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Russian Universities Reports. Mathematics, 2021, Volume 26, Issue 135, 315–336

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ISSN 2686-9667. Вестник российских университетов. Математика

Том 26, № 135 2021

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DOI 10.20310/2686-9667-2021-26-135-315-336

Homogeneous spaces yielding solutions of the $k[S]$-hierarchy and its strict version

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Abstract. The $k[S]$-hierarchy and its strict version are two deformations of the commutative algebra $k[S]$, $k = \mathbb{R}$ or $\mathbb{C}$, in the $\mathbb{N}\times\mathbb{N}$-matrices, where S is the matrix of the shift operator. In this paper we show first of all that both deformations correspond to conjugating $k[S]$ with elements from an appropriate group. The dressing matrix of the deformation is unique in the case of the $k[S]$-hierarchy and it is determined up to a multiple of the identity in the strict case. This uniqueness enables one to prove directly the equivalence of the Lax form of the k[S]-hierarchy with a set of Sato–Wilson equations. The analogue of the Sato–Wilson equations for the strict $k[S]$-hierarchy always implies the Lax equations of this hierarchy. Both systems are equivalent if the setting one works in, is Cauchy solvable in dimension one. Finally we present a Banach Lie group $G(\mathcal{S}_2)$, two subgroups $P_+(H)$ and $U_+(H)$ of $G(\mathcal{S}_2)$, with $U_+(H) \subset P_+(H)$, such that one can construct from the homogeneous spaces $G(\mathcal{S}_2)/P_+(H)$ resp. $G(\mathcal{S}_2)/U_+(H)$ solutions of respectively the $k[S]$-hierarchy and its strict version.

Keywords: homogeneous spaces, integrable hierarchies, Lax equations, Sato–Wilson form, wave matrices

Mathematics Subject Classification: 22E65, 35Q58, 37K10, 58B25.

For citation: Helminck G.F., Weenink J.A. Odnorodnyye prostranstva, porozhdayushchiye resheniya iyerarkhii $k[S]$ i eye strogoy versii [Homogeneous spaces yielding solutions of the $k[S]$-hierarchy and its strict version]. *Vestnik rossiyskikh universitetov. Matematika – Russian Universities Reports. Mathematics*, 2021, vol. 26, no. 135, pp. 315–336. DOI 10.20310/2686-9667-2021-26-135-315-336.

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DOI 10.20310/2686-9667-2021-26-135-315-336
УДК 512.71, 512.56, 517.95

Однородные пространства, порождающие решения иерархии $k[S]$ и ее строгой версии

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Аннотация. Иерархия $k[S]$ и ее строгая версия представляют собой две деформации коммутативной алгебры $k[S]$, с $k = \mathbb{R}$ или $k = \mathbb{C}$, в пространстве $\mathbb{N} \times \mathbb{N}$ матриц, где S — матрица оператора сдвига. В работе показано, что обе деформации соответствуют сопряжению $k[S]$ элементами подходящей группы. При этом одевающая матрица деформации единственна в случае иерархии $k[S]$ и определяется с точностью до умножения на единичную в случае строгой иерархии $k[S]$. Эта единственность позволяет непосредственно доказать, что форма Лакса иерархии $k[S]$ равносильна семейству уравнений Сато–Вильсона. Аналог уравнений Сато–Вильсона для строгой иерархии $k[S]$ всегда приводит к уравнениям Лакса этой иерархии. Эти системы эквивалентны, если окружение, в котором рассматривается иерархия, разрешимо по Коши в одномерном пространстве. В работе также представлена банахова группа Ли $G(\mathcal{S}_2)$ и две ее подгруппы $P_+(H)$ и $U_+(H)$, где $U_+(H) \subset P_+(H)$, такие, что однородные пространства $G(\mathcal{S}_2)/P_+(H)$ и $G(\mathcal{S}_2)/U_+(H)$ дают решения иерархии $k[S]$ и ее строгой версии, соответственно.

Ключевые слова: однородные пространства, интегрируемые иерархии, уравнения Лакса, форма Сато–Вильсона, волновые матрицы

Для цитирования: *Хельминк Г.Ф., Вининк Д.А.* Однородные пространства, порождающие решения иерархии $k[S]$ и ее строгой версии // Вестник российских университетов. Математика. 2021. Т. 26. № 135. С. 315–336. DOI 10.20310/2686-9667-2021-26-135-315-336. (In Engl., Abstr. in Russian)

Dedicated to Alexander Ivanovich Bulgakov
at the anniversary of his 70-th birthday

Introduction

In [1] we introduced a wide collection of integrable hierarchies in the $\mathbb{N}\times\mathbb{N}$-matrices corresponding to deformations of various commutative Lie subalgebras of the algebra $LT_{\mathbb{N}}(k)$ of all $\mathbb{N}\times\mathbb{N}$-matrices with coefficients from k that possess only a finite number of nonzero diagonals above the central diagonal. In the present paper we focus on the subalgebra $k[S]=\{\sum_{i=0}^{N}a_iS^i \mid a_i\in k\}$ and prove some additional results for the $k[S]$-hierarchy and its strict version. We start by showing that $k[S]$ is a maximal commutative subalgebra of $LT_{\mathbb{N}}(k)$ and that each of the two deformations of S corresponds to conjugations with elements from its appropriate group, where the deforming group is larger for the strict version. In both cases the evolution equations of the deformed generator is a set of Lax equations for this generator and this deformed generator together with the set of Lax equations it satisfies forms an integrable hierarchy. The dressing matrix of the deformation turns out to be unique in the case of the $k[S]$-hierarchy and it is determined up to a multiple of the identity in the strict case. The uniqueness of the dressing matrix enables one to prove directly the equivalence of the Lax form of the k[S]-hierarchy with a set of Sato–Wilson equations. There exists an analogue of the Sato–Wilson equations for the strict $k[S]$-hierarchy. It always implies the Lax equations of this hierarchy. Both systems can be shown to be equivalent if the setting one works in, is Cauchy solvable in dimension one.

