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A. S. Fokas

INTEGRABILITY AND BEYOND

§1. Introduction

In the last twenty years or so a great number of nonlinear equations has been solved exactly. These equations take the form of partial differential equations (PDE's), of ordinary differential equations (ODE's), of differential difference equations, of difference-difference equations, etc. The method of solution of these equations, which we shall refer to as the **inverse spectral method** (ISM), is based on the association of the given nonlinear equation with a pair of linear equations known as Lax pair [1]. Regarding continuous equations it has been established that: (a) nonlinear PDE's in x, y, t (like the Kadomtsev-Petviashvili equation), (b) nonlinear PDE's in x, t (like the Korteweg - de Vries equation), (c) nonlinear ODE's in t with explicit t- dependence (like the Painlevé transcendents) and (d) nonlinear ODE's in t (like the Arnold top) lead to investigating inverse problems for: (a) linear PDE's in x, y, (b) linear ODE's in x, (c) linear ODE's in the spectral parameter λ, and (d) λ-matrices, i.e., linear algebraic equations depending on the spectral parameter λ. These inverse problems can be solved in terms of local [2] and nonlocal [3] Riemann–Hilbert problems, of $\bar{\partial}$ problems [4, 5] and of Riemann - theta functions [6–11].

In this note we shall summarize the following recent developments:

(a) There exist functional equations which can also be solved using an appropriate version of the inverse spectral method.

(b) The combination of the inverse spectral method and general PDE techniques can be used to linearize initial-boundary value problems of the nonlinear Schrödinger (NLS) and other integrable equations.

§2. Inverse spectral method, algebraic geometry and functional equations

We give a summary of the results obtained in [12].

Consider the functional equations

$$\frac{q(x,y)q(y,z)}{q(x,z)} - r(x,y) + r(z,y) = \sum_{l=1}^{m} \gamma^l(y) p^l(x,y), \qquad (2.1)$$

for the unknown functions $q(x,y)$, $r(x,y)$ and $p^1(x,y),\dots,p^m(x,y)$, where the functional parameters $\gamma^1(x),\dots,\gamma^m(x)$ are given. We notice that, in the case $m=1$, $\gamma(y)=1$, $q(x,y)=q(x-y)$, $r(x,y)=r(x-y)=-r(y-x)$, $p(x,y)=p(x-y)$, this equation reduces to the functional equation

$$\frac{q(x)q(y)}{q(x+y)}-r(x)-r(y)=p(x+y). \tag{2.2}$$

This equation was introduced and solved by Calogero and Bruschi in connection with integrable many-body problems [13].

It is possible to solve equation (2.1) by considering the following discrete approximation

$$\frac{q_{ij}q_{jk}}{q_{ik}}=r_{ij}-r_{kj}+\sum_{l=1}^{m}p_{ik}^{l}\gamma_j^l,\quad i,j,k \text{ distinct}, \tag{2.3}$$

in which the n-dimensional vectors γ_j^l, $l=1,\dots,m$ are given and the $n\times n$ matrices q_{ij}, r_{ij} and p_{ij}^l are the unknowns. Any solution of the functional equations (2.1) determines a solution of the algebraic equations (2.3) by choosing arbitrary interpolation points $x_1,\dots,x_n$ and putting

$$\gamma_j^l=\gamma^l(x_j),\quad q_{ij}=q(x_i,x_j),\quad r_{ij}=r(x_i,x_j),\quad p_{ij}^l=p^l(x_i,x_j).$$

In the simplest case $m=0$ the algebraic equation (2.3) becomes

$$\frac{q_{ij}q_{jk}}{q_{ik}}=r_{ij}-r_{kj}. \tag{2.4a}$$

We show that the general solution of this algebraic system has the form

$$q_{ij}=\frac{v_i\lambda_j}{v_j(z_i-z_j)},\quad r_{ij}=\frac{\lambda_j}{z_i-z_j}+f_j, \tag{2.4b}$$

where $\lambda_j\neq 0$, $v_j\neq 0$, z_j, f_j, $j=1,\dots,n$ are arbitrary constants ($z_i\neq z_j$ for $i\neq j$).

