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Clusters of algebra

by

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By a cluster of algebras of a given species Ω we shall understand a class $\underline{\underline{V}}$ that consists of all Ω-algebras satisfying a set Σ of (possibly infinitely long) sentences of the form

$$(I) \qquad \forall\, x_1,\ldots,x_n[(w_1 = w_1' \vee \ldots \vee w_\ell = w_\ell' \quad \ldots) \vee (v_1 \neq v_1' \vee \ldots \vee v_p \neq v_p' \vee \ldots)] ,$$

where w_ℓ , w_ℓ' , v_p , v_p' are Ω-words in (a finite number n of variables) $x_1,\ldots,x_n$. The part $(w_1 = w_1' \vee \ldots \vee w_\ell = w_\ell' \vee \ldots)$ of the sentence (I) is called its positive part and the part $(v_1 \neq v_1' \vee \ldots \vee v_p \neq v_p' \vee \ldots)$ is called its negative part. If positive part of every sentence in Σ is finite then the cluster defined by Σ is called positively elementary. A negatively elementary cluster is defined similarly. An elementary cluster is often called a universal class.

In this paper we give a categorical characterization of clusters and prove that a positively (negatively) elementary cluster is closed with respect to inverse (direct) limits of its inverse (direct) systems [1].

For every class $\underline{\underline{V}}$ of Ω-algebras we write $\underline{\underline{K}}(\underline{\underline{V}})$ for the categories of algebras of $\underline{\underline{V}}$ and homomorphisms between them. Write $\underline{\underline{K}}(\Omega)$ for the category of all Ω-algebras and Ω-homomorphisms.

A direct (inverse) system in an arbitrary category is called mono or epi according as all of its morphisms are monomorphisms or epimorphisms.

Theorem 1. A class $\underline{\underline{V}}$ of Ω-algebras is a cluster if and only if:

(1.1) For every monomorphism $\underline{A}_1 \longrightarrow \underline{A}_2$ in $\underline{\underline{K}}(\Omega)$ if $\underline{A}_2 \in \underline{\underline{K}}(\underline{\underline{V}})$ then $\underline{A}_1 \in \underline{\underline{K}}(\underline{\underline{V}})$.

(1.2) Direct limits in $\underline{\underline{K}}(\Omega)$ of mono direct systems in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$.

Proof. Conditions (1.1) and (1.2) are equivalent to saying that $\underline{\underline{V}}$ is abstract, hereditary and local. When $\underline{\underline{V}}$ is a cluster $\underline{\underline{V}}$ is obviously abstract, hereditary and local. This proves one part of the theorem. Now assume that $\underline{\underline{V}}$ satisfies (1.1) and (1.2). Let $\Sigma(\underline{\underline{V}})$ be the set of all sentences of the form (I) that hold in every algebra of $\underline{\underline{V}}$. Let $\underline{A}$ be an Ω-algebra satisfying $\Sigma(\underline{\underline{V}})$. We have to show that $\underline{A} \in \underline{\underline{V}}$. By locality we can assume that $\underline{A}$ is finitely generated, say, by $\{a_1,\ldots,a_n\}$. Let σ be the sentence obtained by taking first the negation of the conjunction of 'relations' of the form

$w(a_1,\ldots,a_n) = w'(a_1,\ldots,a_n)$ or $w(a_1,\ldots,a_n) \neq w'(a_1,\ldots,a_n)$

that hold in $\underline{A}$, then replacing a_1 by $x_1,\ldots,a_n$ by x_n and finally prefixing $\forall x_1,\ldots,x_n$; here w , w' denote arbitrary Ω-words. Clearly σ is of the form (I) and does not hold in $\underline{A}$. This means that $\sigma \notin \Sigma(\underline{\underline{V}})$. Hence there is an algebra $\underline{A}' \in \underline{\underline{V}}$ which fails to satisfy σ . Let $a_1',\ldots,a_n' \in \underline{A}'$ be such that σ is false when $x_1 = a_1',\ldots,x_n = a_n'$. Then the subalgebra $\underline{A}''$ of $\underline{A}'$ generated by $a_1',\ldots,a_n'$ is isomorphic to $\underline{A}$. Thus $\underline{A}$ is embeddable in an algebra of $\underline{\underline{V}}$ and by abstractness and heredity we conclude that $A \in \underline{\underline{V}}$. This completes the proof of Theorem 1.

