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Commuting ordinary differential operators of rank 3 corresponding to an elliptic curve

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We consider a pair (M, L) of commuting operators of orders 9 and 6 connected by the relation $M^2 = 4L^3 - g_2L - g_3$, $g_2^3 - 27g_3^2 \neq 0$; Γ is the elliptic curve $\mu^2 = 4\lambda^3 - g_2\lambda - g_3$; $\Psi_i(x, P;\ x_0)$, $0 \leqslant i \leqslant 2$, $P \in \Gamma$ are common eigenfunctions of M and L for which $\widetilde{\Psi}(x_0, P;\ x_0) = E$, where $\widetilde{\Psi}$ is the Wronskian matrix. According to [1], the functions $\Psi_i(x, P;\ x_0)$ are determined by Γ with the distinguished point $P_0 = \infty$, the "Tyurin parameters" $(\gamma_i, \alpha_i, \beta_i)$, and the scalar functions $(u_0(x), u_1(x))$. In [2] a method of "deformation of the Tyurin parameters" was proposed and the coefficients of the operators in the case of rank 2 (that is, orders 4 and 6) were computed. In the same paper it was pointed out that the problem might be solved completely for rank 3. We show here that this is the case (based as a communication to the author by S.P. Novikov), and we solve the problem of separation and complete classification of commuting pairs with rational coefficients of genus $g = 1$ and rank $l = 3$. For $l = 2$ this problem was solved in [3], where the answer is much simpler.

Using [4] one can prove Lemmas 1 and 2.

***Lemma* 1.** *An operator of order* 6 *occuring in a commuting pair has the form* (1), (2):

$$L = L_6 = \left(\frac{d^3}{dx^3} - u_1\frac{d}{dx} - u_0\right)^2 - 2\varphi_2\frac{d^2}{dx^2} - (2\varphi_1 + 3\varphi_2')\frac{d}{dx} - (2\varphi_0 + 3\varphi_1' + 3\varphi_2''), \tag{1}$$

$$\begin{cases} \varphi_0 = -\alpha_1c_1\wp(\gamma_1) - \alpha_2c_2\wp(\gamma_2) - \alpha_3c_3\wp(\gamma_3), \\ \varphi_1 = -\beta_1c_1\wp(\gamma_1) - \beta_2c_2\wp(\gamma_2) - \beta_3c_3\wp(\gamma_3), \\ \varphi_2 = -c_1\wp(\gamma_1) - c_2\wp(\gamma_2) - c_3\wp(\gamma_3). \end{cases} \tag{2}$$

***Lemma* 2.** *For* $g = 1$, $l = 3$, *the Novikov-Krichever system of equations for "the deformation of the Tyurin parameters" has the form* (3). *Here* $\Phi_{ij} = \Phi(\gamma_i, \gamma_j) = \wp(\gamma_j - \gamma_i) + \wp(\gamma_i) - \wp(\gamma_j)$, *where* $\wp(P)$ *is the Weierstrass function* [5]. *The parameters* α_i *and* β_i *satisfy the additional relations* (4):

$$\begin{cases} \gamma_i' = -c_i, \qquad i = i \pmod 3, \\ \alpha_i' = \alpha_i\beta_i + (\alpha_i - \alpha_{i+1})\gamma_{i+1}'\Phi_{i\,i+1} + (\alpha_{i-1} - \alpha_i)\gamma_{i-1}'\Phi_{i-1\,i} - u_0, \\ \beta_i' = \beta_i^2 - \alpha_i + (\beta_i - \beta_{i+1})\gamma_{i+1}'\Phi_{i\,i+1} + (\beta_{i-1} - \beta_i)\gamma_{i-1}'\Phi_{i-1\,i} - u_1; \end{cases} \tag{3}$$

$$\begin{cases} \alpha_1c_1 + \alpha_2c_2 + \alpha_3c_3 = -1, \\ \beta_1c_1 + \beta_2c_2 + \beta_3c_3 = 0, \\ c_1 + c_2 + c_3 = 0. \end{cases} \tag{4}$$

***Lemma* 3.** *All quantities in* (3) *and* (4) *can be expressed as two-valued functions of* $\gamma_1(x)$ *and* $\gamma_2(x)$.