Solutions of both hierarchies are constructed by producing wave matrices in the linearization module of each hierarchy. Therefore we recall the essentials of this approach. We conclude by presenting a Banach Lie group $G(\mathcal{S}_2)$, the two subgroups $P_+(H)$ and $U_+(H)$ of $G(\mathcal{S}_2)$, $U_+(H)\subset P_+(H)$, and by giving the construction from the flag variety $G(\mathcal{S}_2)/P_+(H)$ of a wave matrix of the $k[S]$-hierarchy and from its cover $G(\mathcal{S}_2)/U_+(H)$ of a wave matrix of the strict $k[S]$-hierarchy.

The content of the various sections is as follows: Section 1. describes the scene of the deformations, the algebra $LT_{\mathbb{N}}(R)$, the maximality of $k[S]$ and the properties required later on. The next section is devoted to the description of the two deformations, it contains a discussion of the Lax equations they have to satisfy and we describe there the link with their Sato–Wilson form. The form of the relevant $LT_{\mathbb{N}}(R)$-module, the equations of the linearization and a characterization of the special vectors, called wave matrices, can all be found in Section 3. In the last section we present the homogeneous spaces $G(\mathcal{S}_2)/P_+(H)$ and $G(\mathcal{S}_2)/U_+(H)$ and show how to construct wave matrices of the $k[S]$-hierarchy and its strict version from them.

1. The algebra $LT_{\mathbb{N}}(R)$

Let R be a commutative k-algebra over the field k. We write $M_n(R)$ for the $n\times n$-matrices with coefficients from R and similarly $M_{\mathbb{N}}(R)$ for the space of $\mathbb{N}\times\mathbb{N}$-matrices with coefficients from R. The transpose A^T of any matrix $A\in M_{\mathbb{N}}(R)$ is defined as in the finite dimensional case. Let V be the R-module of all $1\times\mathbb{N}$-matrices with coefficients from R, i. e.

$$V=R^{\mathbb{N}}=\{\vec{x}=(x_j)=\begin{pmatrix}x_0 & x_1 & x_2 & \cdots\end{pmatrix}\mid x_j\in R \text{ for all } j\in\mathbb{N}\}.$$

Inside V we consider for each $i\in\mathbb{N}$ the R-submodules

$$V_{\leqslant i}=\{\vec{x}=(x_j)\mid x_j=0 \text{ for all } j>i\} \text{ and } V_{\text{fin}}=\cup_{i\in\mathbb{N}}V_{\leqslant i}.$$

The space V_{fin} is a free R-module with basis the $\{\vec{e}(i) \mid i \in \mathbb{N}\}$, where $\vec{e}(i)$ is the vector with the i-th coordinate equal to one and the remaining ones equal to zero. For each $\vec{x} \in V_{\text{fin}}$ and each $A \in M_{\mathbb{N}}(R)$ the product $\vec{x}A$ is well-defined and determines a vector in V. Hence, if we write

$$M_A(\vec{x}) := \vec{x}A,$$

then M_A is an R-linear map in $\operatorname{Hom}_R(V_{\text{fin}}, V)$. The subspace of $M_{\mathbb{N}}(R)$ that is central in this paper is the space $LT_{\mathbb{N}}(R)$ of all $A \in M_{\mathbb{N}}(R)$ that possess the property that there is an $m \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ there holds

$$M_A(V_{\leqslant i}) \subset V_{\leqslant i+m}. \tag{1.1}$$

Property (1.1) implies that $LT_{\mathbb{N}}(R)$ is an algebra w.r.t. matrix multiplication. Inside $LT_{\mathbb{N}}(R)$ there are some classes of basic matrices with their own notation: first of all there are the basic matrices $E_{(i,j)}$, i and $j \in \mathbb{N}$, whose matrix entries, in Kronecker notation, are given by

$$(E_{(i,j)})_{mn} = \delta_{im}\delta_{jn}.$$

It is convenient to use the notation $A = \sum_{n,m} a_{(i,j)}E_{(i,j)}$ for an $A = (a_{(i,j)}) \in LT_{\mathbb{N}}(R)$. The second class of matrices for which we introduce a special notation are the diagonal matrices. Let $\{d(s) \mid s \in \mathbb{N}\}$ be a set of of elements in R. Then the diagonal matrix $\operatorname{diag}(\,d(s)\,)$ in $M_{\mathbb{N}}(R)$ is given by

$$\operatorname{diag}(d(s)) := \sum_{s\in\mathbb{N}} d(s)E_{(s,s)} = \begin{pmatrix} d(0) & 0 & 0 & \dots \\ 0 & d(1) & 0 & \ddots \\ 0 & 0 & d(2) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The algebra of all diagonal matrices in $M_{\mathbb{N}}(R)$ is denoted by

$$\mathcal{D}_{\mathbb{N}}(R) = \{d = \operatorname{diag}(d(s)) \mid d(s) \in R \text{ for all } s \in \mathbb{N}\}.$$

One has a diagonal embedding i_1 from R into $\mathcal{D}_{\mathbb{N}}(R)$ by taking all diagonal coefficients of $i_1(r)$ equal to r, i. e.

$$i_1(r) = \begin{pmatrix} r & 0 & 0 & \dots \\ 0 & r & 0 & \ddots \\ 0 & 0 & r & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

A central role in this paper is played by the shift matrix S, its transpose S^T and their powers, where S is the matrix corresponding to the operator $M_S : V \to V$ defined by

$$M_S(\begin{pmatrix} x_0 & x_1 & x_2 & \cdots \end{pmatrix}) = \begin{pmatrix} 0 & x_0 & x_1 & x_2 & \cdots \end{pmatrix}.$$

Note that we have

$$SS^T = \mathrm{Id} \text{ and } S^TS = \sum_{i\geqslant 1} E_{(i,i)}. \tag{1.2}$$

