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ESTIMATING POLYNOMIALS AND ENTIRE FUNCTIONS BY USING THEIR LOGARITHMIC SUMS OVER COMPLEX SEQUENCES

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Introduction

Let us consider any sequence of distinct complex numbers ζ_n going out towards ∞. Usually the index n will be taken to run through the values $1, 2, \ldots$, and then we will suppose the ζ_n enumerated so as to have $|\zeta_1| \leqslant |\zeta_2| \leqslant \ldots$; in that event we will always assume that $|\zeta_1| \geqslant 1$. Sometimes we will use an index n running through $\mathbb{Z}$ or $\mathbb{Z} \sim \{0\}$, and then we will require that $\zeta_n \to \infty$ for $n \to \pm\infty$.

Given a sequence $\{\zeta_n\}$, we would like to know whether or not there are weights $w_n \geqslant 0$ (depending, *a priori*, on $\{\zeta_n\}$) such that $\sum_n w_n(1 + \log^+ |\zeta_n|) < \infty$, and for which the *polynomials* $p(z)$ *with* $\sum_n w_n \log^+ |p(\zeta_n)| \leqslant \eta$, a sufficiently small quantity > 0, *form a normal family* in the complex plane.

For certain *real* sequences $\{\zeta_n\}$, such weights w_n are known to exist. When, for instance, $\zeta_n = n$ with n running through $\mathbb{Z}$, we may take $w_n = 1/(n^2+1)$. See p. 520 in [1] and, for a more accessible proof of this (difficult) result, Pedersen's article [2], or [3] and [4]. In a strong sense, the result is best possible; see [1, p. 446].

A similar statement is valid for two-sided real sequences $\{\zeta_n\}$ symmetric about 0 and having at least one member in each real interval of some given length L. This follows from Remark 2 on p. 518 of [1] and the discussion on pp. 519–522 therein. Pedersen obtains the same result more easily in [5] and, in [6], he shows that the *symmetry condition* can be dropped. A new, simpler, and more encompassing version of the work in [6]—based, in part, on some material from [8]—is given by him in [7]. All the results just mentioned hold for entire functions $p(z)$ of sufficiently small exponential type as well as for polynomials, and those given in [4] and [7] include specifications of the precise upper limit of that type.

In the present article we obtain extensions of the above material to fairly general complex sequences $\{\zeta_n\}$. It will turn out that one can always extract a *subsequence* $\{\underline{\zeta}_n\}$ from $\{\zeta_n\}$ such that the polynomials $p(z)$ with

$$\sum_n \frac{1}{n^2+1} \log^+ |p(\underline{\zeta}_n)|$$

small enough from a normal family. In many instances, this criterion cannot be sensibly improved.

These generalizations will be deduced from the original result in [1] (and in [2]) by carrying out what amounts to a simple *change of variables*. On that account, the development to follow cannot be looked on as very deep. The results themselves, however, seem to be of interest and to lend themselves to application. I also found them to be a bit unexpected.

What surprised me was the ubiquitous rôle of the weight $1/(n^2+1)$. Without a prior inkling that this was (always) the one to use, I would have been hard put to know where to start, and it was only after I had an idea of what kind of results to expect that I could begin looking for proofs. I think, therefore, that it is not amiss to begin this paper with an account of how I came to guess that the weight $1/(n^2+1)$ might be general, even though that information will play no rôle in the deduction of the actual results to be given later on.

The weight in question was suggested by the considerations outlined in the next section. Those involve various computations, one of which—the estimate of a certain measure—may be of independent interest. But the details of these computations do make that section somewhat long, and for that reason we advise the reader who wishes only to learn of the main results in this paper that he or she can just as well proceed directly to §2.

1. The weight $1/(n^2+1)$ appearing in the initial result quoted from [1] was suggested by the evaluation of a certain harmonic measure. Taking $0 < \rho < 1/2$, we consider the domain

$$\mathcal{D}_\rho = \mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} [n-\rho,\ n+\rho]$$

and the harmonic measure $\omega_{\mathcal{D}_\rho}(E_n, z)$ of the boundary component $E_n = [n-\rho,\ n+\rho]$ in $\mathcal{D}_\rho$. It can be proven that

$$\omega_{\mathcal{D}_\rho}(E_n, i) \leqslant \frac{C_\rho}{n^2+1} \tag{1}$$

with a constant C_ρ depending on ρ; see [1, problem 26, pp. 545–547] and, for a more general result, [1, p. 394]. With Harnack's inequality, this relation yields one of the form

$$\omega_{\mathcal{D}_\rho}(E_n, z) \leqslant \frac{K(z,\rho)}{n^2+1}, \quad z \in \mathcal{D}_\rho,$$

from which we get

$$\log|p(z)| \leqslant K(z,\rho) \sum_{n=-\infty}^{\infty} \frac{\max\{\log^+|p(t)|;\ n-\rho \leqslant t \leqslant n+\rho\}}{n^2+1} \tag{2}$$

for *polynomials* $p(z)$. That makes us think that *perhaps* we could then also have

$$\log|p(z)| \leqslant K(z) \sum_{n=-\infty}^{\infty} \frac{\log^+|p(n)|}{n^2+1} \tag{3}$$

for $z \in \mathbb{C}$.

Unfortunately, the best value of $C_\rho = K(i,\rho)$ tends to ∞ like $\log(1/\rho)$ when $\rho \to 0$ ([1, pp. 546–547]), and nobody has yet been able to deduce anything like (3) from (2). Nevertheless, a suitable variant of (3) *is true* provided that the sum figuring on its right side is *small enough*, and that is the result stated on p. 520 of [1].

Around 1980, Benedicks looked at harmonic measure in domains

$$\mathcal{D}'_\rho = \mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} [|n|^q \operatorname{sgn} n - \rho,\ |n|^q \operatorname{sgn} n + \rho]$$

where $q > 1$ and $0 < \rho < 1/2$. For the harmonic measure $\omega_{\mathcal{D}'_\rho}(E'_n, z)$ of $E'_n = [|n|^q \operatorname{sgn} n - \rho,\ |n|^q \operatorname{sgn} n + \rho]$ in $\mathcal{D}'_\rho$ he found that

$$\omega_{\mathcal{D}'_\rho}(E'_n, i) \leqslant \frac{C(q,\rho)}{|n|^{1+q}+1}$$

(see [1, problem 19, p. 443 *et seq.*]). The quantity on the left actually lies between *two constant multiples* (depending of course on q and ρ) of $1/(|n|^{1+q}+1)$; for the lower bound one may consult §3 of [9].

In view of (4) one finds it natural to expect, by analogy with the (merely suggestive) passage from (1) and (2) to (3), that a relation like

$$\log|p(z)| \leqslant K_1(z) \sum_{n=-\infty}^{\infty} \frac{\log^+|p(|n|^q \operatorname{sgn} n)|}{|n|^{q+1}+1} \tag{5}$$

should hold for polynomials $p(z)$, at least when the sum on the right is small. *But that is false*! Pedersen ([10]) has observed that when $g > 1$, the sequence of polynomials

$$P_N(z) = (1-z^2)^{k_N} \prod_{n=1}^{N} \left(1 - \frac{z^2}{n^{2q}}\right)$$

satisfies

$$\sum_{n=-\infty}^{\infty} \frac{\log^+|P_N(|n|^q \operatorname{sgn} n)|}{|n|^{q+1}+1} \to_N 0$$

and at the same time

$$P_N(i) \to_N \infty,$$

provided that the integers k_N tend *sufficiently slowly* to ∞. This is rather easy to verify: the above sum actually runs over the $n \in \mathbb{Z}$ with $|n| > N$ and can hence be estimated in terms of an integral. By this example I was made to realize that, in the event that $q > 1$, the unions $\cup_{n=-\infty}^{\infty}[|n|^q \operatorname{sgn} n - \rho\, |n|^q \operatorname{sgn} n + \rho]$ do not approximate the set $\{|n|^q \operatorname{sgn} n;\ n \in \mathbb{Z}\}$ rapidly enough for the question at hand when $\rho \to 0$.

If $q = 1$, however, those unions seem to do that. Here it comes to mind that in that case, any one of the unions is precisely the subset of $\mathbb{R}$ on which $|\sin \pi x| \leqslant \sin \pi\rho$, with

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

the entire function "naturally" associated with $\mathbb{Z}$. And that in turn gives us the idea to take

$$f(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^{2q}}\right) \tag{6}$$

"naturally" associated with the set $\{|n|^q \operatorname{sgn} n;\ n \in \mathbb{Z}\}$ when $q > 1$, and to look at the intervals E_n'' on $\mathbb{R}$ where $|f(x)| \leqslant \pi\rho$, say (with $\rho > 0$ and small). *Maybe* the harmonic measures $\omega_{\mathcal{D}_\rho''}(E_n'', z)$ of those intervals in the complement

$$\mathcal{D}_\rho'' = \mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} E_n''$$

will suggest to us a weight which, when used instead of $1/(|n|^{q+1} + 1)$ in (5), will yield a valid result.

To follow up this hunch—which will indeed prove fruitful—we have to describe the intervals E_n'' and then evaluate the harmonic measures $\omega_{\mathcal{D}_\rho''}(E_n'', z)$. That involves a fair amount of computation so, in order to make the work easier, we first carry out some simplifications in our set-up, without—so we hope—changing the situation essentially.

In the first simplification, we replace the *real* intervals E_n'' where $|f(x)| \leqslant \pi\rho$ by the components, in $\mathbb{C}$, of the set where $|f(z)| \leqslant \pi\rho$. That should probably not make much difference for, when $q = 1$, each of the components of the set in $\mathbb{C}$ on which $|\sin \pi z| \leqslant \tan(\pi\rho/2)$ has (away from the boundary) about the same harmonic measure (in the complement of the union of those components) as the corresponding real interval $[n - \rho, n + \rho]$ has in $\mathcal{D}_\rho$. The former complement is indeed taken onto the domain $\mathcal{D}_\rho$ by the *conformal mapping*

$$z \mapsto \frac{1}{\pi} \arcsin\left(\frac{\sin^2 \pi z + \tan^2(\pi\rho/2)}{(1 + \tan^2(\pi\rho/2)) \sin \pi z}\right).$$

A further simplification consists in our *dropping the point* $n = 0$ *altogether*—it is already known that this makes no real difference when $q = 1$. We therefore *replace* the function $f(z)$, given by (6), by

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^{2q}}\right),$$

and look at the set

$$\widetilde{E} = \{z;\ |F(z)| \leqslant a\}, \tag{8}$$

where a is some *fixed* small positive quantity. In order to describe the components of $\widetilde{E}$, we need a reasonably good approximation to $F(z)$. Very precise estimates of the latter are available (see, for instance, the last part of Appendix 2 in [11]), but we do not have to resort to them here and can obtain satisfactory information directly.

For that purpose we compare $F(z)$ with $\sin(\pi z^{1/q})$ in the right half-plane, where $z^{1/q}$ is given its principal determination. The *ratio* $F(z)/\sin(\pi z^{1/q})$ is analytic and of order < 1 in that half-plane (we are taking $g > 1$ throughout this discussion); it has, moreover, no *zeros* for $\Re z > 0$, so its behaviour there is governed by how it acts *on the imaginary axis*. For y real we have

$$\log|F(iy)| = \int_0^{\infty} \log\left(1 + \frac{y^2}{t^2}\right) d[t^{1/q}] = \int_0^{\infty} \frac{2y^2}{y^2+t^2} \cdot \frac{[t^{1/q}]}{t} dt$$

$$= \int_0^{\infty} \frac{2y^2}{y^2+t^2} \cdot \frac{t^{1/q}}{t} dt - \int_0^{1} \frac{2y^2}{yu^2+t^2} \cdot \frac{t^{1/q}}{t} dt + O(1) \int_1^{\infty} \frac{y^2}{y^2+t^2} \cdot \frac{dt}{t}.$$

The last two terms on the right amount to

$$-2|y|^{1/q} \int_0^{1/|y|} \frac{\tau^{1/q}}{\tau(\tau^2+1)} d\tau + O(1) \int_{1/|y|}^{\infty} \frac{d\tau}{\tau(\tau^2+1)} = -2q + O(1/y^2) + O(\log|y|)$$

for large $|y|$, and the main contribution is

$$\int_0^{\infty} \frac{2y^2}{y^2+t^2} \cdot \frac{t^{1/q}}{t} dt = |y|^{1/q} \cdot \int_0^{\infty} \frac{2\tau^{1/q}}{\tau(\tau^2+1)} d\tau.$$

Thus,

$$\log|F(iy)| = K_q|y|^{1/q} + O(\log^+|y|) + O(1), \tag{9}$$

where

$$K_q = 2\int_0^\infty \frac{\tau^{1/q}}{\tau(\tau^2+1)}d\tau. \tag{10}$$

On the imaginary axis we have, for the principal determination,

$$(iy)^{1/q} = |y|^{1/q}\cos\frac{\pi}{2q} \pm i|y|^{1/q}\sin\frac{\pi}{2q},$$

which, with (9), shows that

$$\left|\frac{(iy)^{1/q}F(iy)}{\sin\pi(iy)^{1/q}}\right|$$

lies there between $\text{const}.\frac{e^{L_q|y|^{1/q}}}{1+|y|^c}$ and $\text{const}.(1+|y|^c)e^{L_q|y|^{1/q}}$, where $L_q = K_q - \pi\sin\frac{\pi}{2q}$ and c is some constant. It thus suffices to take

$$A_q = \pi\tan\frac{\pi}{2q} - K_q\sec\frac{\pi}{2q} \tag{11}$$

to ensure that

$$\frac{\text{const}.}{(1+|y|^c)} \leqslant \frac{|e^{A_q(iy)^{1/q}}(iy)^{1/q}F(iy)|}{|\sin\pi(iy)^{1/q}|} \leqslant \text{const}.(1+|y|^c)$$

for $-\infty < y < \infty$.