The determinant $\Delta(x) = \det\begin{Vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 1 & 1 & 1 \end{Vmatrix}$ *satisfies the equation*

$$\Delta^2\gamma_1'\gamma_2'\gamma_3' + 2\Delta\gamma_1'\gamma_2'\gamma_3'[\Phi_{21} + \Phi_{13} + \Phi_{32}] + \Delta(\gamma_1''\gamma_2' - \gamma_2''\gamma_1') + 1 = 0. \tag{5}$$

***Theorem* 1.** *For* α_i *and* β_i *the formulae* (6), (7), *and* (8) *hold, which together with the formulae* (1), (2), *and* (5) *give an explicit form of the operator* $L = L_6$ *of rank* 3 *and genus* 1:

$$\beta_i = \frac{1}{3}\left[\frac{\Delta'}{\Delta} + \Delta(\gamma_{i-1}' - \gamma_{i+1}') + (\gamma_1' - \gamma_2')\Phi_{12} + (\gamma_2' - \gamma_3')\Phi_{23} + (\gamma_3' - \gamma_1)\Phi_{31}\right] \tag{6}$$

$$\alpha_i - \alpha_{i+1} = -\Delta\gamma_{i-1}'' - \beta_{i-1}\gamma_{i-1}'\Delta - 2\Delta\gamma_i'\gamma_{i-1}'\Phi_{i-1\,i} + 2\Delta\gamma_{i+1}'\gamma_{i-1}'\Phi_{i+1\,i-1}, \tag{7}$$

(8) $$\alpha_i \Delta\gamma_i' = (\alpha_{i+1} - \alpha_{i+2})' + \beta_{i+1}(\alpha_i - \alpha_{i+1}) + \beta_{i-1}(\alpha_{i-1} - \alpha_i) + (\alpha_{i+1} - \alpha_{i+2})(\gamma_{i+1}' - \gamma_{i+2}')\Phi_{i+1 i+2} - (\alpha_i - \alpha_{i+1})\gamma_i'\Phi_{i i+1} + (\alpha_{i-1} - \alpha_i)\gamma_i'\Phi_{i-1 i}.$$

***Theorem* 2.** *Let* $B(x)$ *and* $C(x)$ *be rational functions such that* $C(x) \neq -aB(x) \pm \sqrt{(4a^3 - g_2 a - g_3)}$, $a =$ const; *the* $\lambda_i(x)$ *are the roots of* (9). *The function* $\gamma_i(x)$ *correspond to a commuting pair* L_6 *of the form* (1) *and* L_9 *with rational coefficients if and only if* (10) *holds and the roots of the polynomial* (11) *are rational functions*:

(9) $$[B(x)\lambda + C(x)]^2 = 4\lambda^3 - g_2\lambda - g_3,$$

(10) $$\gamma_i(x) = \gamma_0 - \int_\infty^{\lambda_i(x)} \frac{dt}{\sqrt{(4t^3 - g_2 t - g_3)}},$$

(11) $$R^2(\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2\gamma_1'\gamma_2'\gamma_3' + 1 + R(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)[2\gamma_1'\gamma_2'\gamma_3'(\Phi_{21} + \Phi_{13} + \Phi_{32}) + (\gamma_1''\gamma_2' - \gamma_2''\gamma_1')].$$

Example. We consider Γ: $\{\mu^2 = \lambda^3 - \alpha\}$ and $B(x) = 0$. Then $\mu_i(x) = c(x)$, $\lambda^3 = c^2(x) + \alpha$. The condition (11) takes the form $c'(x) = -u^2(x)$, where $u(x)$ is a rational function. Then for the coefficients of the operator (1) we have

(12) $$\begin{cases} \varphi_0(x) = u'(x), \quad \varphi_1(x) = -u(x), \quad \varphi_2(x) = 0, \\ u_0(x) = \dfrac{(u'(x))^3}{u^3(x)} - 2\dfrac{u'(x)\,u''(x)}{u^2(x)} + \dfrac{u'''(x)}{u(x)} - \dfrac{c^2(x) + \alpha}{u^3(x)}, \\ u_1(x) = \dfrac{2u(x)\,u''(x) - (u'(x))^2}{u^2(x)}. \end{cases}$$

Dixmier's example [6] corresponds to $u(x) = -1$, $c(x) = -x$.

Remark. In [7] particular solutions are obtained for the case of the elliptic curve $\mu^2 = 4\lambda^3 - g_2\lambda - g_3$, depending on a single arbitrary function and three constants. These solutions are a slight generalization of the example given above.

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