1} \hat{R}_{ji,pq} \partial_j \mathcal{L}_i, \tag{92}$$

$$\mathbf{i}_p \partial_q = q \hat{R}_{ji,pq} \partial_j \mathbf{i}_i, \tag{93}$$

$$\mathcal{L}_p \mathcal{L}_q = q^{-1} \hat{R}_{ji,pq} \mathcal{L}_j \mathcal{L}_i, \tag{94}$$

$$\mathcal{L}_p \mathbf{i}_q = q^{-1} \hat{R}_{ji,pq} \mathbf{i}_j \mathcal{L}_i, \tag{95}$$

$$\mathbf{i}_p \mathbf{i}_q = -q \hat{R}_{ji,pq} \mathbf{i}_j \mathbf{i}_i. \tag{96}$$

The explicit form, for  $n = 2$ , of this extended calculus is given in the appendix.

#### §4. Realizations Via Pseudodifferential Operators

We wish now to remark briefly on the case of other quantum groups. We start with  $SL_q(n)$  and follow the approach and conventions of [9]. It will be convenient in the following to introduce the matrix  $Y$  of vector fields [8, 10] on  $GL_q(n)$  through

$$Y = 1 - \lambda X. \tag{97}$$

One can define a determinant-like quantity  $Det Y \equiv q^{2H_0}$ , the precise definition of which can be found in [10, 16], which is homogeneous of degree  $n$  in the  $Y_{ij}$ 's and commutes with them. Vector fields  $V$  on the group manifold of  $SL_q(n)$  can now be defined via

$$1 - \lambda V \equiv Z \equiv q^{-2H_0/n} Y = q^{-2H_0/n} (1 - \lambda X). \tag{98}$$

The determinant of  $Z$ , using the same definition as for that of  $Y$ , is found to be equal to 1 and this restricts the number of generators to  $n^2 - 1$ . Non-homogeneous constraint equations like this, do not allow realizations of the action of the  $V_{ij}$ 's on the plane bilinear in the coordinates and the derivatives. However, (98) offers a realization in terms of pseudodifferential operators. We will need the relation

$$q^{2H_0} A = q^2 A q^{2H_0} \tag{99}$$

to make this precise. Indeed, the above equation implies

$$q^{2H_0} x_i = q^{-2} x_i q^{2H_0}. \tag{100}$$

On the other hand, one easily verifies that

$$(1 - q^{-1} \lambda x \cdot \partial) x_i = q^{-2} x_i (1 - q^{-1} \lambda x \cdot \partial), \tag{101}$$

which permits the realization

$$\begin{aligned} V_{ij} &= \lambda^{-1} (1 - q^{-2H_0/n}) \delta_{ij} + q^{-2H_0/n} X_{ij} \\ &\sim \lambda^{-1} (1 - (1 - q^{-1} \lambda x \cdot \partial)^{-\frac{1}{n}}) \delta_{ij} \\ &\quad + (1 - q^{-1} \lambda x \cdot \partial)^{-\frac{1}{n}} x_i D_{jr} \partial_r. \end{aligned} \tag{102}$$

The above approach clearly relies on the fact that  $V_{ij}$  can be obtained in terms of the unconstrained  $X_{ij}$  which admit simple bilinear realizations. Such a construction is not known for other quantum groups like, for example,  $SO_q(n)$ . Nevertheless, it is not hard to outline a general procedure for obtaining realizations, valid for those groups as well. We will use  $SL_q(2)$  as an illustrative example. In this case, starting from (39), one can

compute the action of the  $V$ 's on monomials in the coordinates, ordered according to some standard ordering. We use the notation

$$V = \begin{pmatrix} v_+ & v_- \\ v_- & v_+ \end{pmatrix}, \quad x_1 \equiv x, \quad x_2 \equiv y, \tag{103}$$

and find, for example,

$$v_- \triangleright x^k y^l = q^{-2-l} [k]_q x^{k-1} y^{l+1}, \tag{104}$$

where  $[k]_q \equiv (1 - q^{2k}) / (1 - q^2)$ . Regard, for the moment, the above equation as referring to classical variables. Then it is easy to realize  $v_-$  in terms of classical coordinates and derivatives,

$$v_- \sim y' \partial'_x q^{-2-L} \frac{[K]_q}{K}, \tag{105}$$

where  $K \equiv x' \partial'_x$ ,  $L \equiv y' \partial'_y$  and the primes are there to remind us that we are dealing with classical quantities. It is well known that quantum derivatives (acting on quantum coordinates) can be expressed in terms of classical pseudodifferential operators (acting on the corresponding classical coordinates) via invertible maps [4]. For example, in the case we are considering

$$\partial_x \sim \partial'_x \frac{[K]_q}{K} q^{-2L}, \quad \partial_y \sim \partial'_y \frac{[L]_q}{L} q^{-K}. \tag{106}$$

Inverting these and substituting in (105) we find, in agreement with (102),

$$v_- \sim q^{-3} (1 - q^{-1} \lambda (x \partial_x + y \partial_y))^{-1/2} y \partial_x. \tag{107}$$

Maps like the above are known for  $SL_q(n)$  and  $SO_q(n)$  [4]. Clearly then, a computation along the same lines can provide realizations for the Lie algebra generators of them as well.