According to the previous observation, such an inequality must persist (Phragmén–Lindelöf!) in the right half-plane, and we will have

$$\frac{\text{const}.}{|z+1|^c} \leqslant \left|\frac{e^{A_q z^{1/q}}z^{1/q}F(z)}{\sin\pi z^{1/q}}\right| \leqslant \text{const}.|z+1|^c$$

there. The integral in (10) can be worked out by contour integration, yielding $K_q = \pi/\sin\frac{\pi}{2q}$ which, substituted into the preceding relation, gives

$$\frac{\text{const}.|e^{\pi\cot\frac{\pi}{2q}\cdot z^{1/q}}\sin(\pi z^{1/q})|}{|z+1|^c}$$
$$\leqslant |z^{1/q}F(z)| \leqslant \text{const}.|z+1|^c|e^{\pi\cot\frac{\pi}{2q}\cdot z^{1/q}}\sin(\pi z^{1/q})|, \quad \Re z \geqslant 0. \tag{12}$$

From this we see that when the quantity $a > 0$ figuring in (8) is *small*, the set $\widetilde{E}$ where $|F(z)| \leqslant a$ consists, $F(z)$ being *even*, of disjoint components $\widetilde{E}_n$ looking very much like the disks

$$|z - |n|^q \operatorname{sgn} n| \leqslant \rho_n, \quad n = \pm 1, \pm 2, \ldots, \tag{13}$$

where

$$\rho_n \approx e^{-\pi |n| \cot \frac{\pi}{2q}}. \tag{14}$$

In (14), which is only intended as a crude approximation, we are deliberately *leaving out* a factor proportional to $a|n|^{c'}$ on the right, where $-q(1-c) \leqslant c' \leqslant q(1+c)$. As it turns out, the main parameters involved here are the values of $\log(1/\rho_n)$, so that omission makes very little difference.

Following the idea guiding our work, we should now take the domain $\mathcal{E}$ consisting of the points outside all of the little circles

$$\gamma_n = \{z;\ |z - |n|^q \operatorname{sgn} n| = \rho_n\}, \quad n = \pm 1, \pm 2, \ldots \tag{15}$$

and look at the harmonic measures $\omega_{\mathcal{E}}(\gamma_n, z)$. We would like to have

$$\log |p(z)| \leqslant \sum_{n=-\infty}^{\infty}{}' \omega_{\mathcal{E}}(\gamma_n, z) \max\{\log^+ |p(\zeta)|;\ \zeta \in \gamma_n\} \tag{16}$$

for $z \in \mathcal{E}$ and polynomials $p(z)$, but that is not automatically guaranteed. For (16) to hold, ∞ has to be a regular point for the Dirichlet problem in $\mathcal{E}$, and that is far from evident since the radii ρ_n of the circles γ_n go to zero so rapidly.

To verify that ∞ is a regular point, one may use Wiener's criterion. If, for $n = \pm 1, \pm 2, \ldots$, we denote the disk (13) by Δ_n and then form the unions

$$R_k = \bigcup_{4^k \leqslant n^q < 4^{k+1}} (\Delta_n \cup \Delta_{-n}),$$

the criterion is that

$$\sum_{k=k_0}^{\infty} \frac{k}{\log^+(2/\operatorname{Cap} R_k)} \tag{17}$$

should diverge for all sufficiently large k_0, where $\operatorname{Cap} R_k$ is the logarithmic capacity of R_k. (See problem 1 on p. 152 of [12] where the formulation is not, however, quite correct.) When $k \geqslant 2$, formula (5.5) on p. 130 of [12] is applicable, and gives

$$\frac{1}{\log^+(2/\operatorname{Cap} R_k)} \geqslant \sum_{\Delta_n \subseteq R_k} \frac{1}{\log^+(2/\operatorname{Cap} \Delta_n)}.$$

Referring to (14) we see that $\log(2/CAP\Delta_n)$ is proportional to $|n|$, making the right side of the last relation greater than or equal to some positive constant. Using this we see that the sum (17) *is* divergent; that ensures the regularity of ∞ and thus the validity of (16).

In this way we come to the problem of estimating the harmonic measures $\omega_{\mathcal{E}}(\gamma_n, z)$; thanks to Harnack their values are only needed for $z = 0$. And since the quantities $\omega_{\mathcal{E}}(\gamma_n, 0)$ can in turn be read off from the Riesz representation of the Green's function $G_{\mathcal{E}}(z, 0)$, it is to the latter that we direct our attention.

Some eleven years ago I arrived at a fairly precise evaluation of such a Green's function after having gone through some long semiheuristic and rather laborious calculations. Although this work had in the meantime been almost completely forgotten, I have been able to locate a record of it, and its conclusion is simple enough.

According to that conclusion, one can adjust the constants $\alpha > 0$ and C in the formula

$$G_{\mathcal{E}}(z, 0) \simeq \log \frac{1}{|z|} + {\sum_{\nu=-\infty}^{\infty}}' \frac{C \log |z - |\nu|^q \operatorname{sgn} \nu|}{|\nu|^{1+\alpha}} \tag{18}$$

so as to obtain a very good approximation. Once known, this fact can simply be *used* and the accuracy of (18) then confirmed directly after determination of the right values of α and C. When that is done, (18) will imply

$$\omega_{\mathcal{E}}(\gamma_n, 0) \approx \frac{C}{|n|^{1+\alpha}}, \tag{19}$$

at least in *some* sense. Let us proceed with this approach.

Suppose that z is on the circle γ_n, where $n \geqslant 1$ is large. Then each of the terms of the summation in (18), *save the one with* $\nu = n$, is very nearly equal to $(C \log |n^q - |\nu|^q \operatorname{sgn} \nu|)/|\nu|^{1+\alpha}$, and the term with $\nu = n$ comes to $-\frac{\pi C}{n^\alpha} \cot \frac{\pi}{2q}$ by (14) and (15). We can thus write

$$\begin{aligned} &{\sum_{\nu=-\infty}^{\infty}}' \frac{\log |z - |\nu|^q \operatorname{sgn} n\nu|}{|\nu|^{1+\alpha}} \\ &\qquad \simeq -\frac{\pi}{n^\alpha} \cot \frac{\pi}{2q} + \sum_{\nu \geqslant 1, \nu \neq n} \frac{\log |n^{2q} - \nu^{2q}|}{\nu^{1+\alpha}} + \frac{\log(2n^q)}{n^{1+\alpha}} \quad \text{for} \quad z \in \gamma_n \end{aligned} \tag{20}$$

when n is large; the discrepancy is bounded above by $(2 \sum_{\nu \geqslant 1} 1/\nu^{1+\alpha}) e^{-\pi n \cot \frac{\pi}{2q}}$.

Here we have

$$\sum_{\nu\geqslant 1,\nu\neq n}\frac{\log|n^{2q}-\nu^{2q}|}{\nu^{1+\alpha}}$$
$$=\sum_{\nu=1}^{\infty}\frac{2q}{\nu^{1+\alpha}}\log n-\frac{2q\log n}{n^{1+\alpha}}+\frac{1}{n^{\alpha}}\sum_{\nu\geqslant 1,\nu\neq n}\frac{1}{(\nu/n)^{1+\alpha}}\log\left|1-\frac{\nu^{2q}}{n^{2q}}\right|\cdot\frac{1}{n}, \tag{21}$$

and in the expression

$$\sum_{\nu\geqslant 1,\nu\neq n}\frac{1}{(\nu/n)^{1+\alpha}}\log\left|1-\frac{\nu^{2q}}{n^{2q}}\right|\cdot\frac{1}{n} \tag{22}$$

we recognize a Riemann sum for the integral

$$J_q=\int_0^{\infty}\frac{1}{t^{1+\alpha}}\log|1-t^{2q}|dt. \tag{23}$$

It is necessary to see how close (22) is to J_q when n is large. We have

$$\sum_{\nu=1}^{n-1}\frac{\log(1-(\nu/n)^{2q})}{(\nu/n)^{1+\alpha}}\cdot\frac{1}{n}-\int_0^{\frac{n-1}{n}}\frac{\log(1-t^{2q})}{t^{1+\alpha}}dt$$
$$=\sum_{\nu=1}^{n-1}\int_{\frac{\nu-1}{n}}^{\frac{\nu}{n}}\left(s-\frac{\nu-1}{n}\right)\frac{d}{ds}\left(\frac{\log(1-s^{2q})}{s^{1+\alpha}}\right)ds \tag{24}$$

and

$$\sum_{\nu=n+1}^{\infty}\frac{\log((\nu/n)^{2q}-1)}{(\nu/n)^{1+\alpha}}\cdot\frac{1}{n}-\int_{\frac{n+1}{n}}^{\infty}\frac{\log(t^{2q}-1)}{t^{1+\alpha}}dt$$
$$=-\sum_{\nu=n+1}^{\infty}\int_{\frac{\nu}{n}}^{\frac{\nu+1}{n}}\left(\frac{\nu+1}{n}-s\right)\frac{d}{ds}\left(\frac{\log(s^{2q}-1)}{s^{1+\alpha}}\right)ds. \tag{25}$$

The sums on the right are bounded respectively by

$$\frac{1}{n}\int_0^{\frac{n-1}{n}}\left|\frac{d}{ds}\left(\frac{\log(1-s^{2q})}{s^{1+\alpha}}\right)\right|ds\quad\text{and}\quad\frac{1}{n}\int_{\frac{n+1}{n}}^{\infty}\left|\frac{d}{ds}\left(\frac{\log(s^{2q}-1)}{s^{1+\alpha}}\right)\right|ds$$

in absolute value. Making now the working assumption that $1+\alpha < 2q$ (which will turn out to be the case), we note that $s^{-1-\alpha}\log(1-s^{2q})$ is *zero* for $s=0$ and *decreasing* for $0 \leqslant s < 1$, whereas $\frac{d}{ds}\left(\frac{\log(s^{2q}-1)}{s^{1+\alpha}}\right)$ *vanishes only one* for $s > 1$, at the place where $u = s^{2q-1}$ satisfies the equation $u + 1 = \frac{1+\alpha}{2q} u \log u$. These properties ensure that each of the preceding two expressions is $O(\log n/n)$ for large n. At the same time,

$$\int_{\frac{n-1}{n}}^{\frac{n+1}{n}} \frac{\log|1-t^{2q}|}{t^{1+\alpha}} dt = O\left(\frac{\log n}{n}\right),$$

so (24) and (25) imply that *the sum* (22) *differs from* J_q *by* $O(\log n/n)$ *when* n *is large.*

By (20) and (21) we therefore have

$$\sum_{\nu=-\infty}^{\infty}{}' \frac{\log|z-|\nu|^q \operatorname{sgn}\nu|}{|\nu|^{1+\alpha}} = \sum_{\nu=1}^{\infty} \frac{2q}{\nu^{1+\alpha}} \log|n| + \left(J_q - \pi\cot\frac{\pi}{2q}\right)\cdot\frac{1}{|n|^\alpha}$$
$$+ O\left(\frac{\log|n|}{|n|^{1+\alpha}}\right) + O(e^{-\pi|n|\cot\frac{\pi}{2q}}) \quad \text{for} \quad z \in \gamma_n \qquad (26)$$

with n large; the same result also holds for negative n. From this relation we deduce that the right side of (18) *nearly balances out to zero* on the circles γ_n ($|n|$ large), provided that α is chosen properly and C taken equal to $(2\sum_{\nu\geqslant 1} 1/\nu^{1+\alpha})^{-1}$. On the γ_n the term $\log(1/|z|)$ is then nearly cancelled out from one in $\log|n|$ coming from the first right-hand member in (26), and we can ensure that

$$\left(J_q - \pi\cot\frac{\pi}{2q}\right)\cdot\frac{1}{|n|^\alpha} = 0 \qquad (27)$$

by adjusting α to make $J_q = \pi\cot\frac{\pi}{2q}$.