For the next case ($m=1$)

$$\frac{q_{ij}q_{jk}}{q_{ik}}=r_{ij}-r_{kj}+p_{ik}\gamma_j, \tag{2.5a}$$

we show that the general solution is parametrized by elliptic functions

$$q_{ij}=\gamma_j\frac{v_i}{v_j}\frac{\sigma(z_j-z_i-z_0)}{\sigma(z_0)\sigma(z_j-z_i)},\quad r_{ij}=\gamma_j\zeta(z_i-z_j), \tag{2.5b}$$

where z_j, z_0, v_j, $j = 1, \ldots, n$ are arbitrary constants, and σ, ζ are the well-known Weierstrass elliptic functions.

The method of solution of the algebraic equations (2.3) is based on the fact that the equations (2.3) are equivalent to the commutativity conditions

$$[L_{A_i}, L_{A_j}] = 0, \quad i,j = 1, \ldots, n-m \tag{2.6}$$

for a family of $n-m$ linear λ-matrices of the form

$$\begin{aligned} L_{A_i} &= \lambda A_i + U_{A_i}, \quad A_i = \operatorname{diag}(a_1^i, \ldots, a_n^i), \\ U_{A_i} &= (u_{A_i pq})_{1 \le p,q \le n}, \quad i = 1, \ldots, n-m. \end{aligned} \tag{2.7}$$

This allows one to solve (2.3) in terms of theta functions using an appropriate version of the ISM.

We remark that the commutativity conditions for λ-matrices can be used to characterize the stationary points of integrable ODE's (e.g., integrable tops). In particular, the equations (2.4)-(2.7) provide parametrizations of the stationary points of the Euler–Arnold–Manakov top. For $n-m=2$, it was proposed in [14] to consider the commutativity equation (2.6) (which is a system of purely algebraic equations for the entries of the matrices U_{A_1}, U_{A_2}) as "algebraic integrable equations." We note that equations (2.3) are not the first algebraic equations to be solved by the ISM. In fact, Krichever [15] applied ISM to classify two-component solutions of the Yang–Baxter equation. However, because the structure of the underlying linear equations associated with (2.3) and with the Yang–Baxter equation is different, the method used here is substantially different from the one used in [15].

The results regarding equations (2.4) imply that

$$q(x,y) = \frac{v(x)\lambda(y)}{v(y)(z(x)-z(y))}, \quad r(x,y) = \frac{\lambda(y)}{z(x)-z(y)} + f(y) \tag{2.8}$$

solve the functional equation (2.1) for $\gamma^l(x) \equiv 0$, where $\lambda(x)$, $v(x)$, $z(x)$, and $f(x)$ are arbitrary functions. Similarly

$$\begin{aligned} q(x,y) &= \gamma(y)\frac{v(x)}{v(y)}\frac{\sigma(z(y)-z(x)-z_0)}{\sigma(z_0)\sigma(z(y)-z(x))}, \\ r(x,y) &= \gamma(y)\zeta(z(x)-z(y)) \end{aligned} \tag{2.9}$$

solve the functional equation (2.1) for $m=1$, where $z(x)$ and $v(x)$ are also arbitrary functions.

§3. THE LINEARIZATION OF AN INITIAL-BOUNDARY VALUE PROBLEM FOR THE NLS

We give a summary of the results obtained in [17].

For integrable equations, a method exists for solving the initial-value problem on the infinite line for decaying initial data. For evolution equations in one spatial variable, this method reduces the solution of the Cauchy problem to the formulation of a Riemann-Hilbert (RH) problem. This RH problem is essentially determined by the x-part of the associated Lax pair; the t-part of the Lax pair plays only an auxiliary role. In the case of the nonlinear Schrödinger (NLS) equation, the relevant RH problem is formulated in the complex k-plane with a jump on $\mathrm{Im}(k) = 0$, and is depicted in Fig. 3.1.

$$\begin{array}{c} + \\ \hline - \end{array} \qquad \begin{pmatrix} 1 & -b(k)e^{-\theta} \\ \lambda\overline{b(k)}e^{\theta} & 1-\lambda|b(k)|^2 \end{pmatrix}.$$

Fig. 3.1

The function $b(k)$ is called scattering data and can be computed in terms of initial data. The x, t dependence of the RH of Fig. 3.1 enters through $\theta(x,t) = 2i(kx + 2k^2t)$.