The above proof is an application of the well-known 'method of diagrams'.

Theorem 2. Let $\underline{\underline{V}}$ be a negatively elementary cluster. Then direct limits in $\underline{\underline{K}}(\Omega)$ of direct systems in $\underline{\underline{K}}(\underline{\underline{V}})$ are in $\underline{\underline{K}}(\underline{\underline{V}})$.

Proof. Let I be a set upward directed by the partial ordering $\leqq$. Let $\{f_{ji} : \underline{A}_i \longrightarrow \underline{A}_j\}_{i \leqq j \text{ in } I}$ be a direct system of $\underline{\underline{K}}(\underline{\underline{V}})$. We shall assume that Ω does not contain nullary operators, the proof being essentially the same in the general case. To construct the direct

limit of $\{f_{ji}\}$ we first construct the sum $\underline{S}$ of $\{\underline{A}_i\}_{i\in I}$ as follows. We can assume without loss of generality that the carriers A_i of the algebras $\underline{A}_i$ are disjoint. Let

$$S_o = \bigcup_{i\in I} A_i ,$$

$$S_1 = S_o \cup \{s_1,\dots,s_{h(\omega)}\omega;\ \omega \in \Omega,\ s_1,\dots,s_{h(\omega)} \in S_o ,$$

but $s_1,\dots,s_{h(\omega)}$ do not belong to the same $A_i\}$

$$S_{n+1} = \{s_1,\dots,s_{h(\omega)}\omega\ ;\ \omega \in \Omega\ ,\ s_1,\dots,s_{h(\omega)} \in S_n\} \cup S_n ,$$

$$S = \bigcup_{n=o}^{\infty} S_n ,$$

where for every $\omega \in \Omega$ we write $h(\omega)$ for the arity of ω and $s_1\dots s_{h(\omega)}\omega$ is just a sequence. Clearly S is a set of Ω-words in S_o We take S to be the carrier of $\underline{S}$. For every $\omega \in \Omega$ we now define an operation $\underline{S}(\omega)$ over S. Let $\underline{A}_i(\omega)$ be the operation over A_i determined by ω in $\underline{A}_i$. We define $\underline{S}(\omega)$ by:

$$\langle s_1,\dots,s_{h(\omega)}\rangle \underline{S}(\omega) = \langle s_1,\dots,s_{h(\omega)}\rangle \underline{A}_i(\omega) \ , \quad \text{if } i \in I ,$$

$$s_1,\dots,s_{h(\omega)} \in A_i ,$$

$$= s_1\dots s_{h(\omega)}\omega , \quad \text{otherwise.}$$

The algebra $\underline{S}$ so defined is the sum of $\{\underline{A}_i\}_{i\in I}$ in $\underline{\underline{K}}(\Omega)$. Over $\underline{S}$ we define a congruence κ as follows. For every set r of ordered pairs write $\mathrm{sym}(r)$ for the set $\{\langle s,t\rangle\ ;\ \langle t,s\rangle \text{ or } \langle s,t\rangle \in r\}$ and $\mathrm{tran}(r)$ for the set of ordered pairs $\langle s,t\rangle$ such that there exists a sequence $s_o,\dots,s_m$ satisfying $s_o = s$, $s_m = t$ and $\langle s_o,s_1\rangle,\dots,\langle s_{m-1},s_m\rangle \in r$. Let

$$r_o = \{\langle a_i, f_{ji}(a_i)\rangle\ ;\ i \in I\ ,\ a_i \in A_i\} ,$$

$$\kappa_o = \mathrm{tran}(\mathrm{sym}(r_o))$$

$$r_{n+1} = \kappa_n \cup \{\langle \langle s_1,\dots,s_{h(\omega)}\rangle \underline{S}(\omega),\ \langle s_1',\dots,s_{h(\omega)}'\rangle \underline{S}(\omega)\rangle\ ;$$

$$\omega \in \Omega,\ \langle s_1,s_1'\rangle,\dots,\langle s_{h(\omega)},s_{h(\omega)}'\rangle \in \kappa_n\} ,$$

$$\kappa_{n+1} = \mathrm{tran}(\mathrm{sym}(r_{n+1})) ,$$

$$\kappa = \bigcup_{n=0}^{\infty} \kappa_n .$$

Thus κ is the least congruence over $\underline{S}$ containing r_o. It is not difficult to verify that $\underline{S}/\kappa$ is the direct limit of the direct system $\{f_{ji}\}_{i \leq j}$. We have to prove that $\underline{S}/\kappa \in \underline{\underline{K}}(\underline{\underline{V}})$. Before doing this we need to learn few facts about κ.