Referring to (23) we see that for this we need

$$\int_0^\infty \frac{\log|1-t^{2q}|}{t^{1+\alpha}} dt = \pi\cot\frac{\pi}{2q}.$$

The integral on the left, equal to $(1/q)\int_0^\infty s^{-1-\alpha/q}\log|1-s^2|ds$, works out, for $0<\alpha<q$, to $\frac{\pi}{\alpha}\cot\frac{\pi\alpha}{2q}$, so (27) *is satisfied when* $\alpha = 1$ (whatever be the value of $q > 1$!). Thanks to (14) and (15) we thus have, by (26) and (27),

$$\sum_{\nu=1}^{\infty} \frac{2}{\nu^2}\log\frac{1}{|z|} + \sum_{\nu=-\infty}^{\infty}{}' \frac{\log|z-|\nu|^q\operatorname{sgn}\nu|}{\nu^2} = O\left(\frac{\log|n|}{n^2}\right) \quad \text{for} \quad z\in\gamma_n, \qquad (28)$$

as long as $|n|$ is large.

Denote the left side of (28) by $U(z)$; then we see by (14) and (15) that $U(z)$, for sufficiently large $|n|$, *vanishes exactly* on a simple closed curve γ_n' lying in an annulus

$$|n|^{-K}e^{-\pi|n|\cot\frac{\pi}{2q}} \leqslant |z - |n|^q \operatorname{sgn} n| \leqslant |n|^K e^{-\pi|n|\cot\frac{\pi}{2q}} \tag{29}$$

and encircling the point $|n|^q \operatorname{sgn} n$—here K is a certain constant. If only we could be sure that such *disjoint* curves γ_n' were forthcoming for *all* $n \neq 0$, $2\sum_{\nu\geqslant 1} 1/\nu^2)^{-1}U(z)$ would, by (28), be the *precise* Green's function $G_{\mathcal{E}'}(z,0)$ for the domain $\mathcal{E}'$ consisting of the points outside of all of those γ_n', and then (18) would represent a very good approximation to the Green's function of $\mathcal{E}$, especially since we have (deliberately) left out a factor proportional to a power of $|n|$ in (14). But we are unable to ensure existence of the γ_n' for small $|n|$ without resorting to more elaborate computations, and that is why we fall back on the following argument.

Suppose that the disjoint curves γ_n' lying in the annuli (29) and on which $U(z) = 0$ can be obtained for $|n| > N$, say. Taking N so large as to also have $|z^2/(z^2-N^{2q})| > 1$ on each γ_n' with $|n| > N$, which is possible by (29), we then put

$$U_N(z) = \sum_{\nu=N+1}^{\infty} \frac{2}{\nu^2} \log\frac{1}{|z|} + \sum_{|\nu|>N} \frac{\log|z - |\nu|^q \operatorname{sgn}\nu|}{\nu^2}, \tag{30}$$

making

$$U_N(z) = U(z) + \sum_{\nu=1}^{N} \frac{1}{\nu^2} \log\left|\frac{z^2}{z^2 - \nu^{2q}}\right|, \tag{31}$$

since $U(z)$ is the left-hand expression in (28).

Due to our choice of N we have, $U_N(z) > U(z) = 0$ on each of the γ_n' with $|n| > N$, so $U_N(z)$ can then, by (30), be *diminished to zero* by moving z *closer to* $|n|^q \operatorname{sgn} n$ from any initial position on γ_n'. It does not have to be moved very much when $|n| > N$ is *large*. In that event,

$$\sum_{\nu=1}^{N} \frac{1}{\nu^2} \log\left|\frac{z^2}{z^2 - \nu^{2q}}\right| = O\left(\frac{N^{2q}}{|z|^2}\right)$$

for z near $|n|^q \operatorname{sgn} n$; this quantity behaves there like $O(N^{2q}/|n|^{2q})$ which, in order to bring $U_N(z)$ down to 0, needs, according to (31) and the left side of (28), to be *compensated by a decrease* in $n^{-2}\log|z - |n|^q \operatorname{sgn} n|$ when z is moved closer to $|n|^q \operatorname{sgn} n$ from a point on γ_n'. (When z is so moved the net change of all of the *remaining* terms on the left side of (28) will be much, much *smaller*—of the order of magnitude of the change in z.) Since $g > 1$, the desired counterbalancing can be achieved by diminishing $|z - |n|^q \operatorname{sgn} n|$ in the ratio 1: $\exp(-O(1/|n|^{2q-2}))$; which for all intents and purposes is nearly 1:1 when $|n|$ is large.

By going through this process for each n, $|n| > N$, we obtain simple closed curves $\tilde{\gamma}_n$ about the points $|n|^q \operatorname{sgn} n$, $|n| > N$, lying in annuli of the form

$$A|n|^{-K} e^{-\pi|n| \cot \frac{\pi}{2q}} \leqslant |z - |n|^q \operatorname{sgn} n| \leqslant |n|^K e^{-\pi|n| \cot \frac{\pi}{2q}}, \tag{32}$$

and on which $U_N(z) = 0$. Denoting by $\mathcal{E}''$ the domain consisting of the z outside of these $\tilde{\gamma}_n$, we have by (30),

$$G_{\mathcal{E}''}(z, 0) = \Big(\sum_{\nu=N+1}^{\infty} 2/\nu^2 \Big)^{-1} U_N(z)$$

for the Green's function of $\mathcal{E}''$. From this relation—and (30)—we can read off the harmonic measures $\omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0)$, and see in that way that

$$\omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0) = \frac{C}{n^2}, \quad |n| > N, \tag{33}$$

where $C = (\sum_{\nu=N+1}^{\infty} 2/\nu^2)^{-1}$.

About each of the $2N$ points $|n|^q \operatorname{sgn} n$, $n = \pm 1, \pm 2, \ldots, \pm N$, we now take *very small* circles $\tilde{\gamma}_n$, and let $\tilde{\mathcal{E}}$ be the domain consisting of the points outside *all* the $\tilde{\gamma}_n$, including those described above. Writing

$$\Gamma_N = \bigcup_{n=-N}^{N}{}' \, \tilde{\gamma}_n,$$

we have, by the principle of maximum,

$$\omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0) - \omega_{\tilde{\mathcal{E}}}(\Gamma_N, 0) \cdot \max_{z \in \Gamma_N} \omega_{\mathcal{E}''}(\tilde{\gamma}_n, z) \leqslant \omega_{\tilde{\mathcal{E}}}(\tilde{\gamma}_n, 0) \leqslant \omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0) \quad \text{for} \quad |n| > N,$$

where $\omega_{\tilde{\mathcal{E}}}(\,,\,)$ denotes harmonic measure for the domain $\tilde{\mathcal{E}}$. Harnack makes, however, $\max_{z \in \Gamma_N \omega_{\mathcal{E}''}}(\tilde{\gamma}_n, z) \leqslant 4N\omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0)$ (say), so the left side of the preceding relation is $\geqslant (1 - 4N\omega_{\tilde{\mathcal{E}}}(\Gamma_N, 0))\omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0)$. By taking the individual circles $\tilde{\gamma}_n$, $|n| \leqslant N$, *small enough* we can ensure that $4N\omega_{\tilde{\mathcal{E}}}(\Gamma_N, 0) \leqslant 1/2$ and thus that

$$\frac{1}{2}\omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0) \leqslant \omega_{\mathcal{E}''}(\tilde{\gamma}_n, 0) \quad \text{for} \quad |n| > N.$$

With (33), this yields

$$\frac{C}{2n^2} \leqslant \omega_{\tilde{\mathcal{E}}}(\tilde{\gamma}_n, 0) \leqslant \frac{C}{n^2}, \quad |n| > N.$$

A similar relation of course holds for each of the $2N$ circles $\tilde{\gamma}_n$ with $|n| \leqslant N$ provided that we replace $C/2$ by a (small) constant B and C by a large one, B'. We will then have

$$\frac{B}{n^2} \leqslant \omega_{\tilde{\mathcal{E}}}(\tilde{\gamma}_n, 0) \leqslant \frac{B'}{n^2}, \quad n = \pm 1, \pm 2, \ldots. \tag{34}$$

And *each* of the $\tilde{\gamma}_n$ will lie in a corresponding annulus of the form (32), provided that the constant $A > 0$ appearing there is small enough.

We have practically achieved what we originally had in mind to do. True, the $\tilde{\gamma}_n$ are not exactly the circles γ_n described by (15) and $\tilde{\mathcal{E}}$ is not the domain $\mathcal{E}$ figuring in (18) and (19). Comparison if (14) and (32) shows, however, that the $\tilde{\gamma}_n$ are very close (in an appropriate sense) to the γ_n, and that the approximation (18) should thus be rather good if we take in it $\alpha = 1$ and use an appropriate value of C. This should be enough to give us (19), at least in a form like (34).

I am confident that such an estimate of the $\omega_{\mathcal{E}}(\gamma_n, 0)$ could in fact be *deduced* from (34) by repeated application of the maximum principle coupled with the theorem on 3 circles. For the present study, however, that hardly seems worthwhile, and we content ourselves with (34).

An appropriate version of (16) holds in $\tilde{\mathcal{E}}$ for polynomials $p(z)$; according to (34) together with Harnack, it reads

$$\log |p(z)| \leqslant \tilde{K}(z) \sum_{n=-\infty}^{\infty}{}' \frac{\max\{\log^+ |p(\zeta)|;\ \zeta \in \tilde{\gamma}_n\}}{n^2}. \tag{35}$$

Here the function $\tilde{K}(z)$ of course *depends* on the exponent $q > 1$, *but the weight* $1/n^2$ *figuring in the summation does not.*

The initial idea guiding all of the above work has been that the relation (35), once found, might *suggest*—although it cannot prove—a valid result involving the quantities $\log^+ |p(|n|^q \operatorname{sgn} n)|$ in place of $\max\{\log^+ |p(\zeta)|;\ \zeta \in \tilde{\gamma}_n\}$. And one indeed sees almost immediately that such a result *does hold* when $q = 2$. Then it follows directly from the original one on p. 520 of [1] on making the change of variable $z^2 = w$.

For arbitrary $q > 1$ a simple artifice enables us to deduce a similar result from that in [1]. It is thus *true* that for any $q > 1$ (as well as for $q = 1$), *the polynomials* $p(z)$ with

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^2} \log^+ |p(|n|^q \operatorname{sgn} n)|$$

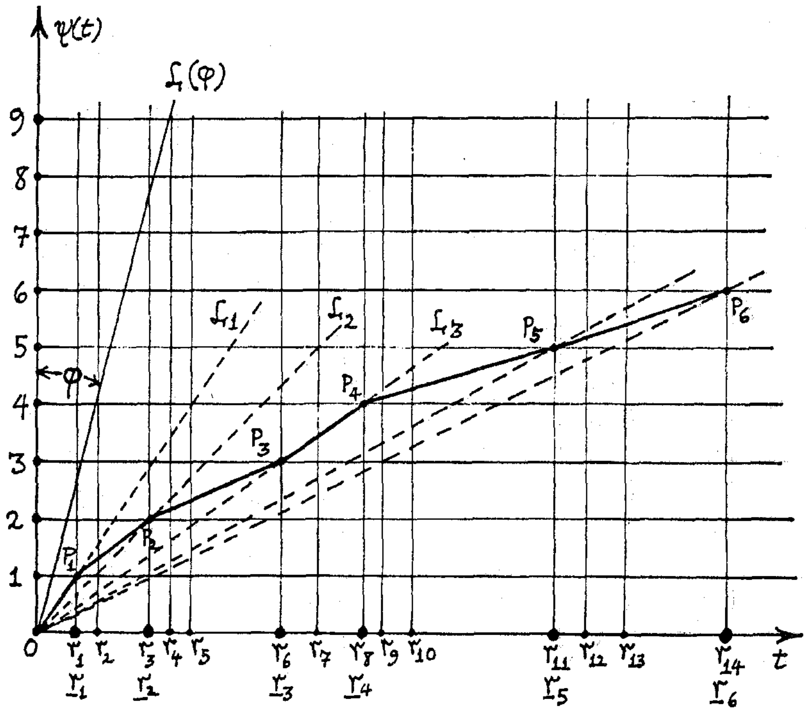

Figure.

sufficiently small form a normal family. Simple examples also show that the weight $1/n^2$ appearing here is (essentially) the best possible one.

Generalization of the artifice used in obtaining the last result leads to the procedure to be employed in the following sections. In that way we will arrive at results concerning various complex sequences, of the kind mentioned in the Introduction.