We have developed a new method for studying initial-boundary value problems on the half-infinite line for decaying initial and boundary data. This formalism also reduces the solution of the initial-boundary value problem to the solution of a single RH problem. However, for the formulation of this RH problem, both the $x-$ and the $t-$parts of the Lax pair play an important role. Actually, it is the t-part which determines where, in the complex k-plane, the jumps occur. In the case of the NLS the jumps occur on $\mathrm{Im}(k^2) = 0$, which is a reflection of the fact that the t-part of the Lax pair contains k^2, which in turn is a consequence of the fact that the NLS involves a second derivative in x. We have found that the analysis of this problem, in addition to techniques from exact integrability, it also requires the essential use of more general PDE techniques. In the cases studied so far, the exact methods could be used to establish existence of global solutions as well as to study the properties of these solutions. In contrast, in the problem studied here exact methods are used only to study the properties of solutions. We believe that this hybrid between exact methods and general PDE techniques, can provide a powerful approach for analyzing problems of mathematical and physical significance. We expect that a wide class of problems can be analyzed in a similar manner.

We consider the NLS equation

$$iq_t + q_{xx} - 2\lambda|q|^2 q = 0, \quad x, t\epsilon[0,\infty); \quad \lambda = \pm 1, \tag{3.1}$$

where $q(x,0) = u(x)$ and $q(0,t) = v(t)$ are given. We assume that

$$\begin{aligned} u(x)\epsilon H_2(\mathbb{R}^+), v(t)\epsilon C_2(\mathbb{R}^+), u(0) = v(0), xu(x) \\ \text{and } \; x^2u(x)\epsilon L_2(\mathbb{R}^+), v(t)\epsilon L_1 \cap L_2(\mathbb{R}^+), \end{aligned} \tag{3.2}$$

$$v'(t), \quad tv(t), \quad tv'(t), \quad tv''(t)\epsilon L_1(\mathbb{R}^+),$$

where H_2 denotes that a function and its first two derivatives belong to L_2, C_2 denotes that a function is twice differentiable, and prime denotes differentiation.

The cases $\lambda = 1$ and $\lambda = -1$ are usually referred to as the defocusing and focusing cases, respectively. Equation (3.1) is the compatibility condition of the following Lax pair for the 2×2 matrix $w(x,t,k)$ [2],

$$w_x + ik\sigma_3 w = Qw, \tag{3.3a}$$

$$w_t + Uw = wC(t), \quad U(x,t,k) \doteq 2ik^2\sigma_3 + i\lambda|q|^2\sigma_3 - 2kQ + iQ_x\sigma_3, \tag{3.3b}$$

where $\sigma_3 = \text{diag}(1,-1)$, the 2×2 matrix $C(t)$ is an arbitrary function of t, and $Q(x,t)$ is an off-diagonal matrix with 12 and 21 entries given by q and $\lambda\bar{q}$, respectively.

We have developed the following linearization scheme for the solution of the initial-boundary value problem of the NLS. Given $q(x,0)$ construct s_1^+ and s_2^+ by $s_1^+(k) \doteq \psi_1(0,k)$, $s_2^+(k) \doteq \psi_2(0,k)$, where $(\psi_1(x,k), \psi_2(x,k))^T$ is the solution of (3.3a) with $q(x,t)$ replaced by $q(x,0)$, satisfying the boundary condition $\lim_{x\to\infty}[(\psi_1,\psi_2)^T \exp(-ikx)] = (0,1)^T$. Define $b(k)$ by $b = s_1^+/\overline{s_2^+}$. Let $c(k)$, $k\epsilon\mathbb{R}^- \cup i\mathbb{R}^+$ be the boundary value of a function meromorphic for $k\epsilon II$ (I, II, III, IV denote the first, second, third, and fourth quadrants of the complex k-plane), with poles at the zeros of $s_2^+(k)$ and at the points $\{k_j\}_1^N$, $k_j\epsilon II$ which are assumed to be different than the zeros of $s_2^+(k)$ (generic case); let c_j denote the residues of $c(k)$ at k_j; also $c(k) \to 0$ as $k \to \infty$. Having $s_1^+(k)$, $s_2^+(k)$ and $c(k)$ solve a RH problem for a 2×2 meromorphic function $\widehat{Z}_p$ with possible poles only at $\{k_j\}_1^N$. This RH problem is depicted in Fig. 3.2. Finally determine $q(x,t)$ by $q(x,t) = 2i\lim_{k\to\infty}(k\widehat{Z}_p(x,t,k))_{12}$, $k\epsilon I$, where the subscript 12 denotes the 12 components of the matrix $\widehat{Z}_p$.