(2.1) The relation κ_o consists of ordered pairs $\langle a_i, a_j \rangle$, $a_i \in A_i$, $a_j \in A_j$ such that there exist a $k \in I$ satisfying $k \geq i,j$ and $f_{ki}(a_i) = f_{kj}(a_j)$.

Let $f_{ki}(a_i) = f_{kj}(a_j) = a_k$, say. Then $\langle a_i, a_k \rangle$, $\langle a_j, a_k \rangle \in r_o$, $\langle a_i, a_k \rangle$, $\langle a_k, a_j \rangle \in \mathrm{sym}(r_o)$, $\langle a_i, a_j \rangle \in \mathrm{tran}(\mathrm{sym}(r_o)) = \kappa_o$. Conversely let $\langle a_i, a_j \rangle \in \kappa_o$. Then if $\langle a_i, a_j \rangle \in \mathrm{sym}(r_o)$ we can take $k = j$ or i according as $\langle a_i, a_j \rangle \in r_o$ or $\langle a_j, a_i \rangle \in r_o$. In the case when $\langle a_i, a_j \rangle \notin \mathrm{sym}(r_o)$ there exist $a_{i_o} \in A_{i_o}, \ldots, a_{i_m} \in A_{i_m}$ such that $a_{i_o} = a_i$, $a_{i_m} = a_j$ and $\langle a_{i_o}, a_{i_1} \rangle; \ldots, \langle a_{i_{m-1}}, a_{i_m} \rangle \in \mathrm{sym}(r_o)$. We can find $k_1, \ldots, k_m$ such that $f_{k_1 k_o}(a_{i_o}) = f_{k_1 i_1}(a_{i_1}), \ldots$ $\ldots, f_{k_{m-1} i_{m-1}}(a_{i_{m-1}}) = f_{k_m i_m}(a_{i_m})$. Find $k \in I$ such that $k \geq k_1, \ldots, k_m$. Then $f_{ki}(a_i) = f_{kj}(a_j)$ because $f_{kk_1} f_{k_1 i_o} = f_{ki_o}, \ldots,$ $f_{kk_m} f_{k_m i_m} = f_{ki_m}$. This proves (2.1).

(2.2) For every non-negative integer n the relation κ_n is a 'congruence over the partial subalgebra S_n' of $\underline{S}$, that is, κ_n is an equivalence relation over S_n and if $\langle s_1, s_1' \rangle, \ldots, \langle s_{h(\omega)}, s_{h(\omega)}' \rangle \in \kappa_n$ and $s = \langle s_1, \ldots, s_{h(\omega)} \rangle \underline{S}(\omega) \in S_n$, $s' = \langle s_1', \ldots, s_{h(\omega)}' \rangle \underline{S}(\omega) \in S_n$ then $\langle s, s' \rangle \in \kappa_n$.

We use induction to prove (2.2). It follows directly from (2.1) and the definition of $\underline{S}$ that κ_o is a congruence over S_o. Assume (2.2) for all $n_1 \leq n$. The symmetry and reflexivity of κ_n implies the symmetry and reflexivity of r_{n+1} as may be directly seen by the definition of r_{n+1}. This shows that $\kappa_{n+1} = \mathrm{tran}(r_{n+1})$ and that κ_{n+1} is a reflexive relation over S_{n+1}. Transitivity of κ_{n+1} is immediate. For symmetry let $\langle s_o, s_{m+1} \rangle \in \kappa_{n+1}$ so that there exist $s_1, \ldots, s_m$ such that $\langle s_o, s_1 \rangle, \ldots, \langle s_m, s_{m+1} \rangle = r_{n+1}$. But then

$\langle s_{m+1}, s_m \rangle, \ldots, \langle s_1, s_o \rangle \in r_{n+1}$, since r_{n+1} is symmetric. Hence $\langle s_{m+1}, s_o \rangle \in \text{tran}(r_{n+1}) = \kappa_{n+1}$. This proves that κ_{n+1} is an equivalence relation over S_{n+1} .