Besides the expression in the basic matrices it is also convenient to have at one's disposal the decomposition of a matrix $A = (a_{ij}) \in M_{\mathbb{N}}(R)$ in its diagonals. If $m \geqslant 0$, then the m-th diagonal of A is by definition the matrix

$$d_m(A)S^m = \operatorname{diag}(a_{(s,s+m)})S^m = \sum_{i \geqslant 0} a_{(i,i+m)}E_{(i,i+m)}$$

and those diagonals are called *positive*. Similarly, for $m \leqslant 0$, the m-th diagonal of A is defined as the matrix

$$(S^T)^{-m}d_m(A) = (S^T)^{-m}\operatorname{diag}(a_{(s-m,s)}) = \sum_{i \geqslant 0} a_{(i-m,i)}E_{(i-m,i)}$$

and they are called *negative*. So each matrix $A \in M_{\mathbb{N}}(R)$ decomposes uniquely as

$$A = \sum_{m \geqslant 0} d_m(A)S^m + \sum_{m<0}(S^T)^{-m}d_m(A). \tag{1.3}$$

We use the decomposition (1.3) to assign a degree to elements of $LT_{\mathbb{N}}(R)$. For a nonzero $A \in LT_{\mathbb{N}}(R)$ the degree is equal to m if its highest nonzero diagonal is the m-th and the degree of the zero element is $-\infty$.

Lemma 1.1. *The centralizer in $LT_{\mathbb{N}}(R)$ of the matrix S consists of the*

$$\{\sum_{j \geqslant 0} i_1(r_j)S^j \mid r_j \in R\}.$$

Proof. Let $A = (a_{(i,j)})$ belong to $LT_{\mathbb{N}}(R)$. Then we have on one hand

$$AS = \begin{pmatrix} 0 & a_{(0,0)} & \cdots & a_{(0,n)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & a_{(n,0)} & \cdots & a_{(n,n)} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix} \tag{1.4}$$

and on the other

$$SA = \begin{pmatrix} a_{(1,0)} & a_{(1,1)} & \cdots & a_{(1,n+1)} & \cdots \\ a_{(2,0)} & a_{(2,1)} & \cdots & a_{(2,n+1)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ a_{(n+1,0)} & a_{(n+1,1)} & \cdots & a_{(n+1,n+1)} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}. \tag{1.5}$$

If the expressions (1.4) and (1.5) are equal then induction w.r.t. i shows first of all that A is uppertriangular, i. e. for all $j < i$, $a_{(i,j)} = 0$. For the remaining coefficients the identity $AS = SA$ yields then that $a_{(i,j)} = a_{(i+1,j+1)}$ for all $i \leqslant j$. This proves the claim. □

A consequence of Lemma 1.1 is the following property of $k[S] = \{\sum_{i=0}^{N} k_i S^i \mid k_i \in k\}$:

Corollary 1.1. *The algebra $k[S]$ is a maximal commutative subalgebra of $LT_{\mathbb{N}}(k)$.*

As any associative algebra, $LT_{\mathbb{N}}(R)$ is a Lie algebra with the commutator as a bracket. We use two decompositions of $LT_{\mathbb{N}}(R)$ into the direct sum of two Lie subalgebras. The first splits elements of $LT_{\mathbb{N}}(R)$ as follows

$$A = \pi_{ut}(A) + \pi_{slt}(A) = \sum_{m\geqslant 0} d_m(A)S^m + \sum_{m<0}(S^T)^m d_m(A)$$

and the second as

$$A = \pi_{sut}(A) + \pi_{lt}(A) = \sum_{m>0} d_m(A)S^m + \sum_{m\leqslant 0}(S^T)^m d_m(A).$$

The first way to split elements of $LT_{\mathbb{N}}(R)$ yields the Lie algebra decomposition

$$\begin{aligned}&LT_{\mathbb{N}}(R) = \pi_{ut}(LT_{\mathbb{N}}(R)) \oplus \pi_{slt}(LT_{\mathbb{N}}(R)), \text{ where}\\ &\pi_{ut}(LT_{\mathbb{N}}(R)) = \{A \in LT_{\mathbb{N}}(R) \mid \pi_{ut}(A) = A\},\\ &\pi_{slt}(LT_{\mathbb{N}}(R)) = \{A \in LT_{\mathbb{N}}(R) \mid \pi_{slt}(A) = A\}.\end{aligned}$$

The second leads to

$$\begin{aligned}&LT_{\mathbb{N}}(R) = \pi_{sut}(LT_{\mathbb{N}}(R)) \oplus \pi_{lt}(LT_{\mathbb{N}}(R)), \text{ where}\\ &\pi_{sut}(LT_{\mathbb{N}}(R)) = \{A \in LT_{\mathbb{N}}(R) \mid \pi_{sut}(A) = A\},\\ &\pi_{lt}(LT_{\mathbb{N}}(R)) = \{A \in LT_{\mathbb{N}}(R) \mid \pi_{lt}(A) = A\}.\end{aligned}$$

Inside $LT_{\mathbb{N}}(R)$ we associate a group to each of the Lie subalgebras $\pi_{slt}(LT_{\mathbb{N}}(R))$ and $\pi_{lt}(LT_{\mathbb{N}}(R))$. Note that on $\pi_{slt}(LT_{\mathbb{N}}(R))$ the exponential map is well-defined and yields elements in

$$\mathcal{U}_- = \mathcal{U}_-(R) = \{\mathrm{Id} + Y \mid Y \in \pi_{slt}(LT_{\mathbb{N}}(R))\}.$$

One easily verifies that $\mathcal{U}_-$ is a group w.r.t. multiplication. We see $\mathcal{U}_-$ as the group corresponding to $\pi_{slt}(LT_{\mathbb{N}}(R))$. If the exponential map is well-defined on $\pi_{lt}(LT_{\mathbb{N}}(R))$, then the resulting elements belong to the group

$$\mathcal{P}_- = \mathcal{P}_-(R) = \{A = \sum_{m\leqslant 0}(S^T)^m d_m(A) \mid d_0(A) = \mathrm{diag}(d(s)), \text{ all } d(s) \in R^*\}$$

and therefore we see $\mathcal{P}_-$ as the group associated with $\pi_{lt}(LT_{\mathbb{N}}(R))$.