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It is a pleasure to dedicate this paper to Ludwig D. Faddeev

### Appendix A. Extended Calculus On The 2-D Quantum Plane

The  $\hat{R}$ -matrix for  $GL_q(n)$  is as follows:

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \lambda \equiv q - q^{-1}. \tag{108}$$

Using this and the definitions

$$\xi \equiv dx - xd, \quad \eta \equiv dy - yd \tag{109}$$

(with  $d\xi + \xi d = d\eta + \eta d = 0$ ), the general formulas of §3 give (we remind the reader that the calculus presented here differs from that of [12] by the substitution  $q \mapsto q^{-1}$ ):

$$xy = q^{-1}yx,$$

$$x\xi = q^{-2}\xi x,$$

$$x\eta = q^{-1}\eta x - q^{-1}\lambda\xi y,$$

$$y\xi = q^{-1}\xi y,$$

$$y\eta = q^{-2}\eta y,$$

$$\partial_x x = 1 + q^{-2}x\partial_x - q^{-1}\lambda y\partial_y,$$

$$\partial_x y = q^{-1}y\partial_x,$$

$$\partial_y x = q^{-1}x\partial_y,$$

$$\partial_y y = 1 + q^{-2}y\partial_y;$$

$$\mathcal{L}_x x = 1 + q^{-2}x\mathcal{L}_x - q^{-1}\lambda y\mathcal{L}_y - q^{-1}\lambda\xi i_x + \lambda^2\eta i_y,$$

$$\mathcal{L}_x y = q^{-1}y\mathcal{L}_x - \lambda\eta i_x,$$

$$\mathcal{L}_y x = q^{-1}x\mathcal{L}_y - \lambda\xi i_y,$$

$$\mathcal{L}_y y = 1 + q^{-2}y\mathcal{L}_y - q^{-1}\lambda\eta i_y;$$

$$i_x x = q^{-2}x i_x - q^{-1}\lambda y i_y,$$

$$i_x y = q^{-1}y i_x,$$

$$i_y x = q^{-1}x i_y,$$

$$i_y y = q^{-2}y i_y;$$

$$\xi\xi = 0,$$

$$\eta\eta = 0,$$

$$\xi\eta = -q\eta\xi;$$

$$\partial_x \xi = q^2\xi\partial_x,$$

$$\partial_x \eta = q\eta\partial_x,$$

$$\partial_y \xi = q\xi\partial_y,$$

$$\partial_y \eta = q^2\eta\partial_y + q\lambda\xi\partial_x;$$

$$\mathcal{L}_x \xi = \xi\mathcal{L}_x - q\lambda\eta\mathcal{L}_y,$$

$$\mathcal{L}_x \eta = q\eta\mathcal{L}_x,$$

$$\mathcal{L}_y \xi = q\xi\mathcal{L}_y,$$

$$\mathcal{L}_y \eta = \eta\mathcal{L}_y;$$

$$i_x \xi = 1 - \xi i_x + q\lambda\eta i_y,$$

$$i_x \eta = -q\eta i_x,$$

$$i_y \xi = -q\xi i_y,$$

$$i_y \eta = 1 - \eta i_y;$$

$$\partial_x \partial_y = q\partial_y \partial_x;$$

$$\mathcal{L}_x \partial_x = \partial_x \mathcal{L}_x,$$

$$\mathcal{L}_x \partial_y = q^{-1}\partial_y \mathcal{L}_x + q^{-1}\lambda\partial_x \mathcal{L}_y,$$

$$\mathcal{L}_y \partial_x = q^{-1}\partial_x \mathcal{L}_y,$$

$$\mathcal{L}_y \partial_y = \partial_y \mathcal{L}_y;$$

$$i_x \partial_x = q^2\partial_x i_x,$$

$$i_x \partial_y = q\partial_y i_x + q\lambda\partial_x i_y,$$

$$\begin{aligned}
i_y \partial_x &= q \partial_x i_y, \\
i_y \partial_y &= q^2 \partial_y i_y; \\
\mathcal{L}_x \mathcal{L}_y &= q \mathcal{L}_y \mathcal{L}_x; \\
\mathcal{L}_x i_x &= i_x \mathcal{L}_x, \\
\mathcal{L}_x i_y &= q^{-1} i_y \mathcal{L}_x + q^{-1} \lambda i_x \mathcal{L}_y, \\
\mathcal{L}_y i_x &= q^{-1} i_x \mathcal{L}_y, \\
\mathcal{L}_y i_y &= i_y \mathcal{L}_y; \\
i_x i_x &= 0, \\
i_y i_y &= 0, \\
i_x i_y &= -q^{-1} i_y i_x.
\end{aligned}$$

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