Before proving the congruence property for κ_{n+1} we first show that the restriction of κ_{n+1} to S_n is κ_n . Let $\langle s, s' \rangle \in \kappa_{n+1}$, $s, s' \in S_n$. Let $s_o, \ldots, s_m$ be one of the shortest sequences such that $s_o = s$, $s_m = s'$, $\langle s_o, s_1 \rangle, \ldots, \langle s_{m-1}, s_m \rangle \in r_{n+1}$. Suppose $s_\ell \notin S_n$ for some ℓ , $1 < \ell < m$. Then we can write

$$s_{\ell-1} = \langle s_{\ell-1,1}, \ldots, s_{\ell-1,h(\omega)} \rangle \underline{S}(\omega) ,$$

$$s_\ell = \langle s_{\ell,1}, \ldots, s_{\ell,h(\omega)} \rangle \underline{S}(\omega) = \langle s'_{\ell,1}, \ldots, s'_{\ell,h(\omega')} \rangle \underline{S}(\omega') ,$$

$$s_{\ell+1} = \langle s_{\ell+1,1}, \ldots, s_{\ell+1,h(\omega')} \rangle \underline{S}(\omega') ,$$

where ω , $\omega' \in \Omega$ and $\langle s_{\ell-1,p}, s_{\ell,p} \rangle \in \kappa_n$ for $1 \leq p \leq h(\omega)$ and $\langle s'_{\ell,p}, s_{\ell+1,p} \rangle \in \kappa_n$ for $1 \leq p \leq h(\omega')$. This is immediate from the definition of r_{n+1} since $s_\ell \notin S_n$ and $\langle s_{\ell-1}, s_\ell \rangle$, $\langle s_\ell, s_{\ell+1} \rangle \in r_{n+1}$. Now, by the definition of $\mathfrak{S}$, we find that $s_\ell \notin S_o$ and

$$s_\ell = \langle s_{\ell,1}, \ldots, s_{\ell,h(\omega)} \rangle \underline{S}(\omega) = \langle s'_{\ell,1}, \ldots, s'_{\ell,h(\omega')} \rangle \underline{S}(\omega')$$

together imply that $\omega = \omega'$, $s_{\ell,p} = s'_{\ell,p}$ for $1 \leq p \leq h(\omega)$. Thus $s_{\ell-1} = \langle s_{\ell-1,1}, \ldots, s_{\ell-1,h(\omega)} \rangle \underline{S}(\omega)$, $s_{\ell+1} = \langle s_{\ell+1,1}, \ldots, s_{\ell+1,h(\omega)} \rangle \underline{S}(\omega)$, and $\langle s_{\ell-1,p}, s_{\ell+1,p} \rangle \in \kappa_n$ for $1 \leq p \leq h(\omega)$. Hence $\langle s_{\ell-1}, s_{\ell+1} \rangle \in r_{n+1}$. This contradicts the assumption that $s_o, \ldots, s_m$ is one of the shortest sequences of a certain type. Hence $s_\ell \notin S_n$ is false and $s_o, \ldots, s_m \in S_n$. Again $\langle s_o, s_1 \rangle \in r_{n+1} - \kappa_n$ implies that

$$s_o = \langle s_{o,1}, \ldots, s_{o,h(\omega)} \rangle \underline{S}(\omega)$$

$$s_1 = \langle s_{1,1}, \ldots, s_{1,h(\omega)} \rangle \underline{S}(\omega)$$

for some $\omega \in \Omega$ and $\langle s_{o,1}, s_{1,1} \rangle, \ldots, \langle s_{o,h(\omega)}, s_{1,h(\omega)} \rangle \in \kappa_n$. Since s_o, s_1 have been shown to be in S_n we have by the congruence property of κ_n that $\langle s_o, s_1 \rangle \in \kappa_n$. Hence and similarly $\langle s_o, s_1 \rangle, \ldots, \langle s_{m-1}, s_m \rangle \in \kappa_n$ which implies that $\langle s_o, s_m \rangle = \langle s, s' \rangle \in \kappa_n$, as was to be proved.