2. The $k[S]$-hierarchy and its strict version

In this section we discuss the two deformations of $k[S]$ that we consider and the evolution equations we want the deformations of S to satisfy. At the first deformation each $\sum_{i\geqslant 0} k_i S^i$ in $k[S]$ is deformed into $\sum_{i\geqslant 0} k_i \mathcal{L}^i$, where $\mathcal{L} \in LT_{\mathbb{N}}(R)$ is an element of the form

$$\mathcal{L} = S + \sum_{i\leqslant 0}(S^T)^i \ell_i, \ell_i \in \mathcal{D}_{\mathbb{N}}(R). \tag{2.1}$$

One directly checks that any element USU^{-1}, with $U \in \mathcal{U}_-$, has this form and we call USU^{-1} a *$\mathcal{U}_-$-deformation* of S. We call U also the *dressing matrix* of USU^{-1}. At the second deformation we transform each matrix $\sum_{i\geqslant 0} k_i S^i \in k[S]$ into $\sum_{i\geqslant 0} k_i \mathcal{M}^i$, where $\mathcal{M} \in LT_{\mathbb{N}}(R)$ is an element of the form

$$\mathcal{M} = m_1 S + \sum_{i\leqslant 0} (S^T)^i m_i, m_i \in \mathcal{D}_{\mathbb{N}}(R), m_1 \in \mathcal{D}_{\mathbb{N}}(R)^*. \tag{2.2}$$

Also in this case one easily verifies that any matrix PSP^{-1}, with $P \in \mathcal{P}_-$, possesses the form (2.2) and therefore it is called a $\mathcal{P}_-$*-deformation* of S. Likewise we call P also the *dressing matrix* of the deformation PSP^{-1}. Moreover, we have

Lemma 2.1. *Reversely there holds for the deformations* (2.1) *and* (2.2)

(a) Any $\mathcal{L}$ of the form (2.1) *can uniquely be written in the form $\mathcal{L} = USU^{-1}$ with $U \in \mathcal{U}_-$, i.e. $\mathcal{L}$ is a $\mathcal{U}_-$-deformation of S.*

(b) Any $\mathcal{M}$ of the form (2.2) *can be written in the form $\mathcal{M} = dUSU^{-1}d^{-1}$, where $U \in \mathcal{U}_-$ is unique and $d \in \mathcal{D}_{\mathbb{N}}(R)^*$ is determined up to a factor from $i_1(R^*)$. In particular, $\mathcal{M}$ is a $\mathcal{P}_-$-deformation of S.*

Proof. We start with a proof of statement (a). So, given an $\mathcal{L}$ of the form (2.1), we have to find an $U = \mathrm{Id} + \sum_{j\geqslant 1} (S^T)^j u_j, u_j \in \mathcal{D}_{\mathbb{N}}(R)$, such that $\mathcal{L}U = US$. For any $U \in \mathcal{U}_-$ the matrix $\mathcal{L}U - US$ has no strict positive diagonals. So, it suffices to show that for all $n \geqslant 0$ the equations

$$d_{-k}(\mathcal{L}U) = d_{-k}(US), 0 \leqslant k \leqslant n,$$

determine the $\{u_1, \cdots, u_{n+1}\}$ uniquely. Hereby we use the relations in (1.2) and the following two relations between S, S^T and the diagonal matrices:

$$S^T \mathrm{diag}(d(s)) S = S^T S \ \mathrm{diag}(1, d(0), d(1), \cdots) = \mathrm{diag}(0, d(0), d(1), \cdots), \tag{2.3}$$

$$\mathrm{diag}(d(s)) S^T = S^T \mathrm{diag}(d(s+1)). \tag{2.4}$$

By applying (2.3) one gets for US the expression

$$\begin{aligned} US &= S + \sum_{j\geqslant 1} (S^T)^j u_j S = S + \sum_{j\geqslant 1} (S^T)^j S \ \mathrm{diag}(1, u_j(0), u_j(1), \cdots) \\ &= S + \sum_{j\geqslant 1} (S^T)^{j-1} \ \mathrm{diag}(0, u_j(0), u_j(1), \cdots). \end{aligned}$$

From this expression we conclude for each $n \geqslant 0$ that

$$d_{-n}(US) = \mathrm{diag}(0, u_{n+1}(0), u_{n+1}(1), \cdots). \tag{2.5}$$

Now we apply the first relation in (1.2) and repeatedly relation (2.4) to the product $\mathcal{L}U$ and get

$$\begin{aligned} \mathcal{L}U &= S + \sum_{i\geqslant 0} (S^T)^i \ell_i + S \sum_{j\geqslant 1} (S^T)^j u_j + \sum_{\substack{i \geqslant 0 \\ j \geqslant 1}} (S^T)^i \ell_i (S^T)^j u_j \\ &= S + \sum_{i\geqslant 0} (S^T)^i \ell_i + \sum_{j\geqslant 1} (S^T)^{j-1} u_j + \sum_{\substack{i \geqslant 0 \\ j \geqslant 1}} (S^T)^{i+j} \ \mathrm{diag}(\ell_i(s+j)) u_j. \end{aligned}$$

Thus we get $d_0(\mathcal{L}U) = \ell_0 + u_1$ and for the remaining diagonal components of $\mathcal{L}U$

$$d_{-n}(\mathcal{L}U) = \ell_n + u_{n+1} + \sum_{k=1}^{n} \operatorname{diag}(\ell_{k-1}(s+n+1-k))u_{n+1-k}, n \geqslant 1. \tag{2.6}$$