2. Given a sequence of distinct complex numbers ζ_n, $n = 1, 2, \ldots$, with $1 \leqslant |\zeta_1| \leqslant |\zeta_2| \leqslant \ldots$ and $\zeta_n \to_n \infty$, we put $r_n = \sqrt{|\zeta_n|}$ (sic!) and shall mainly be interested in subsequences $\{r_{n_k}\}$ of $\{r_n\}$ for which r_{n_k} *is increasing* (i.e., nondecreasing). $\{r_n\}$ always has a *minimal* subsequence, henceforth denoted by $\{\underline{r}_n\}$, with that property, and it will—without loss of generality—be involved in the formulation of the main results in this paper. That minimal subsequence is obtained by a simple geometric construction, which we now describe.

Taking a rectangular coordinate system, we first locate the numbers r_n on the horizontal axis, henceforth called the t-axis, and then mark all of the points (r_n, m) on each of the horizontal lines of ordinate $m = 1, 2, \ldots$. For $0 \leqslant \varphi < \pi/2$ we let $\mathcal{L}(\varphi)$ be the ray out from the origin, lying in the first quadrant and making an angle φ with the *vertical* axis (sic!). As φ increases, starting from 0, $\mathcal{L}(\varphi)$ will turn in the *clockwise* direction until it encounters the point $P_1 = (r_1, 1)$ for $\varphi = \varphi_1$, say. Taking $\underline{r}_1 = r_1$, we then look to see whether $\mathcal{L}_1 = \mathcal{L}(\varphi_1)$ passes through any point $(r_n, 2)$, $n > 1$, or not. If it does we denote that point by P_2 and its *abscissa* r_n by $\underline{r}_2$; otherwise we let φ increase beyond φ_1 util $\mathcal{L}(\varphi)$ first passes through such a point, for $\varphi = \varphi_2$, say. That point is then denoted by P_2 and its abscissa by $\underline{r}_2$. The procedure is now repeated, taking P_3 as $(r_n, 3)$ if there is such a point on $\mathcal{L}_2 = \mathcal{L}(\varphi_2)$ (or on $\mathcal{L}_1$ if P_2 is on that ray) and, if there is not, increasing φ from φ_2 (or from φ_1, as the case may be) until $\mathcal{L}(\varphi)$ first encounters such a point, which we then take as P_3. In either event, the abscissa of P_3 is called $\underline{r}_3$.

Continuing in this fashion we obtain, one after another, the points P_n whose respective abscissae $\underline{r}_n$ all belong to the original sequence $\{r_n\}$. Since φ continually increases—or at least never decreases—in this construction, it is clear that

$$\frac{\underline{r}_1}{1} \leqslant \frac{\underline{r}_2}{2} \leqslant \frac{\underline{r}_3}{3} \leqslant \dots . \tag{36}$$

We denote by $\psi(t)$ the function whose graph consists of the straight segments joining successively 0 to P_1, P_1 to P_2 to P_3 and so forth; we have $\psi(\underline{r}_n) = n$ for $n = 1, 2, \dots$, and the ratio $\psi(t)/t$ is *decreasing* (in the wide sense) by (36). The function $\psi(t)$ is of course continuous and strictly increasing, so it has a continuous increasing *inverse* $\phi(s)$. The ratio $\phi(s)/s$ is then *increasing* (in the wide sense), and we have $\phi(n) = \underline{r}_n$. The functions ψ and ϕ will play an important rôle in the following work.

It was stated at the beginning of this article that $\{\underline{r}_n\}$ is the *minimal* subsequence of $\{r_n\}$ having the property (36); let us prove that fact. Given, then, any subsequence $\{r'_n\}$ of $\{r_n\}$ with $r'_1/1 \leqslant r'_2/2 \leqslant r'_3 \leqslant \dots$, we wish to show that $r'_n \geqslant \underline{r}_n$ for $n = 1, 2, \dots$. For $n = 1$ that statement is manifest since $\underline{r}_1 = r_1$. Assuming, then, that we know that $r'_m \geqslant \underline{r}_m$ for some $m \geqslant 1$, let us verify that $r'_{m+1} \geqslant \underline{r}_{m+1}$.

If $\mathcal{L}(\varphi_m)$ is the ray $\mathcal{L}(\varphi)$ through $P_m = (\underline{r}_m, m)$ we have $\varphi' \geqslant \varphi_m$ for the ray $\mathcal{L}(\varphi')$ through (r'_m, m) because $r'_m \geqslant \underline{r}_m$. Again, the relation $r'_{m+1}/(m+1) \geqslant r'_m/m$ implies that $\varphi'' \geqslant \varphi'$ if $\mathcal{L}(\varphi'')$ passes through $(r'_{m+1}, m+1)$. Therefore $\varphi'' \geqslant \varphi_m$. But $(\underline{r}_{m+1}, m+1)$, by construction, is the point *furthest to the left* among those of the form $(r_n, m+1)$ encountered by rays $\mathcal{L}(\varphi)$ with $\varphi \geqslant \varphi_m$. Therefore $r'_{m+1} \geqslant \underline{r}_{m+1}$ as claimed, and the desired relation holds by induction.

3. For each of the members $\underline{r}_n$ of the subsequence of $\{\sqrt{|\zeta_n|}\}$ just constructed, we select a ζ_m with $\underline{r}_n = \sqrt{|\zeta_m|}$ and denote that ζ_m by $\underline{\zeta}_n$. In that way we obtain a subsequence $\{\underline{\zeta}_n\}$ of $\{\zeta_n\}$, the sequence originally given.

Concerning this subsequence, we have

Theorem 1. *Provided that $\eta > 0$ is sufficiently small, the polynomials $p(z)$ with*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |p(\underline{\zeta}_n)| \leqslant \eta \tag{37}$$

form a normal family in $\mathbb{C}$.

Proof. Unless we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log |\zeta_n| < \infty, \tag{38}$$

the only polynomials $p(z)$ satisfying (37) are (bounded!) constants, and in that event the result is evident. Assuming, then, that (38) *holds*, let us first reduce the general statement of the theorem to one involving polynomials equal to 1 at the origin.

The reduction parallels one made on pp. 519–522 of [1]. Given any polynomial $p(z)$, we have, for $M \geqslant 1$ and the polynomial

$$P(z) = 1 + \frac{z}{M} p(z),$$

$$\log^+ |P(z)| \leqslant \log(1 + |p(z)|) + \log\left(1 + \frac{|z|}{M}\right) \leqslant \log^+ |p(z)| + \log 2 + \log(1 + |z|). \tag{39}$$

When $\eta > 0$ is *also* given, we can choose—and then *fix*—N so large as to have

$$\sum_{n>N} \frac{\log 2}{n^2} < \eta,$$

and also

$$\sum_{n>N} \frac{1}{n^2} \log(1 + |\zeta_n|) < \eta,$$

thanks to (38). According to (39) and the relation following it, that will make

$$\sum_{n>N} \frac{1}{n^2} \log^+ |P(\zeta_n)| < \sum_{n>N} \frac{1}{n^2} \log^+ |p(\zeta_n)| + 2\eta \tag{40}$$

(as long as $M \geqslant 1$).

Assuming (37), we have, however, $|p(\zeta_n)| \leqslant \exp(\eta n^2)$, so, N having been fixed, we can choose $M \geqslant 1$ large enough, *in a way depending only on* η *and* N, to have

$$\sum_{n=1}^{N} \log\left(1 + \frac{|\zeta_n|}{M} |p(\zeta_n)|\right) < \eta. \tag{41}$$

For such a value of M, (37), (40) and (41) then imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P(\zeta_n)| < 4\eta \tag{42}$$

where $P(z)$, the polynomial related to $p(z)$ by (39), takes the value 1 at 0. Since

$$p(z) = \frac{M}{2}(P(z) - 1)$$

in thus suffices, in order to prove the theorem, to show that the polynomials $P(z)$ satisfying (42) with $\eta > 0$ small enough and with $P(0) = 1$ form a normal family in $\mathbb{C}$.

For any such polynomial we have

$$P(z) = \prod_k \left(1 - \frac{z}{a_k}\right) \tag{43}$$

with a finite product and $a_k \neq 0$ on the right. Thence,

$$|P(\underline{\zeta}_n)| \geqslant \prod_k \left|1 - \frac{|\underline{\zeta}_n|}{a_k}\right| \tag{44}$$

for each of the $\underline{\zeta}_n$. Taking

$$\alpha_k = \sqrt{|a_k|} \tag{45}$$

and then putting

$$F(z) = \prod_k \left(1 - \frac{z^2}{\alpha_k^2}\right), \tag{46}$$

we see that the right-hand product in (44) is equal to $|F(\overline{r}_n)|$, and thus that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |F(\underline{r}_n)| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P(\underline{\zeta}_n)|. \tag{47}$$

Using the function $\psi(t)$ described at the end of the last section, we now put

$$\beta_k = \psi(\alpha_k),$$

making

$$\alpha_k = \phi(\beta_k), \tag{48}$$

and then form a new polynomial

$$Q(w) = \prod_k \left(1 - \frac{w^2}{\beta_k^2}\right); \tag{49}$$

it is claimed that

$$|Q(u)| \leqslant |F(\phi(u))| \quad \text{for} \quad u > 0. \tag{50}$$

For each k we have indeed

$$\left|1 - \left(\frac{\phi(u)}{\alpha_k}\right)^2\right| = \left|1 - \left(\frac{\phi(u)}{\beta_k}\right)^2\right|$$

by (48) so, since the ratio $\phi(s)/s$ is *increasing*,

$$\frac{\phi(u)}{u} \geqslant \frac{\phi(\beta_k)}{\beta_k} \quad \text{when} \quad u \geqslant \beta_k,$$

whence

$$\left(\frac{\phi(u)}{\phi(\beta_k)}\right)^2 \geqslant \left(\frac{u}{\beta_k}\right)^2 \geqslant 1$$

and thus

$$\left|1 - \left(\frac{\phi(u)}{\phi(\beta_k)}\right)^2\right| \geqslant \left|1 - \frac{u^2}{\beta_k^2}\right| \tag{51}$$

for $u \geqslant \beta_k$. If, however, $0 < u < \beta_k$, we have on the other hand

$$\frac{\phi(u)}{u} \geqslant \frac{\phi(\beta_k}{\beta_k},$$

so that

$$\left(\frac{\phi(u)}{\phi(\beta_k)}\right)^2 \leqslant \left(\frac{u}{\beta_k}\right)^2 < 1,$$

and then (51) again holds. The relation (50) now follows from (46), (49) and (51).

Since $\phi(n) = \underline{r}_n$, we have, from (50),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |Q(n)| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |F(\underline{r}_n)|,$$

so, *assuming that* (42) *holds*, we have, by (47),

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |Q(n)| \leqslant 4\eta. \tag{52}$$

It is here that we fall back on the material from [1] referred to in the Introduction. According to the theorem on p. 516 of that book, if (52) holds for a polynomial $Q(w)$ of the form (49) (with the $\beta_k > 0$) and $\eta > 0$ is small enough, we must have

$$\frac{N(s)}{s} \leqslant C\eta \quad \text{for all} \quad s > 0, \tag{53}$$

where $N(s)$ denotes the number of β_k in the interval $[0, s]$ and C is a numerical constant. (One can also deduce this using the *corollary* on p. 516 of [1] and Jensen's formula. Pedersen's proof of that corollary in [2] is easier than the one in [1]—see also [3] and [4].)

According to (48), the β_k in $[0, s]$ correspond to the α_k in the interval $[0, \phi(s)]$ and thus, by (45), to the roots a_k of $P(z)$ with $0 < |a_k| < (\phi(s))^2$. Writing $\nu(R)$ for the number of roots of $P(z)$ in the disk $|z| \leqslant R$, we therefore have

$$\frac{\nu(R)}{\psi(\sqrt{R})} \leqslant C\eta \tag{54}$$

for all $R > 0$ by (53), as long as (42) holds with $\eta > 0$ small enough.

Referring now to (43), we see that

$$\log|P(z)| \leqslant \int_0^\infty \log\left(1 + \frac{|z|}{t}\right) d\nu(t) = \int_0^\infty \frac{|z|}{|z|+t} \cdot \frac{\nu(t)}{t} dt.$$

Using (54), we thence get

$$\log|P(z)| \leqslant C\eta \int_0^\infty \frac{|z|}{|z|+t} \cdot \frac{\psi(\sqrt{t})}{t} dt. \tag{55}$$

Here the ratio $\psi(\sqrt{t})/\sqrt{t}$ is bounded and decreasing, so the right side of (55) furnishes a finite bound on $|P(z)|$ depending *only* on η and $|z|$. And that completes the proof of our theorem.

Remark. Since $\psi(\sqrt{t}) \leqslant \text{const}\,.\sqrt{t}$, the estimate (55) shows that *any limit* of polynomials $P(z)$ of the form (43) satisfying (42) with a sufficiently small $\eta > 0$ is *an entire function of order at most* $1/2$ *and, if of order* $1/2$, *of type* $\leqslant \text{const}\,.\eta$.