We conclude the proof of (2.2) by showing that if $s,s' \in S_{n+1}$, $s = \langle s_1,\ldots,s_{h(\omega)}\rangle \underline{S}(\omega)$, $s' = \langle s'_1,\ldots,s'_{h(\omega)}\rangle \underline{S}(\omega)$ and $\langle s_1,s'_1\rangle,\ldots,\langle s_{h(\omega)},s'_{h(\omega)}\rangle \in \kappa_{n+1}$ then $\langle s,s'\rangle \in \kappa_{n+1}$. Clearly $s_1,\ldots,s_{h(\omega)},s'_1,\ldots,s'_{h(\omega)} \in S_n$ and, by what we proved in the last paragraph, $\langle s_1,s'_1\rangle,\ldots,\langle s_{h(\omega)},s'_{h(\omega)}\rangle \in \kappa_n$. Hence $\langle s,s'\rangle \in r_{n+1} \subseteq \kappa_{n+1}$ and (2.2) is proved.

(2.3) The restriction of κ to S_n is κ_n for every non-negative n.

Let $\langle s,s'\rangle \in \kappa$, $s,s' \in S_n$. Let m be the least integer such that $\langle s,s'\rangle \in \kappa_m$. If $m > n$ then $s,s' \in S_{m-1}$ and since by the proof of (2.2) the restriction of κ_m to S_{m-1} is κ_{m-1} we conclude that $\langle s,s'\rangle \in \kappa_{m-1}$. This contraction proves that $m \leq n$ and that $\langle s,s'\rangle \in \kappa_n$. Hence (2.3).

(2.4) For every finite subset $\{s_1,\ldots,s_m\}$ of S there exist i, $a_1,\ldots,a_m$ such that $i \in I$, $a_1,\ldots,a_m \in A_i$ and $\langle s_1,a_1\rangle,\ldots,\langle s_m,a_m\rangle \in \kappa$.

Again we use induction. Let $s_1,\ldots,s_m \in S_o$ so that $s_1 \in A_{i_1},\ldots,s_m \in A_{i_m}$ for some $i_1,\ldots,i_m \in I$. Find $i \in I$ such that $i \geq i_1,\ldots,i_m$. Let $a_1 = f_{ii_1}(s_1),\ldots,a_m = f_{ii_m}(s_m)$. Then $\langle s_1,a_1\rangle,\ldots,\langle s_m,a_m\rangle \in \kappa$ and $a_1,\ldots,a_m \in A_i$. Now assume (2.4) for all finite subsets of S_n and let $s_1,\ldots,s_m \in S_{n+1}$. If $s_1 \in S_n$ we can find $i_1 \in I$, $a'_1 \in A_i$, such that $\langle s_1,a'_1\rangle \in \kappa$. Otherwise we can write $s_1 = \langle s_{1,1},\ldots,s_{1,h(\omega)}\rangle \underline{S}(\omega)$ for some $\omega \in \Omega$ and $s_{1,1},\ldots,s_{1,h(\omega)} \in S_n$. By the induction hypothesis $\langle s_{1,1},a_{1,1}\rangle,\ldots,\langle s_{1,h(\omega)},a_{1,h(\omega)}\rangle \in \kappa$ for some $i_1 \in I$, $a_{1,1},\ldots,a_{1,h(\omega)} \in A_{i_1}$. Since κ is a congruence $\langle s_1,a'_1\rangle \in \kappa$ where $a'_1 = \langle a_{1,1},\ldots,a_{1,h(\omega)}\rangle \underline{S}(\omega) \in A_{i_1}$. Similarly $\langle s_2,a'_2\rangle,\ldots,\langle s_m,a'_m\rangle \in \kappa$ for some $a'_2 \in A_{i_2},\ldots,a'_m \in A_{i_m}$, $i_2,\ldots,i_m \in I$. Find i such that $i \geq i_1,\ldots,i_m$ and let $a_1 = f_{ii_1}(a'_1),\ldots,a_m = f_{ii_m}(a_m)$. Then $\langle s_1,a_1\rangle,\ldots,\langle s_m,a_m\rangle \in \kappa$, $a_1,\ldots,a_m \in A_i$. This completes the proof of (2.4).