The equality $d_0(\mathcal{L}U) = d_0(US)$ is in terms of the diagonal components of $\mathcal{L}$ and U the identity

$$\ell_0 + u_1 = \operatorname{diag}(0, u_1(0), u_1(1), \cdots).$$

The $(0,0)$-entry of this matrix identity yields $u_1(0) = -\ell_0(0)$, the $(1,1)$-entry gives $u_1(1) = u_1(0) - \ell_0(1)$ and continuing in this fashion, one gets that any $u_1(s)$ is a linear combination of the matrix coefficients of ℓ_0. By induction w.r.t. n we may assume that the equations

$$d_{-k}(\mathcal{L}U) = d_{-k}(US), 0 \leqslant k \leqslant n-1,$$

determine the $\{u_1, \cdots, u_n\}$ and each $u_k(s), s \in \mathbb{N}$ and $, 0 \leqslant k \leqslant n-1$, is a polynomial expression in the matrix coefficients of the $\{\ell_k, 0 \leqslant k \leqslant n-1\}$. By combining the expressions (2.5) and (2.6) we get for $n \geqslant 1$ from $d_{-n}(US) = d_{-n}(\mathcal{L}U)$ the relation

$$u_{n+1} = \operatorname{diag}(0, u_{n+1}(0), u_{n+1}(1), \cdots) - \ell_n - \sum_{k=1}^{n} \operatorname{diag}(\ell_{k-1}(s+n+1-k))u_{n+1-k}.$$

Again we look successively at the diagonal entries of this matrix identity, starting with the $(0,0)$-entry and recalling that all $u_{n+1-k}(s)$ are known. This yields us

$$u_{n+1}(0) = -\ell_n(0) - \sum_{k=1}^{n} \ell_{k-1}(n+1-k)u_{n+1-k}(0).$$

Next we consider the $(1,1)$-entry and that gives us

$$u_{n+1}(1) = u_{n+1}(0) - \ell_n(1) - \sum_{k=1}^{n} \ell_{k-1}(n+2-k)u_{n+1-k}(1).$$

Continuing in this fashion, one gets that any $u_{n+1}(s)$ is a polynomial expression in the matrix coefficients of $\ell_0, \cdots, \ell_n$. This proves the claim in item (a).

The proof of statement (b) can be reduced to that of (a) by the following observation: take an arbitrary element $k \in \mathcal{D}_{\mathbb{N}}(R)$ and an $d = \operatorname{diag}(d(s)) \in \mathcal{D}_{\mathbb{N}}(R)^*$. Then there holds $d^{-1}kSd = \operatorname{diag}(\frac{d(s+1)}{d(s)})kS$. Given any $\mathcal{M}$ of the form (2.2), choose a $d \in \mathcal{D}_{\mathbb{N}}(R)^*$ such that for all $s \in \mathbb{N}$ the element $\frac{d(s+1)}{d(s)}$ equals $m_1(s)^{-1}$. Then the matrix $d^{-1}\mathcal{M}d$ has the form (2.1) and, hence there is a unique $U \in \mathcal{U}_-$ such that $d^{-1}\mathcal{M}d = USU^{-1}$. Then $\mathcal{M}$ equals PSP^{-1} with $P = dU \in \mathcal{P}_-$. In this case d is not unique, because any element $i_1(a)$, with $a \in R^*$, is in the center of $LT_{\mathbb{N}}(R)$ and there also holds $\mathcal{M} = i_1(a)dUSU^{-1}d^{-1}i_1(a^{-1})$. □

Next we discuss the evolution equations that an $\mathcal{U}_-$-deformation $\mathcal{L}$ of S has to satisfy and those for a $\mathcal{P}_-$-deformation $\mathcal{M}$ of S. Hereby each $S^i, i \geqslant 1$, is seen as an infinitesimal generator of a flow. In that light we assume in both cases that R is equipped with a set of commuting k-linear derivations $\{\partial_i : R \to R \mid i \geqslant 1\}$, where each ∂_i should be seen as an algebraic substitute for the derivative w.r.t. the flow parameter corresponding to the flow generated by each S^i. By letting each ∂_i act coefficient wise on matrices in $LT_{\mathbb{N}}(R)$, we get a set of derivations of $LT_{\mathbb{N}}(R)$, also denoted by $\{\partial_i\}$. The data $(R, \{\partial_i \mid i \geqslant 1\})$ we call a *setting* for both deformations.

For each $\mathcal{U}_-$-deformation $\mathcal{L}$ of S and all $i \geqslant 1$ we consider the cut-off's

$$\mathcal{B}_i(S) := \pi_{ut}(\mathcal{L}^i). \tag{2.7}$$

Note that, since all $\{\mathcal{L}^i\}$ commute, the $\{\mathcal{B}_i(S) \mid i \geqslant 1\}$ satisfy for all $m \geqslant 1$

$$[\mathcal{B}_i(S), \mathcal{L}^m] = -[\pi_{slt}(\mathcal{L}^i), \mathcal{L}^m],$$

where the right hand side is of degree $m-1$ or lower, like $\partial_i(\mathcal{L})$. This shows that it makes sense to unite the following set of Lax equations for the $\mathcal{L}^m$ in one combined system, the so-called $k[S]$-hierarchy:

$$\partial_i(\mathcal{L}^m) = [\mathcal{B}_i(S), \mathcal{L}^m] = -[\pi_{slt}(\mathcal{L}^i), \mathcal{L}^m]. \tag{2.8}$$

It suffices to prove the equations just for $m = 1$. For, since ∂_i and $\mathrm{ad}(\mathcal{B}_i(S))$ are both k-linear derivations of $LT_{\mathbb{N}}(R)$, all basis elements $\{\mathcal{L}^m \mid m \geqslant 1\}$ of the deformation $k[\mathcal{L}]$ of $k[S]$ satisfy the same Lax equations. The equations (2.8) itself are called the *Lax equations of the* $k[S]$*-hierarchy*. Note that the Lax equations (2.8) show that the action of each ∂_i on the coefficients of $\mathcal{L}$ expresses each of them in a polynomial expression of the coefficients of $\mathcal{L}$. Any $\mathcal{U}_-$-deformation $\mathcal{L}$ of S in $LT_{\mathbb{N}}(R)$ that satisfies all the equations (2.8), $i \geqslant 1$, is called a *solution* of the $k[S]$-hierarchy in the setting $(R, \{\partial_i\})$. Note that in each setting there is at least one solution of the $k[S]$-hierarchy, namely $\mathcal{L} = S$, the *trivial solution* of the $k[S]$-hierarchy. We can express the conditions when a $\mathcal{U}_-$-deformation $\mathcal{L} = USU^{-1}$ is a solution of the $k[S]$-hierarchy, in terms of U. For there holds