4. The sum (37) figuring in the statement of Theorem 1 always involves weights equal to $1/n^2$, no matter what the sequence $\{\zeta_n\}$. It is perhaps somewhat astonishing that this assignment of weights cannot be significantly improved if the quantities $|\zeta_n|$ do not grow too rapidly with n. What is essentially required for that is *polynomial* growth, together with some regularity.

Theorem 2. *Suppose that* $\{\underline{\zeta}_n\}$ *is the subsequence extracted from* $\{\zeta_n\}$ *in* §2, *and that* $\phi(s)/s^K$ *is decreasing for some* $K > 1$, *where* $\psi(s)$ *is the function described near the end of that section. Then there is a sequence of polynomials* $p_l(z)$ *for which the sums*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |p_l(\underline{\zeta}_n)| \tag{56}$$

remain bounded as $l \to \infty$, *while*

$$p_l(z) \to \infty \tag{57}$$

at any z *different from the* $\underline{\zeta}_n$ *with* $|z+1| > 1$.

Proof (Cf. [1, pp. 446–447]). We start by taking the polynomials

$$P_l(z) = \prod_{k=1}^{l} \left(1 - \frac{z}{\underline{\zeta}_k}\right) \tag{58}$$

for which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P_l(\underline{\zeta}_n)| = \sum_{n=l+1}^{\infty} \frac{1}{n^2} \log^+ |P_l(\underline{\zeta}_n)| \tag{59}$$

and proceed to estimate the sum on the right.

Continuing with the notation of the preceding two sections we can write, for $n > l$,

$$|P_l(\underline{\zeta}_n)| \leqslant \prod_{k=1}^{l} \left(1 + \left|\frac{\underline{\zeta}_n}{\underline{\zeta}_k}\right|\right) = \prod_{k=1}^{l} \left(1 + \left(\frac{\underline{r}_n}{\underline{r}_k}\right)^2\right) = \prod_{k=1}^{l} \left(1 + \left(\frac{\phi(n)}{\phi(k)}\right)^2\right). \tag{60}$$

Here n is strictly greater than each k figuring in the last product, so

$$\frac{\phi(n)}{n^K} \leqslant \frac{\phi(k)}{k^K}$$

by hypothesis for those k, and

$$1 + \left(\frac{\phi(n)}{\phi(k)}\right)^2 \leqslant 1 + \left(\frac{n}{k}\right)^{2K}. \tag{61}$$

Because $K > 1$, we have

$$1 + u^{2K} < (1+u)^{2K}$$

for $u > 0$ which, with (60) and (61), makes

$$|P_l(\underline{\zeta}_n)| < \prod_{k=1}^{l}\left(1+\frac{n}{k}\right)^{2K} \quad \text{for} \quad n > l,$$

and finally

$$\sum_{n=l+1}^{\infty} \frac{1}{n^2} \log^+ |P_l(\underline{\zeta}_n)| \leqslant 2K \sum_{n=l+1}^{\infty} \sum_{k=1}^{l} \frac{1}{n^2} \log\left(1+\frac{n}{k}\right). \tag{62}$$

For s and $t > 0$ the expression $\log\left(1+\frac{s}{t}\right)$ is a decreasing function of t and increases with s. The right side of (62) is thus

$$\leqslant 2K\left(\frac{l+2}{l+1}\right)^2 \int_l^{\infty}\int_0^{l} \frac{1}{s^2} \log\left(1+\frac{s}{t}\right) dt\, ds$$

which, after the change of variables $s = l\sigma$, $t = l/\tau$, goes over to

$$2K\left(\frac{l+2}{l+1}\right)^2 \int_1^{\infty}\int_1^{\infty} \frac{\log(1+\sigma\tau)}{\sigma^2\tau^2} d\tau\, d\sigma,$$

and obviously finite quantity remaining bounded for $l \to \infty$. From (59) and (62) we thus have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P_l(\underline{\zeta}_n)| \leqslant O(1), \quad t \to \infty \tag{63}$$

Using again the property that $\phi(n)/n^K$ decreases and the relation $|\underline{\zeta}_n| = \underline{r}_n^2 = (\phi(n))^2$, we see that (38) holds; we can therefore choose a sequence of integers $m_l > 0$ going to ∞ slowly enough to make

$$\sum_{n=l+1}^{\infty} \frac{1}{n^2} \log(1+|\underline{\zeta}_n|)^{m_l} \leqslant O(1) \tag{64}$$

for $l \to \infty$. Putting then

$$p_l(z) = (1+z)^{m_l} P_l(z) \tag{65}$$

so that $p_l(z)$, like $P_l(z)$, vanishes for $z = \underline{\zeta}_1, \underline{\zeta}_2, \ldots, \underline{\zeta}_l$, we see from (63) and (64) that the sum (56) does remain bounded when $l \to \infty$.

The ratio $|\underline{\zeta}_k|/k^2 = (\phi(k)/k)^2$ is, on the other hand, *increasing*, and thus the infinite product

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{\underline{\zeta}_k}\right)$$

is certainly *convergent* to an entire function $\Phi(z)$ vanishing *only* at the points $\underline{\zeta}_k$. From (58) we have $P_l(z) \to \Phi(z)$ as $l \to \infty$ so, since $m_l \to \infty$ in (65), (57) *must hold* at any z with $|z+1| > 1$ different from the zeros of $\Phi(z)$. We are done.

5. Different variants of Theorem 1 (§3) are of course valid, and some of them should perhaps be mentioned now.

It is, in the first place, possible to replace the subsequence $\{\underline{\zeta}_n\}$ of $\{\zeta_n\}$ figuring in (37) by any other one, $\{\zeta_n'\}$, having the property that $\sqrt{|\zeta_n'|}/n$ increases. The proof, making use of functions $\psi(t)$ and $\phi(s)$ appropriate to the new subsequence, is exactly like the one given for Theorem 1. I have preferred to formulate that result in terms of $\{\underline{\zeta}_n\}$ on account of the fact, verified at the end of §2, that $|\underline{\zeta}_n'| \geqslant |\underline{\zeta}_n|$ for any subsequence ζ_n' with the abovementioned property. For *polynomials* $p(z)$, that has a tendency to make the sum (37) *larger* when the $\underline{\zeta}_n$ are replaced by the ζ_n'. (This of course *need* not happen for *particular* polynomials $p(z)$.)

We can, in the second place, extract the subsequence $\{\underline{\zeta}_n\}$ involved in Theorem 1 by using, in place of the *horizontal lines* of ordinate $m = 1, 2, \ldots$ in the construction of §2, *other* such lines of ordinate $\lambda_1, \lambda_2, \ldots$, where

$$L \leqslant \lambda_1 < 2L \leqslant \lambda_2 < 3L \leqslant \ldots$$

for some given $L > 0$. Thus modified, the construction yields functions $\psi(t)$, $\phi(s)$ having the properties of those obtained in §2, save that now $\psi(\underline{r}_n) = \lambda_n$ and $\phi(\lambda_n) = \underline{r}_n$. Proof of the corresponding analogue of Theorem 1 parallels the argument of §3, but with the quantities λ_n now playing the role of the integers n involved there. The only change consists in the appeal, when dealing with the polynomial $Q(w)$, to the result referred to in Remark 2 on p. 518 of [1] (or to the one in [5]), rather than to the theorem from p. 615 of [1] used in §3. Here, $Q(w)$ will satisfy

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\log^+|Q(\lambda_n)| < 4\eta$$

instead of (52). For sufficiently small $\eta > 0$ (depending on L) the estimate (55) is again obtained, but now the constant C also depends on L.

For another related application of the generalization mentioned on p. 518 of [1] see the next section.

Let us end this section by giving one more result.

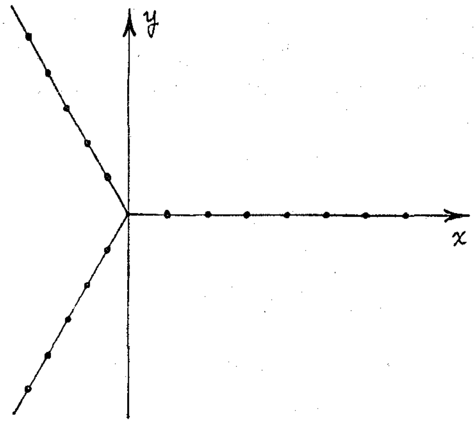

Figure.

Theorem 3. *With* $\omega = e^{2\pi i/3}$, *the polynomials* $p(z)$ *such that*

$$\sum_{n=1}^{\infty} \frac{\log^+ |p(n)| + \log^+ |p(\omega n)| + \log^+ |p(\omega^2 n)|}{n^2} \leqslant \eta$$

form a normal family in $\mathbb{C}$ *when* $\eta > 0$ *is small enough.*

Proof. It is enough to show that the polynomials $P(z)$ with $P(0) = 1$ and

$$\sum_{n=1}^{\infty} \frac{\log^+ |P(n)| + \log^+ |P(\omega n)| + \log^+ |P(\omega^2 n)|}{n^2} \leqslant 4\eta \tag{66}$$

form a normal family for sufficiently small $\eta > 0$, the reduction to this special case being made as in the proof of Theorem 1.

Given $P(z)$, we have, for *the product* $P(z)P(\omega z)P(\omega^2 z)$,

$$P(z)P(\omega z)P(\omega^2 z) = T(z^3), \tag{67}$$

where $T(w)$ is a certain *polynomial* in w with $T(0) = 1$. And from (66), we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |T(n^3)| \leqslant 4\eta. \tag{68}$$

This relation is the same as (42), but with $T(\underline{\zeta}_n)$ standing in place of $P(\underline{\zeta}_n)$ and $\underline{\zeta}_n = \zeta_n = n^3$. According to the reasoning of §3 it implies (see (54)) that for small $\eta > 0$,

$$\frac{\mu(R)}{\psi(\sqrt{R})} \leqslant C\eta, \quad R > 0, \tag{69}$$

where $\mu(R)$ denotes the number of zeros of $T(w)$ in the disk $|w| \leqslant R$ and $\psi(t)$ is the function associated, as in §2, with the sequence $\underline{\zeta}_n = n^3$.

Let $\nu(R)$ be the number of zeros of $P(z)$ in the disk $|z| \leqslant R$; since $P(0) = T(0) = 1$ we then have

$$3\nu(R) = \mu(R^3)$$

by (67), so (69) becomes

$$\frac{\nu(R)}{\psi(R^{3/2})} \leqslant \frac{1}{3}C\eta, \quad R > 0. \tag{70}$$

Here $\psi(n^3) = n$ and $\psi(t)$ is linear on each interval $[(n-1)^3, n^3]$, making

$$\psi(t) \leqslant t^{1/3}, \quad t \geqslant 0.$$

Using (70) instead of (54), we thus obtain an estimate like (55):

$$\log |P(z)| \leqslant \frac{1}{3}C\eta \int_0^\infty \frac{|z|}{|z|+t} \cdot \frac{dt}{\sqrt{t}}.$$

And this immediately implies the desired result.

6. In the construction of §2, $|\underline{\zeta}_n| = (\phi(n))^2$ with $\phi(n)/n$ increasing, so Theorem 1 is limited to sequences of nodes $\underline{\zeta}_n$ whose moduli grow *at least* as rapidly as const $.n^2$. One may well ask whether one can obtain any results involving nodes going out to ∞ *more slowly*, and that is indeed possible. In some such situations one can make direct use of the generalization from p. 518 of [1] already invoked in the last section; then the construction of §2 is not needed and the weight $1/n^2$ is no longer the "right" one.

Without wishing to dwell too much on the results that can be established in such fashion, let us give the following one because of its importance.

Theorem 4. *The polynomials* $p(z)$ *with*

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \log^+ |p(n)| \leqslant \eta$$

for $\eta > 0$ *small enough form a normal family in* $\mathbb{C}$.

Proof. As in the proof of Theorem 1 we first reduce the general statement to one involving polynomials $P(z)$ with $P(0) = 1$ and

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \log^+ |P(n)| < 4\eta. \tag{71}$$

That having been done, we note that for integers $m \geqslant 1$ we have

$$\sum_{m^2 \leqslant n < (m+1)^2} \frac{1}{n^{3/2}} \geqslant \int_{m^2}^{(m+1)^2} \frac{dt}{t^{3/2}} = \frac{2}{m(m+1)},$$

which ensures that for each such m there is an integer n_m, $m^2 \leqslant n_m < (m+1)^2$, with

$$\frac{2}{m(m+1)} \log^+ |P(n_m)| \leqslant \sum_{m^2 \leqslant n < (m+1)^2} \frac{1}{n^{3/2}} \log^+ |P(n)|.$$

Taking

$$\lambda_m = \sqrt{n_m},$$

we then have

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \log^+ |P(\lambda_m^2)| < 4\eta$$

by (71).