The points $\{k_j\}_1^N$ (which have the meaning of the discrete spectrum of equation (3.3b) evaluated at $x=0$ and supplemented with the boundary condition $w_1 s_2^+(k) - w_2 s_1^+(k) = 0$ at $t=0$) give rise to solitons which always move away from the boundary.

$$
\begin{array}{ccccc}
 & & \begin{pmatrix} 1 & 0 \\ c(k)e^{\theta} & 1 \end{pmatrix} & & \\
 & II & & I & \\
\begin{pmatrix} 1 & -\lambda\bar{c}e^{-\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -be^{-\theta} \\ \lambda\bar{b}e^{\theta} & 1-\lambda|b|^{2-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ ce^{\theta} & 1 \end{pmatrix} & & \begin{matrix} - & + \\ + & - \end{matrix} & & \begin{pmatrix} 1 & -b(k)e^{-\theta} \\ \lambda\overline{b(k)}e^{\theta} & 1-\lambda|b(k)|^2 \end{pmatrix} \\
 & III & & IV & \\
 & & \begin{pmatrix} 1 & -\lambda\overline{c(\bar{k})}e^{-\theta} \\ 0 & 1 \end{pmatrix} & &
\end{array}
$$

Fig. 3.2

The RH problem associated with the initial-boundary value problem of NLS. The x, t dependence enters only through $\theta(x,t) = 2i(kx + 2k^2t)$.

Unfortunately, although we have a complete characterization of the analytic properties of $c(k)$, we have not found an effective way of computing $c(k)$ in terms of $q(0,t)$ and $q(x,0)$. (For more details see the discussion below). In spite of this fact we can give an effective description of the long time behavior ($\lambda = -1$):

$$
\begin{aligned}
q(x,t) &= -2\eta_j \frac{\exp[-2i\xi_j x - 4i(\xi_j^2 - \eta_j^2)t - i\varphi_j]}{\cosh[2\eta_j(x + 4\xi_j t) - \Delta_j]} + 0(t^{-\frac{1}{2}}), \\
t &\to \infty, \quad -\frac{x}{4t} = \xi_j + 0(\frac{1}{t}), \quad j = 1, ..., N,
\end{aligned}
\tag{3.4}
$$

where

$$
\eta_j = \mathrm{Im}(k_j), \quad \xi_j = \mathrm{Re}(k_j), \tag{3.5a}
$$

$$
\begin{aligned}
\varphi_j = -\frac{\pi}{2} + \arg c_j + \sum_{l=1_{l\neq j}}^{N} [\mathrm{sign}(\xi_l - \xi_j) - 1] \arg(\frac{k_j - k_l}{k_j - \bar{k}_l}) + \\
+\frac{1}{\pi} \int_{-\infty}^{-x/4t} \frac{\log[1 + |b(\mu) + \overline{c(\mu)}|^2]}{(\mu - \xi_j)^2 + \eta_j^2} (\mu - \xi_j) d\mu,
\end{aligned}
\tag{3.5b}
$$

$$\Delta_j = -\log 2\eta_j + \log|c_j| + \sum_{l=1_{l\neq j}}^{N} [\text{sign}(\xi_l - \xi_j) - 1] \log \left| \frac{k_j - k_l}{k_j - \bar{k}_l} \right| -$$

$$- \frac{\eta_j}{\pi} \int_{-\infty}^{-x/4t} \frac{\log[1 + |b(\mu) + \overline{c(\mu)}|^2]}{(\mu - \xi_j)^2 + \eta_j^2} d\mu. \tag{3.5c}$$

All $k_j \epsilon II$, thus all $\xi_j < 0$ and the solitons move away from the boundary. The summation terms in the above equations describe the interaction among solitons, while the integration terms describe the interaction between solitons and the dispersive part.