Lemma 2.2. *Any* $\mathcal{L} = USU^{-1}$*, with* $U \in \mathcal{U}_-$ *is a solution of the* $k[S]$*-hierarchy, if and only if* U *satisfies the relations: for all* $i \geqslant 1$

$$\partial_i(U)U^{-1} = -\pi_{slt}(\mathcal{L}^i). \tag{2.9}$$

Proof. Since $\partial_i(U^{-1}) = -U^{-1}\partial_i(U)U^{-1}$, we get for $\mathcal{L} = USU^{-1}$ that

$$\partial_i(\mathcal{L}) = [\partial_i(U)U^{-1}, \mathcal{L}].$$

If U satisfies (2.9), then $[-\pi_{slt}(\mathcal{L}^i), \mathcal{L}] = [\mathcal{B}_i(S), \mathcal{L}]$ yields the Lax equations for $\mathcal{L}$. Reversely, if $\mathcal{L}$ is a solution of the $k[S]$-hierarchy, then $\partial_i(U)U^{-1} + \pi_{slt}(\mathcal{L}^i)$ commutes with $\mathcal{L}$ and thus $\hat{U} = U^{-1}(\partial_i(U)U^{-1} + \pi_{slt}(\mathcal{L}^i))U$ commutes with S. The element $\hat{U}$ only has strict negative diagonals and Lemma 1.1 implies that $\hat{U} = 0$ and this proves the claim. $\square$

Since the relations (2.9) are the analogue of the Sato–Wilson equations for the KP hierarchy [3], we call them the *Sato–Wilson form of the* $k[S]$*-hierarchy*. Still another equivalent form of the $k[S]$-hierarchy was proven in [1]:

Proposition 2.1. *Let* $\mathcal{L}$ *be a* $\mathcal{U}_-$*-deformation of* S *with the* $\{\mathcal{B}_i(S)\}$ *as in* (2.7). *If* $\mathcal{L}$ *is a solution of the* $k[S]$*-hierarchy, then they satisfy the zero curvature relations*

$$\partial_{i_1}(\mathcal{B}_{i_2}(S)) - \partial_{i_2}(\mathcal{B}_{i_1}(S)) - [\mathcal{B}_{i_1}(S), \mathcal{B}_{i_2}(S)] = 0. \tag{2.10}$$

Reversely, if the projections $\{\mathcal{B}_i(S)\}$ *of a* $\mathcal{U}_-$*-deformation* $\mathcal{L}$ *satisfy the zero curvature relations* (2.10), *then* $\mathcal{L}$ *satisfies the Lax equations* (2.8).

Remark 2.1. By the equivalence in Proposition 2.1 the set of equations (2.10) is also called *the zero curvature form of the* $k[S]$ *-hierarchy*. Let $d_0(\mathcal{L}^i)$ the zero-th diagonal of $\mathcal{L}^i$, then the equations (2.10) imply that the commuting diagonal matrices $\{d_0(\mathcal{L}^i)\}$ satisfy the compatibility conditions

$$\partial_{i_1}(d_0(\mathcal{L}^{i_2})) = \partial_{i_2}(d_0(\mathcal{L}^{i_1})), \text{ for all } i_1 \geqslant 1, \text{ and } i_2 \geqslant 1.$$

Next we treat the evolution equations for the $\mathcal{P}_-$-deformations $\{\mathcal{M}^m \mid m \geqslant 1\}$. In that case we consider for each $i \geqslant 1$ the strict cut-off

$$\mathcal{C}_i(S) := \pi_{sut}(\mathcal{M}^i). \tag{2.11}$$

Since all the $\mathcal{M}^i$ commute, there holds

$$[\mathcal{C}_i(S), \mathcal{M}^m] = -[\pi_{lt}(\mathcal{M}^i), \mathcal{M}^m],$$

which shows that the $\{\mathcal{C}_i(S) \mid i \geqslant 1\}$ have the common property that the commutator with $\mathcal{M}^m$ has degree m or lower. The same holds for the matrix $\partial_i(\mathcal{M}^m)$, so it makes sense to unite the following set of Lax equations for the $\{\mathcal{M}^m\}$ in one combined system

$$\partial_i(\mathcal{M}^m) = [\mathcal{C}_i(S), \mathcal{M}^m] = -[\pi_{lt}(\mathcal{M}^i), \mathcal{M}^m]. \tag{2.12}$$

Because of the form of the $\{\mathcal{C}_i(S)\}$ and the similarity with the Lax equations (2.8) we call this system the *strict* $k[S]$ *-hierarchy*. The equations (2.12) itself are called the *Lax equations of the strict* $k[S]$ *-hierarchy*. Note that also in the strict case the Lax equations (2.12) show that the action of each ∂_i on the coefficients of $\mathcal{M}$ expresses each of them in a polynomial expression of the matrix coefficients of $\mathcal{M}$. Any $\mathcal{P}_-$-deformation $\mathcal{M}$ of S in $LT_{\mathbb{N}}(R)$ that satisfies all the equations (2.12) is called a *solution* of the strict $k[S]$-hierarchy in the setting $(R, \{\partial_i\})$. By the same argument as for the $k[S]$–case, it suffices to prove the equations (2.12) for $m = 1$, since all basis elements $\{\mathcal{M}^m \mid m \geqslant 1\}$ of the wider deformation $k[\mathcal{M}]$ of $k[S]$ satisfy the same Lax equations. Note that in each setting there is at least one solution of the strict $k[S]$-hierarchy, namely $\mathcal{M} = S$, the *trivial solution* of the strict $k[S]$-hierarchy. Also for the strict $k[S]$-hierarchy we found in [1] an equivalent zero curvature form