In terms of the *even* polynomial

$$Q(z) = P(z^2) \tag{72}$$

with $Q(0) = 1$, the last relation reads

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \log^+ |Q(\lambda_m)| < 4\eta. \tag{73}$$

Here the sequence $\{\lambda_m\}$ depends, of course, on the particular polynomial $P(z)$ figuring in (71), but we do have

$$1 \leqslant \lambda_1 < 2 \leqslant \lambda_2 < 3 \leqslant \lambda_3 < 4 \leqslant \dots .$$

If $\eta > 0$ is small enough, (73) therefore implies that

$$\log |Q(z)| \leqslant C\eta |z|$$

(with a *numerical* constant C) according to the corollary on p. 516 of [1] or, rather, its *extension* noted in Remark 2 on p. 518 of that book.

Referring to (72), we see that

$$|P(z)| \leqslant e^{C\eta\sqrt{|z|}}, \tag{74}$$

and the theorem is thus proved.

Remark. I have been unable to construct an example like the one in §4 which would show that the *smallness* condition on η is *necessary* in this last result. I suspect that it is *not*, and that Theorem 4 is valid for any η, but I cannot see how to that using the methods of the present article. I think that this could be done if one followed the more thoroughgoing procedure of [8] based on use of the least superharmonic majorant; however, I have not tried that.

Whether or not Theorem 4 can be improved, it does lead to a complete solution of the Bernstein approximation problem on

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

In that problem we are given a function $W(n) \geqslant 1$ on $\mathbb{N}_0$ with

$$\frac{p(n)}{W(n)} \to 0 \quad \text{as} \quad n \to \infty \tag{75}$$

for all polynomials $p(z)$, and asked whether the expressions $p(n)/W(n)$ formed from such polynomials are *uniformly dense* in $\mathcal{C}_0(\mathbb{N}_0)$.

Our solution is in terms of the values of W's *Akhiezer minorant*, W_*, on $\mathbb{N}_0$. That object is actually defined for all complex z by means of the formula

$$W_*(z) = \sup\{|p(z)|;\ p(z) \quad \text{a polynomial and} \quad |p(n)| \leqslant W(n) \quad \text{for} \quad n \in \mathbb{N}_0\}. \tag{76}$$

We note that since 1 is a polynomial (!), the relation $W_*(z) \geqslant 1$ always holds.

Theorem 5. *Assume that the condition* (75) *is satisfied. Then the rations $p(n)/W(n)$ formed from polynomials $p(z)$ are uniformly dense in $\mathcal{C}_0(\mathbb{N}_0)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \log W_*(n) = \infty \tag{77}$$

where W_* is given by (76).

Proof. To show that (77) is *necessary* for the uniform density, we assume that the sum appearing there is *finite* and verify that *then* the expressions $p(n)/W(n)$, with polynomials $p(n)$, *cannot* uniformly approximate the function

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, \ldots \end{cases}$$

on $\mathbb{N}_0$. Here there is no loss of generality in assuming that $W(0) = 1$, *and we do that.*

If the approximation in question *were* possible, we would have a sequence of polynomials $P_l(z)$ with

$$P_l(0) \to 1$$

and

$$\frac{P_l(n)}{W(n)} \to 0 \quad \text{uniformly for} \quad n = 1, 2, \ldots \tag{78}$$

as $l \to \infty$. Then we could just as well take each value $P_l(0)$ to be 1, and would certainly have $|P_l(n)| \leqslant W(n)$ for $n = 0, 1, 2, \ldots$ and large l. That would in turn make $|P_l(n)| \leqslant W_*(n)$ by (76). And this together with (78) would imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \log^+ |P_l(n)| \to 0 \quad \text{as} \quad l \to \infty \tag{79}$$

by the dominated convergence theorem, the sum in (77) being finite.

Theorem 4 and (79) now ensure that the $P_l(z)$ form a normal family in $\mathbb{C}$, so a subsequence of them must tend to a certain entire function $\Phi(z)$ with

$$\Phi(n) = 0 \quad \text{for} \quad n = 1, 2, \ldots \tag{80}$$

by (78), whereas $\Phi(0) = 1$ since each $P_l(0) = 1$. The latter property also enables us to apply (74) to the $P_l(z)$ and that, with (79), shows that

$$|\Phi(z)| \leqslant e^{o(\sqrt{|z|})}.$$

The last relation and (80) now imply that $\Phi(z) \equiv 0$ by Jensen's formula. However, $\Phi(0) = 1$, and we have a contradiction showing that the uniform approximation of $\delta(n)$ on $\mathbb{N}_0$ is impossible.

To establish *sufficiency* of (77) for the uniform density in question, we assume that the $p(n)/W(n)$ formed from polynomials $p(Z)$ are *not* uniformly dense in $C_0(\mathbb{N}_0)$, and show that *then* the sum in (77) *is finite*. That can be deduced from a known result by making a change of variable.

Taking the set

$$E = \{\pm\sqrt{n};\, n \in \mathbb{N}_0\}$$

we put

$$\Omega(\pm\sqrt{n}) = W(n) \tag{81}$$

and consider the Bernstein approximation problem on E involving the fractions $q(\lambda)/\Omega(\lambda)$, $\lambda \in E$, formed from polynomials $q(w)$. It is claimed that under the present circumstances, *these functions cannot be uniformly dense in* $\mathcal{C}_0(E)$.

Here we are *assuming* that there is a function $\theta(n)$ in $\mathcal{C}_0(\mathbb{N}_0)$ which cannot be uniformly approximated on $\mathbb{N}_0$ by ratios $p(n)/W(n)$ formed from polynomials $p(z)$. Take, then, the *even* function $\phi(\lambda) \in \mathcal{C}_0(E)$ defined by the formula

$$\phi(\lambda) = \theta(\lambda^2), \quad \lambda \in E. \tag{82}$$

If $\phi(\lambda)$ *could be* uniformly approximated on E from expressions $q(\lambda)/\Omega(\lambda)$ formed from polynomials $q(w)$, we could, since $\phi(\lambda)$ and $\Omega(\lambda)$ are both even, take those $q(w)$ to *also* be even. But then each of those $q(w)$ would be of the form $p(w^2)$ for some polynomial $p(z)$, making, by (81) and (82),

$$\left|\frac{p(\lambda^2)}{W(\lambda^2)} - \theta(\lambda^2)\right| = \left|\frac{|q(\lambda)}{\Omega(\lambda)} - \phi(\lambda)\right| \quad \text{for} \quad \lambda \in E.$$

Here λ^2 ranges over $\mathbb{N}_0$ when λ runs through E so, if we could get even polynomials $q(\lambda)$ rendering the quantity on the right uniformly small there, the corresponding fractions $p(n)/W(n)$ would approximate $\theta(n)$ uniformly, and as closely as we like, on $\mathbb{N}_0$, contrary to our choice of θ.

Having thus verified that the $q(\lambda)/\Omega(\lambda)$ formed from polynomials $q(w)$ are *not* uniformly dense in $\mathcal{C}_0(E)$, we take Ω's Akhiezer minorant

$$\Omega_*(z) = \sup\{|q(z)|; \quad q(w) \quad \text{a polynomial and} \quad |q(\lambda)| \leqslant \Omega(\lambda) \quad \text{on} \quad E\} \tag{83}$$

and recall that the nondensity implies that

$$\int_{-\infty}^{\infty} \frac{\log \Omega_*(t)}{1+t^2} dt < \infty \tag{84}$$

(see [1,p. 158]). If $p(z)$ is any polynomial with $p(n) \leqslant W(n)$ on $\mathbb{N}_0$, we have $|q(\lambda)| \leqslant \Omega(\lambda)$ on E for the polynomial $q(w) = p(w^2)$ and thence, by (83) and since $\Omega_*(z) \geqslant 1$ (as is the case for W_*),

$$\log^+ |p(w^2)| = \log^+ |q(w)| \leqslant \log \Omega_*(w).$$

From this we get, using Poisson's integral twice and Fubini's theorem (cf. [1, p. 141]),

$$\log|p(n)| = \log|q(\sqrt{n})| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^+|q(x+i)|}{(x-\sqrt{n})^2+1}dx$$

$$\leqslant \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x-\sqrt{n})^2+1}\cdot\frac{\log^+|q(t)|}{(t-x)^2+1}dtdx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2\log^+|q(t)|}{(t-\sqrt{n})^2+4}dt \leqslant \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\log\Omega_*(t)}{(t-\sqrt{n})^2+1}dt$$

for $n \in \mathbb{N}_0$.

Allowing $p(z)$ to range over the polynomials with $|p(n)| \leqslant W(n)$ on $\mathbb{N}_0$, we find from the last relation and (76) that

$$\log W_*(n) \leqslant \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\log\Omega_*(t)}{(t-\sqrt{n})^2+1}dt, \quad n \in \mathbb{N}_0,$$

and thence that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\log W_*(n) \leqslant \frac{2}{\pi}\int_{-\infty}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\cdot\frac{1}{(t-\sqrt{n})^2+1}\right\}\log\Omega_*(t)dt.$$

It is left for the reader to verify that the sum in { } appearing in this last integral is $\leqslant \text{const.}/(t^2+1)$. Finiteness of the sum in (77) is thus a consequence of (84).

We are done.

Corollary. *Suppose that* (75) *holds for the function* $W(n) \geqslant 1$ *defined on* $\mathbb{N}_0$. *In order that there exist a nonzero function*

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

analytic for $|z| < 1$ *and continuous, together with all of its derivatives, up to* $|z| = 1$, *such that*

$$\sum_{n=0}^{\infty} W(n)|a_n| < \infty \tag{85}$$

and

$$F^{(n)}(1) = 0 \quad for \quad n = 0, 1, 2, \ldots, \tag{86}$$

it is necessary and sufficient that the sum in (77) *be finite.*

Proof. According to the theorem, finiteness of the sum in (77) is equivalent, *by duality*, to the existence of a *nonzero* sequence $\{c_n\}$, $n = 0, 1, 2, \ldots$, with

$$\sum_{n=0}^{\infty} |c_n| < \infty$$

and

$$\sum_{n=0}^{\infty} \frac{n^K}{W(n)} c_n = 0 \quad \text{for} \quad K = 0, 1, 2, \ldots.$$

With $a_n = c_n/W(n)$, (85) and (86) taken together, are equivalent to these last two relations.

7. Provided that (38) holds, Theorem 1 remains valid for certain entire functions $p(z)$ of order $\leqslant 1/2$; the actual limit on their rate of growth depends on the behaviour of the function $\psi(t)$ obtained during the extraction of the subsequence $\{\underline{\zeta}_n\}$. For this reason the precise wording of a general version of Theorem 1 applying to such entire functions would be rather cumbersome, and I think it is better here to just explain how the extension is carried out, refraining from concrete formulations.

The reader should refer to §3 and the proof of Theorem 1. With (38) being granted, passage from an entire function $p(z)$ to another, $P(z)$, having practically the same growth as $p(z)$ but with $P(0) = 1$ and for which (42) holds, proceeds exactly as at the beginning of that proof. We may therefore confine out discussion to such functions $P(z)$.

If $P(z)$ is (like $p(z)$) of order $\leqslant 1/2$, it has a representation like (43), but with now a convergent infinite product on the right. With the α_k from (45), the infinite product in (46) also converges, and the entire function $F(z)$ thus yielded will satisfy (47) provided that (42) holds.

If now the quantities $\beta_k = \psi(\alpha_k)$ tend sufficiently rapidly to ∞, the product in (49) will also converge, and for the entire function $Q(w)$ defined by that relation we will again have (50) and thus finally (52), thanks to (47).

In these circumstances, we can invoke a result from [4] if $Q(w)$ is known to be of exponential type $< \pi$. According to that result, the entire functions $Q(w)$ of exponential type $\leqslant A < \pi$ satisfying (52) with $\eta > 0$ sufficiently small (depending, now, on A) *form a normal family in* $\mathbb{C}$. For these we indeed have

$$|Q(w)| \leqslant C_\eta e^{A|\Im w| + \varepsilon(\eta)|w|}, \tag{87}$$

with C_η and $\varepsilon(\eta)$ depending on A and η, and $\varepsilon(\eta)$ tending to 0 with η for fixed A (see [4, Theorem 3, p. 453]).

In order to be sure that $Q(w)$, given by (49), is of exponential type $\leqslant A$ we practically need to have

$$\limsup_{s\to\infty}\frac{N(s)}{s}\leqslant\frac{A}{\pi}, \tag{88}$$

where $N(s)$ is the number of β_k in the interval $[0,s]$. *When this holds with* $A<\pi$, (52) *will, for sufficiently small* $\eta>0$, *imply* (87).