Discussion. The linearization scheme we have developed can be summarized as follows: Given $q(x, 0)$, construct $b(k)$. Then, if $c(k)$ is **any** suitably decaying function meromorphic for $k \epsilon II$, the solution of the RH problem of Fig. 3.2 generates the solution $q(x, t)$ corresponding to initial data $q(x, 0)$ and **some** boundary data $q(0, t)$. The main limitation of our result is that for given $q(x, 0)$ and $q(0, t)$ we cannot construct $c(k)$ by solving a linear problem. Nevertheless, we claim that for **any** given $q(x, 0)$ and $q(0, t)$ satisfying (3.2), the corresponding function $c(k)$ exists; in other words, the RH problem of Fig. 3.2 solves the initial-boundary value problem (3.1) for **general** initial-boundary data.

The RH problem of Fig. 3.2 is quite natural. Comparing the RH problems of Fig. 3.1 and Fig. 3.3 we see that the jumps for $k \epsilon \mathbb{R}^+$ are identical. The jump for $k \epsilon i\mathbb{R}^+$ can not have a nonzero entry in the 12 position since $e^{-\theta}$ is unbounded for $k \epsilon i\mathbb{R}^+$. The jump for $k \epsilon i\mathbb{R}^-$ follows by symmetry considerations. Finally the jump for $k \epsilon \mathbb{R}^-$ follows from the cyclic condition that the product of the jump matrices equals unity (this is a reflection of continuity at $k = 0$). The fact that $c(k)$ has analytic continuation for $k \epsilon II$ can also be easily understood. At $t = 0$, the RH problem of Fig. 3.2 must be reduced to the one that defines $q(x, 0)$. At $t = 0$, the term e^{θ} has analytic continuation in II. Thus the jumps along the imaginary axis can be mapped to a jump on the negative real axis. In this way, at $t = 0$ one finds the RH problem of Fig. 3.1 with θ replaced by $2ikx$. This RH problem corresponds precisely to $q(x, 0)$.

The fact that $q(x, 0)$, $q(0, t)$ and $c(k)$ are related in a nonlinear way is a reflection of the fact that $q_x(0, t)$ depends nonlinearly on $q(x, 0)$ and $q(0, t)$. To appreciate this we first recall the solution of the linearized problem

$$iq_t + q_{xx} = 0, \quad x, t \epsilon [0, \infty), \tag{3.6}$$

where $q(0,t)$ and $q(x,0)$ are given and decaying for large t and large x. This problem can be solved by the sine transform. However, in order to draw comparisons with the nonlinear problem we shall use a Fourier transform

$$\widehat{q}(k,t)=\int_0^\infty dx e^{ikx}q(x,t). \tag{3.7}$$

The evolution of the Fourier data $\widehat{q}(k,t)$ is given by

$$\widehat{q}_t+ik^2\widehat{q}=iq_x(0,t)+kq(0,t). \tag{3.8}$$

In equation (3.8), $q(0,t)$ is known but $q_x(0,t)$ is unknown (the sine transform is precisely used in order to eliminate $q_x(0,t)$). This apparently ominus situation can be bypassed by using the fact that the solution $\widehat{q}(k,t)$ of (3.8) is analytic in the upper half of the k complex plane. It turns out that this requirement implies

$$\widehat{q}(k,0)=-\int_0^\infty dt e^{ik^2t}(iq_x(0,t)+kq(0,t)). \tag{3.9}$$

Given $q(x,0)$ and $q(0,t)$, and using the substitution $k=e^{i\pi/4}\sqrt{\rho}$, $\rho>0$, equation (3.9) yields $q_x(0,t)$. It is important to notice that if $q_x(0,t)$ and $q(0,t)$ are arbitrary functions, then the r.h.s. of equation (3.9) will be analytic for $k\epsilon I\cup III$. However, in order for $q_x(0,t)$ and $q(0,t)$ to be the boundary values of the solution of equation (3.6), it is necessary and sufficient that the r.h.s. of equation (3.9) has analytic continuation across the positive imaginary k axis ($\widehat{q}(k,0)$ is analytic for $k\epsilon I\cup II$).