Proposition 2.2. *Let* $\mathcal{M}$ *be a* $\mathcal{P}_-$ *-deformation of* S *with the* $\{\mathcal{C}_i(S)\}$ *as in* (2.11)*. If* $\mathcal{M}$ *is a solution of the strict* $k[S]$ *-hierarchy, then the* $\{\mathcal{C}_i(S)\}$ *satisfy the zero curvature relations*

$$\partial_{i_1}(\mathcal{C}_{i_2}(S)) - \partial_{i_2}(\mathcal{C}_{i_1}(S)) - [\mathcal{C}_{i_1}(S), \mathcal{C}_{i_2}(S)] = 0. \tag{2.13}$$

On the other hand, if the projections $\{\mathcal{C}_i(S)\}$ *of a* $\mathcal{P}_-$ *-deformation* $\mathcal{M}$ *satisfy the zero curvature relations* (2.13)*, then* $\mathcal{M}$ *satisfies the Lax equations* (2.12)*.*

To discuss a Sato–Wilson form of the strict $k[S]$-hierarchy, requires some care, for the dressing matrix of S is not unique as we saw in part (b) of Lemma 2.1, and therefore we need the following notion:

Definition 2.1. Let $(R, \{\partial_i\})$ be a setting for both hierarchies. This setting is called *Cauchy solvable in dimension one*, if , given a collection of elements $\{a_i \mid i \geqslant 1\}$ in R satisfying the compatibility conditions

$$\partial_{i_1}(a_{i_2}) = \partial_{i_2}(a_{i_1}), \text{ for all } i_1 \geqslant 1 \text{ and } i_2 \geqslant 1,$$

there is an $\alpha \in R^*$ such that there holds for all $i \geqslant 1$, $\partial_i(\alpha) = a_i\alpha$.

E. g. the formal power series $k[[t_i]]$ with all the $\partial_i = \frac{\partial}{\partial t_i}$ is such a setting. Now there holds

Proposition 2.3. *Let the setting* $(R, \{\partial_i\})$ *be Cauchy solvable in dimension one and let* $\mathcal{M}$ *be a* $\mathcal{P}_-$ *-deformation that is a solution of the strict* $k[S]$ *-hierarchy in this setting. Then there is a* $P \in \mathcal{P}_-$ *with* $\mathcal{M} = PSP^{-1}$ *such that*

$$\partial_i(P)P^{-1} = -\pi_{lt}(\mathcal{M}^i). \tag{2.14}$$

Reversely, any $\mathcal{M} = PSP^{-1}$ *with* P *satisfying* (2.14)*, is a solution of the strict* $k[S]$ *-hierarchy*

Proof. Since $\partial_i(P^{-1}) = -P^{-1}\partial_i(P)P^{-1}$, we get for $\mathcal{M} = PSP^{-1}$ that

$$\partial_i(\mathcal{M}) = [\partial_i(P)P^{-1}, \mathcal{M}].$$

If P satisfies (2.14), then $[-\pi_{lt}(\mathcal{M}^i), \mathcal{M}] = [\mathcal{C}_i(S), \mathcal{M}]$ yields the Lax equations for $\mathcal{M}$. Note that the proof of the sufficiency of equations (2.14) does not require R to be Cauchy solvable in dimension 1. This we need in the proof of the reverse statement. Assume $\mathcal{M} = PSP^{-1}$ is a solution of the strict $k[S]$-hierarchy, where $P = d^{-1}U$, with $d \in \mathcal{D}_{\mathbb{N}}(R)^*$ and $U \in \mathcal{U}_-$ and we write . The first thing we notice is that $\partial_i(P)P^{-1} + \pi_{lt}(\mathcal{M}^i)$ commutes with $\mathcal{M}$ and thus $\hat{P} = P^{-1}(\partial_i(P)P^{-1} + \pi_{lt}(\mathcal{M}^i))P$ commutes with S. The element $\hat{P}$ only has negative diagonals and Lemma 1.1 implies that $\pi_{slt}(\hat{P}) = 0$ and the zero-th diagonal of $\hat{P}$ commutes with S. This last fact implies that the zero-th diagonal $d_0(\hat{P})$ belongs to $i_1(R)$ and that lies in the center of $LT_{\mathbb{N}}(R)$. So $P^{-1}d_0(\hat{P})P = d_0(\hat{P})$ and thus

$$P^{-1}(\pi_{slt}(\partial_i(P)P^{-1} + \pi_{lt}(\mathcal{M}^i)))P = \pi_{slt}(\hat{P}) = 0.$$

Hence $\tilde{P} = \pi_{slt}(\partial_i(P)P^{-1} + \pi_{lt}(\mathcal{M}^i)) = 0$. On the other hand we have

$$\tilde{P} = d^{-1}\partial_i(U)U^{-1}d + \pi_{slt}(\mathcal{M}^i) = d^{-1}\partial_i(U)U^{-1}d + d^{-1}\pi_{slt}(\mathcal{L}^i)d = 0,$$

so that according to Lemma 2.2 $\mathcal{L}$ is a solution of the $k[S]$-hierarchy. Let $d_0(\mathcal{L}^i)$ be the zero-th diagonal of $\mathcal{L}^i$. Then the $\{d_0(\mathcal{L}^i)\}$ satisfy the compatibility relations from Remark 2.1. A direct computation yields that

$$d_0(\hat{P}) = \partial_i(d^{-1})d + \ell_0(i) = \ell_0(i) - \partial_i(d)d^{-1} = i_1(a_i), \text{ with } a_i \in R.$$