In terms of $\nu(R)$, the number of zeros a_k of $P(z)$ in the disk $|z|\leqslant R$, (88) is equivalent to the relation

$$\limsup_{R\to\infty}\frac{\nu(R)}{\psi(\sqrt{R})}\leqslant\frac{A}{\pi}. \tag{89}$$

Depending on the function $\psi(t)$ this, with the help of Jensen's formula, can be ensured by *imposing a corresponding limitation on the growth of* $P(z)$. *Something is usually lost* in doing that, and, since $\psi(t)/t$ decreases, the procedure will, *at best*, only work for *certain* entire functions $P(z)$ of order $1/2$.

Suppose, then, that we have precisely worked out an order of growth for our functions $P(z)$ which will guarantee (89), with $\psi(t)$ corresponding to $\{\underline{\zeta}_n\}$ and $A<\pi$. Then, to the $P(z)$ satisfying (42) with sufficiently small $\eta>0$ and having at most *that* order of growth *there will correspond*, according to (43), (45), (48) and (49), functions $Q(w)$ satisfying (87). From (87) we obtain, by use of Jensen's formula, a *pointwise* upper bound on $N(s)/s$, $s>0$, and thus finally *the same* bound on $\nu(R)/\psi(\sqrt{R})$ for $R>0$. And from that bound we readily arrive at an estimate like (55) on the individual $P(z)$, and thus show that they form a normal family.

8. The extensions of Theorem 1 furnished by the procedure of the last section are, as noted there, limited to certain entire functions $p(z)$ *of order at most* $1/2$, and without making some further assumption about the configuration of the $\underline{\zeta}_n$, it is impossible to pass that limit. Consider, for instance, the case where $\underline{\zeta}_n=n^2$, $n=1,2,\ldots$, with the functions

$$p_l(z)=l\frac{\sin\pi\sqrt{z}}{\sqrt{z}},\quad l=1,2,\ldots;$$

here the $p_l(z)$ are entire and of order $1/2$, but $p_l(0)\to\infty$ as $l\to\infty$ while

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\log^+|p_l(\underline{\zeta}_n)|=0$$

for all l.

For initial sequences $\{\zeta_n\}$ possessing some symmetry, it is possible to get results valid for entire functions $p(z)$ of order $> 1/2$. It is already enough to assume, for instance, *that* $\{\zeta_n\}$ *consists of points* $\pm\zeta_m$ *with* $|\zeta_m| \geqslant 1$.

In that situation, we start by repeating the construction of §2, *taking, however,* $r_n = |\zeta_n|$ *instead of* $r_n = \sqrt{|\zeta_n|}$ as we did there. Having then arrived as before at the subsequence $\{\underline{r}_n\}$ of $\{r_n\}$, with

$$1 \leqslant \frac{\underline{r}_1}{1} \leqslant \frac{\underline{r}_2}{2} \leqslant \dots,$$

we choose *two* points ζ_n, *one the negative of the other*, having modulus $\underline{r}_1$, and label them $\underline{\zeta}_1$, $\underline{\zeta}_{-1}$. After that we choose two such points of modulus $\underline{r}_2$ which we label $\underline{\zeta}_2$, $\underline{\zeta}_{-2}$, and then continue in this fashion, obtaining a two-sided sequence $\{\underline{\zeta}_n\}$, $n = \pm 1, \pm 2, \dots$, consisting entirely of points ζ_m from the original sequence and with $\underline{\zeta}_{-n} = -\underline{\zeta}_n$. For the increasing function $\psi(t)$ bearing the same relation to $\{\underline{r}_n\}$ as in §2 *we now have* $\psi(|\underline{\zeta}_n|) = \psi(\underline{r}_n) = |n|$ for $n = \pm 1, \pm 2, \dots$, and, for the inverse function $\psi(s)$ to $\psi(t)$, $\phi(|n|) = |\underline{\zeta}_n|$.

A simple modification of the argument made in §3 can now be applied to the examination of logarithmic sums over the $\underline{\zeta}_n$. *Assuming that* (38) *holds*, we start with an entire function $p(z)$ satisfying

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^2} \log^+ |p(\underline{\zeta}_n)| \leqslant \eta \tag{90}$$

for some $\eta > 0$ and having growth limited in a way to be described later on. From $p(z)$ we then form the entire functions

$$P_+(z) = 1 + \frac{z^2}{M}(p(z) + p(-z))$$

and

$$P_-(z) = 1 + \frac{z}{M}(p(z) - p(-z)),$$

both even, and *both equal to* 1 *at the origin*. The quantity $M \geqslant 1$ is here chosen so large, depending on $\eta > 0$ and $\{\underline{\zeta}_n\}$, as to make *both* sums,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P_+(\underline{\zeta}_n)|, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P_-(\underline{\zeta}_n)|,$$

less than 4 η; thanks to (38) that is possible, and the verification, using (90) and the property $\underline{\zeta}_{-n} = -\underline{\zeta}_n$, proceeds as at the beginning of the proof of Theorem 1. We have

$$p(z) = \frac{M}{2z^2}(P_+(z) - 1) + \frac{M}{2z}(P_-(z) - 1),$$

so it suffices, for our purposes, to obtain estimates involving only η and $|z|$ on $|P_+(z)|$ and $|P_-(z)|$ when each of the preceding two sums is bounded by 4 η.

When $p(z)$ is of order $\leqslant 1$, the same is true for both $P_+(z)$ and $P_-(z)$. Taking, then, *either one of these and denoting it* by $P(z)$ we have, in place of (43) the Hadamard factorization

$$P(z) = \prod_k \left(1 - \frac{z^2}{a_k^2}\right) \tag{91}$$

with certain $a_k \neq 0$ arranged in order of increasing modules.

Put now

$$F(z) = \prod_k \left(1 - \frac{z^2}{|a_k|^2}\right)$$

which makes

$$|F(|\underline{\zeta}_n|) \leqslant |P(\underline{\zeta}_n)|, \tag{92}$$

and then, with

$$\beta_k = \psi(|a_k|), \tag{93}$$

write

$$Q(w) = \prod_k \left(1 - \frac{w^2}{\beta_k^2}\right), \tag{94}$$

assuming, for the moment, that the product on the right *converges*. If that is so, then we can use the (present) relation $\phi(|n|) = |\underline{\zeta}_n|$ to argue, much as in §3, that

$$|Q(n)| \leqslant |F(|\underline{\zeta}_n|)| \quad \text{for} \quad n = 1, 2, \ldots.$$

With (92), this yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |Q(n)| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P(\underline{\zeta}_n)| < 4\eta \tag{95}$$

for $P(z)$ equal to either $P_+(z)$ or $P_-(z)$, whenever our original function $p(z)$ satisfies (90).

The legitimacy of this last step depends on the convergence of the infinite product in (94) and thus on that of the series

$$\sum_{k=1}^{\infty} \frac{1}{\beta_k^2} = \sum_{k=1}^{\infty} \frac{1}{(\psi(|a_k|))^2},$$

for which a certain rate of growth must be required of the a_k appearing in (91). If, moreover, we are to deduce, *as in the preceding* section, an estimate for $Q(w)$ of the form (87) from (95), *we need to know* that $Q(w)$ is of exponential type $\leqslant$ some $A < \pi$, and that can be *assured*, as in that section, *by requiring* (88) *to hold*, where $N(s)$ again denotes the number of β_k in $[0, s]$.

In view of (93), that condition in (now) equivalent to

$$\limsup_{R\to\infty} \frac{\nu(R)}{\psi(R)} \leqslant \frac{A}{\pi}, \tag{96}$$

where $\nu(R)$ denotes the number of a_k with $|a_k| \leqslant R$. Referring to (91), we see that (96) in turn is guaranteed by an appropriate limitation, depending on $\psi(t)$ and A, on the growth of $P(z)$. Finally, with $P(z)$ equal to $P_+(z)$ or $P_-(z)$, *that* limitation can be ensured by making the right one on the growth of our original function $p(z)$ from which those two were formed.

Subject to *such* a condition on the growth of an entire function $p(z)$ satisfying (90) with a sufficiently small $\eta > 0$ we can now, *as in* §7, infer the estimate (87) for the function $Q(w)$ corresponding to *either* of the above two choices of $P(z)$. Thence we obtain, proceeding as indicated in §7, a pointwise upper bound on $N(s)/s$, $s > 0$, and thus, by (93), *the same* pointwise bound on $\nu(R)/\psi(R)$. And that result, combined with the relation

$$\log|P(z)| \leqslant 2\int_0^\infty \frac{|z|^2}{|z|^2+t^2}\cdot\frac{\nu(t)}{t}\,dt,$$

following from (91), yields an estimate for $|P(z)|$ in terms of η, A and $|z|$. Which, holding as it does for both $P_+(z)$ and $P_-(z)$, is all we needed.

Since $\psi(t)/t$ is decreasing, Jensen's formula shows that *the best* we can hope for regarding the permissible growth of a function $P(z)$ for which (96) is to hold is that $P(z)$—and therefore the function $p(z)$ from which it originates—be *of exponential type*, indeed, of *sufficiently small* type. That much growth is only allowed when $\psi(t)/t \geqslant c > 0$, and even then, (90) will hardly be satisfied by any $p(z)$ of greater than *zero* exponential type unless all the $\underline{\zeta}_n$ lie quite close to some straight line.

When $\psi(t)$ increases *less rapidly* than t, the condition (96) obliges us to impose a further restriction on the growth of $p(z)$. Depending on ψ, functions $p(z)$ of order between $1/2$ and 1, not amenable to the treatment of §7, can then still be accommodated.

As in §7, we do not enter here into the detailed formulation, involving $\psi(t)$ and a corresponding growth limitation on the functions $p(z)$, of a general result to which the preceding considerations would lead concerning the $p(z)$ fulfilling (90). We do, however, record a version for *polynomials*. In the case of these we no longer need to assume (38), for the result holds trivially when that condition is violated.

Theorem 6. *Let the sequence* $\{\zeta_n\}$ *and its subsequence* $\{\underline{\zeta}_n\}$, $n = \pm 1, \pm 2, \ldots$, *be as described near the beginning of this section. Then the polynomials* $p(z)$ *which, for* $\eta > 0$ *sufficiently small, satisfy*

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^2} \log^+ |p(\underline{\zeta}_n)| \leqslant \eta$$

form a normal family in $\mathbb{C}$.

Remark 1. On taking $\underline{\zeta}_n = |n|^q \operatorname{sgn} n$ with $q \geqslant 1$ for $n = \pm 1, \pm 2, \ldots$, this theorem gives us the result stated at the end of §1.

Remark 2. When $\phi(s)/s^K$ (with now $\phi(|n|) = |\underline{\zeta}_n|$) is *decreasing* for some $K > 0$, *the smallness condition on* η *cannot be dropped* in Theorem 6. That is seen as in §4, but working now with the polynomials

$$p_l(z) = (1 - z^2)^{m_l} \prod_{k=1}^{l} \left(1 - \frac{z^2}{\underline{\zeta}_n^2}\right),$$

where the integers m_l go to ∞ slowly with l.

9. Thinking of the application made towards the end of §6, one finds it natural to try using the material of the preceding section to study the Bernstein approximation problem on *complex* sequences of points symmetric with respect to the origin, and that is indeed feasible. Unfortunately, this approach usually yields only a *necessary* condition.

With $\{\underline{\zeta}_n\}$, $n = \pm 1, \pm 2, \ldots$, one of the subsequences figuring in the discussion of the last section, let us look at the Bernstein approximation problem on the set $\{0\} \cup \{\underline{\zeta}_n\}$. For this it is convenient—*and we do that throughout this section*—to simply *adjoin* 0 to $\{\underline{\zeta}_n\}$ by putting

$$\underline{\zeta}_n = 0.$$

We take, then, a function $W(\underline{\zeta}_n) \geqslant 1$, defined on the set $\{\underline{\zeta}_n;\ n \in \mathbb{Z}\}$ and such that

$$\frac{P(\underline{\zeta}_n)}{W(\underline{\zeta}_n)} \to 0 \quad \text{as} \quad n \to \pm\infty \tag{97}$$

for every polynomial $p(z)$. We wish to see whether or not the ratios $p(\underline{\zeta}_n)/W(\underline{\zeta}_n)$ formed from such polynomials are uniformly dense in $C_0(\{\underline{\zeta}_n;\ n \in \mathbb{Z}\})$.

This idea is proceed here as in §6, forming first the Akhiezer minorant

$$W_*(z) = \sup\{|p(z)|;\ p(z) \quad \text{a polynomial and} \quad |p(\underline{\zeta}_n)| \quad \text{for} \quad n \in \mathbb{Z}\}. \tag{98}$$

As before, $W_*(z) \geqslant 1$, and we can prove

Theorem 7. *Given* $W(\underline{\zeta}_n) \geqslant 1$ *satisfying* (97), *the expressions* $p(\underline{\zeta}_n)/W(\underline{\zeta}_n)$ *formed from polynomials* $p(z)$ *fail to be uniformly dense in* $\mathcal{C}_0(\{\underline{\zeta}_n;\ n \in \mathbb{Z}\})$ *when*

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^2} \log W_*(\underline{\zeta}_n) < \infty \tag{99}$$

for W_* *defined by* (98).