We now discuss the nonlinear problem. Let $(\widehat{\psi}_1(t,k),\widehat{\psi}_2(t,k))^T$ be the solution of the vector equation

$$\widehat{\psi}_t+(2ik^2\sigma_3+i\lambda|q(0,t)|^2\sigma_3-2kQ(0,t)+iQ_x(0,t)\sigma_3)\widehat{\psi}=0, \tag{3.10}$$

satisfying the boundary condition $\lim_{t\to\infty}[(\widehat{\psi}_1,\widehat{\psi}_2)\exp(-2ik^2t)=(0,1)^T$. Let $r(k)\doteqdot\widehat{\psi}_1(0,k)/\widehat{\psi}_2(0,k)$. It turns out that for arbitrary decaying functions $q(0,t)$ and $q_x(0,t)$, $r(k)$ is a meromorphic function for $k\epsilon I\cup III$ and $r(k)\to 0$ as $k\to\infty$. However, if $q(0,t)$ and $q_x(0,t)$ are the boundary values of the NLS then in addition $r(k)$ satisfies

$$r(k)=\frac{s_1^+(k)}{s_2^+(k)},\quad k\epsilon I, \tag{3.11}$$

where s_1^+ and s_2^+ are determined from $q(x,0)$ and are analytic for $k \epsilon I \cup II$. Equation (3.11) is the analogue of equation (3.9). It shows that although the relationship between $q(x,0)$, $q(0,t)$, and $q_x(0,t)$ is highly nonlinear, its reflection in the k-plane (scattering space) is rather simple: $r(k)$ has analytic continuation across the positive imaginary k axis.

There exists an invertible correspondence between the "potential" $\{q(0,t), q_x(0,t)\}$ and the scattering data $r(k)$: Given $q(0,t)$ and $q_x(0,t)$, equation (3.10) implies $r(k)$. Conversely given a meromorphic function $r(k)$, one can find $q(0,t)$ and $q_x(0,t)$ by solving a RH problem with the jump $r(k)$ for $k^2 \epsilon \mathbb{R}$. This provides an effective way for deriving pairs of functions $q(0,t)$ and $q_x(0,t)$ **compatible** with a given $q(x,0)$. Indeed, given $q(x,0)$ one first computes $r(k)$ for $k \epsilon I$ from equation (3.11). Let $r(k)$, $k \epsilon III$ be any suitably decaying meromorphic function. Then the solution of the above RH problem yields $q(0,t)$ and $q_x(0,t)$.

Unfortunately, given $q(0,t)$ and $q(x,0)$ we cannot compute $r(k)$ for $k \epsilon III$ by solving a linear problem. This is a consequence of the fact that now we have a "mixed" problem where one gives "half" the potential, i.e. $q(0,t)$ and "half" the scattering data, i.e. $r(k)$, for $k \epsilon I$. It turns out that this problem can be formulated as a **nonlinear** RH problem and will be discussed elsewhere.

The study of the large t behavior of $q(x,t)$ reduces to the study of the large t behavior of the RH problem of Fig. 3.2. Because the x,t dependence of this RH problem is rather simple, it is possible to give an effective asymptotic description of $q(x,t)$ as $t \to \infty$.

It was mentioned earlier that the analysis of equation (3.1) requires an essential use of general PDE techniques. This follows from the fact that in order to study the map between $\{q(0,t), q_x(0,t)\}$ and $r(k)$ one needs apriori estimates for $q_x(0,t)$. The uniqueness and existence of a global solution for the NLS on the quarter plane is established in [18]. This result makes fundamental use of certain equations which are the analogues of the first three conserved quantities. Therefore, this theory uses L_2 estimates. However, the methods of exact integrability are based on L_1 estimates. It was therefore crucial for us to extend the results of [18] from L_2 to L_1. This poses significant technical difficulties which are discussed in [17] and [19].

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Department of Mathematics
and Computer Science and
Institute for Nonlinear Studies
Clarkson University

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