We are now ready to prove that $\underline{S}/\kappa \in \underline{\underline{K}}(\underline{\underline{V}})$,

Let the sentence

(II) $\forall x_1,\ldots,x_n[(w_1 = w'_1 \vee \ldots \vee w_\ell = w' \vee \ldots) \vee (v_1 \neq v'_1 \vee \ldots \vee v_m = v'_m)]$

hold in A_i for all $i \in I$. We show that (II) also holds in $\underline{S}/\kappa$. For this we need to show that for all $s_1,\dots,s_n \in S$ either $\langle w_\ell(s_1,\dots,s_n), w'_\ell(s_1,\dots,s_n)\rangle \in \kappa$ for some $\ell \geq 1$ or $\langle v_p(s_1,\dots,s_n), v'_p(s_1,\dots,s_n)\rangle \notin \kappa$ for some p, $1 \leq p \leq m$. Suppose otherwise and let $s_1,\dots,s_n \in S$ be such that $\langle w_\ell(s_1,\dots,s_n), w'_\ell(s_1,\dots,s_n)\rangle \notin \kappa$ for all $\ell \geq 1$ and $\langle v_p(s_1,\dots,s_n), v'_p(s_1,\dots,s_n)\rangle \in \kappa$ for all p, $1 \leq p \leq m$. Using (2.4) find i, $a_1,\dots,a_n$ such that $i \in I$, $a_1,\dots,a_n \in A_i$ and $\langle s_1,a_1\rangle,\dots,\langle s_n,a_n\rangle \in \kappa$. Then $\langle w_\ell(a_1,\dots,a_n), w'_\ell(a_1,\dots,a_n)\rangle \notin \kappa$, $\langle v_p(a_1,\dots,a_n), v'_p(a_1,\dots,a_n)\rangle \in \kappa$ for all ℓ, p, $\ell \geq 1$, $1 \leq p \leq m$. Write $b_p = v_p(a_1,\dots,a_n)$, $b'_p = v'_p(a_1,\dots,a_n)$ so that $\langle b_p,b'_p\rangle \in \kappa$ and $b_p,b'_p \in A_i$ where $1 \leq p \leq m$. By (2.3) and (2.1) we can find $j_p \in I$ such that $f_{j_p i}(b_p) = f_{j_p i}(b'_p)$. Find $j \in I$ such that $j \geq j_p$ for all p, $1 \leq p \leq m$. Then $f_{ji}(b_p) = f_{ji}(b'_p)$, $1 \leq p \leq m$. Let $c_1 = f_{ji}(a_1),\dots,c_n = f_{ji}(a_n)$. Then

$$v_p(c_1,\dots,c_n) = f_{ji}(b_p) = f_{ji}(b'_p) = v'_p(c_1,\dots,c_n), \quad 1 \leq p \leq m$$

$$w_\ell(c_1,\dots,c_n) \neq w'_\ell(c_1,\dots,c_n), \quad \ell \geq 1.$$

Since $c_1,\dots,c_n \in A_j$ this contradicts the assumption that (II) holds in $\underline{A}_j$. This proves that every sentence of the form (II) that holds in every algebra of $\underline{\underline{K}}(\underline{\underline{V}})$ also holds in $\underline{S}/\kappa$. Hence $\underline{S}/\kappa \in \underline{\underline{K}}(\underline{\underline{V}})$ and the theorem is proved.

Theorem 2 was stated for some universal classes of algebras without detailed proof in [2] and [3]. For the simpler case of epi direct systems the theorem is stated in [4].

The construction of inverse limits is dual to that of direct limits and is well-known. Using this construction it is not hard to prove

Theorem 3. Let $\underline{\underline{V}}$ be a positively elementary cluster. Then $\underline{\underline{K}}(\underline{\underline{V}})$ is closed with respect to inverse limits of its inverse systems.

The last two theorems show that elementary clusters (universal classes) are abstract, hereditary and closed with respect to direct and inverse limits. The question arises whether these properties characterise elementary clusters and if not what other necessary or sufficient or necessary and sufficient conditions on $\underline{\underline{K}}(\underline{\underline{V}})$ will ensure that $\underline{\underline{V}}$ is an

elementary cluster.

One result in this direction is given by Theorem 5 of [4] which states that $\underline{\underline{V}}$ is an elementary cluster if $\underline{\underline{V}}$ is abstract, hereditary and closed with respect to products and direct limits of epi direct systems.

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