Proof. The argument here is much like the one made to show necessity in the proof of Theorem 5. Again there is no loss of generality in taking $W(0)$ to be 1.

Supposing, then, the expressions in question *to be* uniformly dense in $\mathcal{C}_0(\{\underline{\zeta}_n;\ n \in \mathbb{Z}\})$, we would get a sequence of polynomials $p_l(z)$ with $p_l(0) = 1$ but

$$\frac{p_l(\underline{\zeta}_n)}{W(\underline{\zeta}_n)} \to 0 \quad \text{uniformly as} \quad l \to \infty \tag{100}$$

for $n = \pm 1, \pm 2, \ldots$. We would then have in particular $|p_l(\underline{\zeta}_n)| \leqslant W(\underline{\zeta}_n)$ for $n \in \mathbb{Z}$ and large l, making

$$|p_l(z)| \leqslant W_*(z) \tag{101}$$

by (98).

For the *even* polynomials

$$P_l(z) = \frac{1}{2}(p_l(z) + p_l(-z)) \tag{102}$$

we would thus have $P_l(0) = 1$ and

$$|P_l(\underline{\zeta}_n)| \leqslant \max(W_*(\underline{\zeta}_n),\quad W_*(\underline{\zeta}_{-n})) \tag{103}$$

for large l, by (101) and the relation $\underline{\zeta}_{-n} = -\underline{\zeta}_n$. This relation and (100) also ensure that

$$P_l(\underline{\zeta}_n) \to 0 \quad \text{as} \quad l \to \infty \tag{104}$$

for each n, $n = \pm 1, \pm 2, \ldots$. Since, finally,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log \max(W_*(\underline{\zeta}_n),\quad W_*(\underline{\zeta}_n)) < \infty$$

by (99), we have, recalling that $W_*(\zeta_{\underline{n}}) \geqslant 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |P_l(\zeta_{\underline{n}})| \to 0 \quad \text{as} \quad l \to \infty \tag{105}$$

by (103), (104) and dominated convergence.

Instead of directly invoking Theorem 6 at this point, we note that each of the even polynomials $P_l(z)$ has a product representation of the form (91), which makes it possible for us to pass, using (93), from $P_l(z)$ to a polynomial $Q_l(w)$ given by a formula like (94) and satisfying

$$|Q_l(n)| \leqslant |P_l(\zeta_{\underline{n}})| \quad \text{for} \quad n = 1, 2, \ldots .$$

When $l \to \infty$ we will therefore have

$$Q_l(n) \to \quad \text{for} \quad n = 1, 2, \ldots$$

by (104), whereas

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \log^+ |Q_l(n)| \to 0$$

by (105). These last two relations, however, are *incompatible* for even polynomials $Q_l(w)$ with, as here, $Q_l(0) = 1$; that follows as on p. 525 of [1] from the corollary given there on p. 516—cf. also the end of the proof of necessity for Theorem 5. This contradiction shows that the ratios $p(\zeta_{\underline{n}})/W(\zeta_{\underline{n}})$ *cannot*, as we had assumed, be uniformly dense in $\mathcal{C}_0(\{\zeta_{\underline{n}};\, n \in \mathbb{Z}\})$. We are done.

For $\zeta_{\underline{n}} = n$, $n \in \mathbb{Z}$, Theorem 7 has a valid converse ([1, p. 523]), but if $|\zeta_n| \to \infty$ more rapidly than $|n|$, that no longer need be true. In this circumstance, passage from the $P_l(z)$ to the $Q_l(w)$ in the preceding argument involves some loss, so failure of the converse is perhaps not so surprising. The matter is nevertheless somewhat puzzling, as is illustrated by the simple example where

$$\zeta_{\underline{n}} = n^3, \quad n \in \mathbb{Z}. \tag{106}$$

We *would like*, in this situation, to have a necessary and sufficient condition expressed, as in Theorem 5, *terms of the values of the Akhiezer minorant $W_*(z)$ at the points n^3 of our sequence*. Here, although the condition (99), which now reads

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^2} \log W_*(n^3) < \infty, \tag{107}$$

implies, by the last theorem, that the $p(n^3)/W(n^3)$ formed from polynomials $p(z)$ are *not* uniformly dense in $\mathcal{C}_0(\{n^3;\ n \in \mathbb{Z}\})$, it is *far from* being *implied* by that property.

To see this let us consider the function

$$W(n^3) = e^{c|n|} \tag{108}$$

where c is a constant > 0; we proceed to estimate $W_*(n^3)$ *from below*. For positive integers k, the *monomials* (!)

$$P_k(\xi) = \left(\frac{ce}{3k}\right)^{3k} \cdot \xi^k$$

satisfy $|P_k(\xi^3)| \leqslant e^{c|\xi|}$ on $\mathbb{R}$, so we surely have $|P_k(n^3)| \leqslant W(n^3)$ for $n \in \mathbb{Z}$ by (108), and therefore

$$|P_k(n^3)| \leqslant W_*(n^3), \quad n \in \mathbb{Z},$$

according to (98). For $\xi = n^3$ with $c|n| \geqslant 3$ we put $k = [c|n|/3]$, yielding

$$|P_k(n^3)| = e^{3[c|n|/3]}\left(\frac{c|n|/3}{[c|n|/3]}\right)^{3[c|n|/3]} > e^{c|n|-3},$$

whence

$$W_*(n^3) > e^{c|n|-3}, \quad |n| \geqslant \frac{3}{c},$$

and *the sum in* (107) *diverges*.

Nevertheless, if $0 < c < \sqrt{3}\pi$ *in* (108), *the expressions* $p(n^3)/W(n^3)$ *are not uniformly dense in* $\mathcal{C}_0(\{n^3;\ n \in \mathbb{Z}\})$. That can be verified using the entire function

$$C(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^6}\right). \tag{109}$$

The asymptotic behaviour of such expressions in the right half-plane was examined in §1; however, the reader who does not wish to refer to that material can simply not that

$$C(x^3) = \frac{\sin \pi x}{\pi x} \cdot \frac{\sin \pi\gamma x}{\pi\gamma x} \cdot \frac{\sin \pi\gamma^2 x}{\pi\gamma^2 x}$$

with $\gamma = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, which follows easily from the Euler product formula for $\sin \pi$. From the last representation one readily deduces that

$$\frac{1}{|C'(n^3)|} \sim 12\pi^2|n|^5 e^{-\sqrt{e\pi|n|}} \tag{110}$$

for $n \to \pm\infty$.

With the help of (110), it is not hard to justify the Lagrange interpolation formula

$$p(z) = C(z) \sum_{n=-\infty}^{\infty}{}' \frac{p(n^3)}{C'(n^3)(z - n^3)}$$

for polynomials $p(z)$ which, on taking successively $p(z) = z, z^2, z^3, \ldots$ and putting each time $z = 0$, yields

$$\sum_{n=-\infty}^{\infty}{}' \frac{n^{3l}}{C'(n^3)} = 0, \quad l = 0, 1, 2, \ldots. \tag{111}$$

From (108) with $0 < c < \sqrt{3}\pi$ and (110), we now have

$$\sum_{n=-\infty}^{\infty} \frac{W(n^3)}{|C'(n^3)|} < \infty,$$

so, since (111) can be rewritten in the form

$$\sum_{n=-\infty}^{\infty}{}' \frac{n^{3l}}{W(n^3)} \cdot \frac{W(n^3)}{C'(n^3)} = 0, \quad l = 0, 1, 2, \ldots,$$

the ratios $p(n^3)/W(n^3)$ *formed from polynomials* $p(z)$ *cannot be uniformly dense in* $\mathcal{C}_0(\{n^3;\ n \in \mathbb{Z}\})$.

In §6 Theorem 4, although probably *not* best possible, *did* give us complete solution furnished by Theorem 5 for the Bernstein approximation problem on $\mathbb{N}_0$. But *here* the argument of §7 which led there to Theorem 6, best possible for $\zeta_n = n^3$, has given only a *partial result* about that problem on $\{n^3;\ n \in \mathbb{Z}\}$. Reason for this curious anomaly must lie in the fact that the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \cdot \frac{1}{(t - \sqrt{n})^2 + 1}$$

figuring at the end of the proof of Theorem 5 is bounded above by const./$(t^2 + 1)$, whereas that is no longer true of the sum

$$\sum_{n=-\infty}^{\infty}{}' \frac{1}{n^2} \cdot \frac{1}{(t - n^3)^2 + 1}.$$

More partial results about the Bernstein approximation problem on $\{n^3;\, n \in \mathbb{Z}\}$ are known. If, for instance, $W(n^3)$ is given by (108) with $c > \sqrt{3}\pi$, the expressions $p(n^3)/W(n^3)$ formed from polynomials $p(z)$ *are* uniformly dense in $\mathcal{C}_0(\{n^3;\, n \in \mathbb{Z}\})$. That follows from a result in §4 of [13]. The discussion in Appendix 2 of [11] and Fryntov's paper [14] are also relevant here.

But for $W(n^3)$ an *arbitrary* function $\geqslant 1$ satisfying (97), we still have no general criterion *in terms of the values* $W_*(n^3)$ for deciding about the uniform density of the $p(n^3)/W(n^3)$ in $\mathcal{C}_0(\{n^3;\, n \in BZ\})$. If $W(n^3)$ is *even* and the expressions in question are *not* uniformly dense, one can show using de Branges'theorem that

$$W_*(n^3) \leqslant \text{const}\,.e^{c'|n|} \tag{112}$$

with a certain numerical constant c'. I do not bother to give here the value of c'; it is at any rate $> \sqrt{3}\pi$, the critical value of c for $W(n^3)$ of the form (108). The above discussion of the latter would *seem* to suggest, with (112), that for the "borderline" $W(n^3)$—at least for the *even* ones—$W_*(n^3)$ should lie between two expressions of the form $\text{const}\,.e^{a|n|}$, $\text{const}\,.e^{b|n|}$, where $a < \sqrt{3}\pi < b$. But that does not give us a very good idea of what a precise condition involving $W_*(n^3)$—*if there is one*—should look like. The state of affairs in this simple situation is far from clear.

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References

[1] Koosis P., *The logarithmic integral*. I, Cambridge Stud. Adv. Math., vol. 12, Cambridge Univ. Press, Cambridge, 1998.

[2] Pedersen H. L., *Uniform estimates of entire functions by logarithmic sums*, J. Funct. Anal. **146** (1997), 517–556.

[3] Koosis P., Pedersen H. L., *Lower bounds on the values of an entire function of exponential type at certain integers, in terms of a least superharmonic majorant*, Алгебра и анализ **10** (1998), №3, 31–44.

[4] Koosis P., *A local estimate, involving the least superharmonic majorant, for entire functions of exponential type*, Алгебра и анализ **10** (1998), №3, 45–64.

[5] Pedersen H. L., *Entire functions having small logarithmic sums over certain discrete subsets*, Ark. Mat. **36** (1998), 119–130.

[6] Pedersen H. L., *Entire functions and logarithmic sums over nonsymmetric sets of the real line*, Ann. Acad. Sci. Fenn. Math. **25** (2000), 351–388.

[7] Pedersen H. L., *Estimates of entire functions of exponential type less than π in terms of logarithmic sums over real Duffin and Schaeffer sequences*, Preprint, Univ. Copenhagen, 2000.

[8] Koosis P., *Use of logarithmic sums to estimate polynomials*, Ann. Acad. Sci. Fenn. Math. **26** (2001) (в печати).

[9] Borichev A., Sodin M., *Krein's entire functions and the Bernstein approximation problem*, Preprint, Univ. Bordeaux and Univ. Tel-Aviv, 1999.

[10] Pedersen H. L., *A sequence of polynomials*, Preprint, Univ. Copenhagen, 2000.

[11] Borichev A., Sodin M., *The Hamburger moment problem and weighted polynomial approximation on discrete subsets of the real line*, J. Anal. Math. **76** (1998), 219–264.

[12] Ransford T., *Potential theory in the complex plane*, London Math. Soc. Stud. Texts, No. 28, Cambridge Univ. Press, Cambridge, 1995.

[13] Anderson J. M., Khavinson D., Shapiro H. S., *Analytic continuation of Dirichlet series*, Rev. Mat. Iberoamericana **11** (1995), 453–476.

[14] Фрынтов А., *Одна теорема единственности, связанная с весовой полиномиальной аппроксимацией на последовательности*, Мат. физ., анализ, геом. **1** (1994), №2, 252–264.

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