Since there holds for all $i_1 \geqslant 1$ and $i_2 \geqslant 1$ that

$$\partial_{i_1}(\partial_{i_2}(d)d^{-1}) = \partial_{i_1}\partial_{i_2}(d)d^{-1} - \partial_{i_2}(d)\partial_{i_1}(d)d^{-2} = \partial_{i_2}(\partial_{i_1}(d)d^{-1}),$$

we get in total that the $\{a_i\}$ satisfy the compatibility conditions in Definition 2.1 and there is an $\alpha \in R^*$ such that for all $i \geqslant 1$, $\partial_i(\alpha)\alpha^{-1} = a_i$. Then conjugating S with $P_\alpha := i_1(\alpha^{-1})d^{-1}U$ also yields $\mathcal{M}$ and there holds $\partial_i(P_\alpha)P_\alpha^{-1} = -\pi_{lt}(\mathcal{M}^i)$ and this concludes the proof of the claims. □

We call the relations (2.14) the *Sato–Wilson equations of the strict* $k[S]$ *-hierarchy.*

Remark 2.2. From the proof of Proposition 2.3 follows that, if $P = d^{-1}U \in \mathcal{P}_-$, $\mathcal{M} = PSP^{-1}$ and $\mathcal{L} = USU^{-1}$, then P satisfies the equations (2.14), if and only if $\mathcal{L}$ is a solution of the $k[S]$-hierarchy and $\partial_i(d) = d_0(\mathcal{L}^i)d$, for all $i \geqslant 1$. This equivalence we meet again in the next section, but in a different form.

3. Wave matrices

Let $(R, \{\partial_i\})$ be a setting where one looks for solutions to both hierarchies. The construction from the homogeneous spaces of solutions of both hierarchies is done by producing special vectors, called wave matrices, in a suitable $LT_{\mathbb{N}}(R)$-module that we briefly recall. We start with the upper triangular matrix

$$\psi_0 = \psi_0(t,S) = \exp(\sum_{i=1}^{\infty} t_i S^i) = \begin{pmatrix} 1 & p_1(t) & p_2(t) & \dots \\ 0 & 1 & p_1(t) & \dots \\ 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where t is the short hand notation for $\{t_i \mid i \geqslant 1\}$ and each $p_j(t)$ is a homogeneous polynomial of degree j in the $\{t_i \mid i \leqslant j\}$, where every t_i has degree i. Note that ψ_0 commutes with S and satisfies for all $i \geqslant 1$, $\frac{\partial}{\partial t_i}(\psi_0) = S^i \psi_0$. Recall that each ∂_i was the algebraic substitute on R for the partial derivative w.r.t. the flow parameter of S^i. Therefore we write $\partial_i(\psi_0) = \frac{\partial}{\partial t_i}(\psi_0)$. The $LT_{\mathbb{N}}(R)$-module that we need consists of formal products of a perturbation factor from $LT_{\mathbb{N}}(R)$ and ψ_0. The products will be formal, for in order to make sense out of the product of a matrix from $LT_{\mathbb{N}}(R)$ and the matrix ψ_0 as a matrix requires convergence conditions and we want to give an algebraic description of the module. Consider therefore the space $\mathcal{O}(S)$ consisting of the formal products

$$\left\{ \{m(S)\}\psi_0 = \left\{ \sum_{i \geqslant 0} m_i S^i + \sum_{i<0} (S^T)^{-i} m_i \right\} \psi_0, m_i \in \mathcal{D}_{\mathbb{N}}(R) \right\}. \tag{3.1}$$

Addition resp. scalar multiplication are defined on $\mathcal{O}(S)$ by adding up the perturbation factors of two elements resp. by applying the scalar multiplication on the perturbation factor. Something similar is done with the $LT_{\mathbb{N}}(R)$-module structure: for each $h(S) \in LT_{\mathbb{N}}(R)$ define

$$h(S).\{m(S)\}\psi_0 := \{h(S)m(S)\}\psi_0.$$

Clearly this makes $\mathcal{O}(S)$ into a free $LT_{\mathbb{N}}(R)$-module with generator ψ_0. However, Besides the $LT_{\mathbb{N}}(R)$-action also each ∂_i acts on $\mathcal{O}(S)$ by the formula

$$\partial_i(\{m(S)\}\psi_0) := \left\{ \sum_{k=0}^{N} \partial_i(m_k) S^k + \sum_{k<0} (S^T)^{-k} \partial_i(m_k) + m(S) S^i \right\} \psi_0.$$

Here we impose a Leibnitz rule on the formal product. Finally there is a right hand action of S on $\mathcal{O}(S)$. Since S and ψ_0 commute, we can define it by

$$\{m(S)\}\psi_0 S := \{m(S)S\}\psi_0.$$

Analogous to the terminology in the function case, see e. g. [2], we call the elements of $\mathcal{O}(S)$ *oscillating* $\mathbb{N}\times\mathbb{N}$*-matrices.* Note that any $\psi = \hat{\psi}\psi_0 = h(S)\psi_0$ with $h(S)$ invertible is a generator of the free $LT_{\mathbb{N}}(R)$-module $\mathcal{O}(S)$. Examples are the choices $h(S) \in \mathcal{P}_-$ resp. $h(S) \in \mathcal{U}_-$ in which case we call ψ an oscillating $\mathbb{N}\times\mathbb{N}$-matrix of type $\mathcal{P}_-$ resp. $\mathcal{U}_-$. With the three actions just defined we can introduce inside $LT_{\mathbb{N}}(R)$ two systems of equations leading to solutions of the two hierarchies.

For the $k[S]$-hierarchy, this system looks as follows: consider a *$\mathcal{U}_-$-deformation* $\mathcal{L}$ of S in $LT_{\mathbb{N}}(R)$ with the set of projections $\{\mathcal{B}_i(S) := \pi_{ut}(\mathcal{L}^i)\}$. The goal is now to find an oscillating $\mathbb{N}\times\mathbb{N}$-matrix $\psi = \{h(S)\}\psi_0$ of type $\mathcal{U}_-$ such that in $\mathcal{O}(S)$ the following set of equations holds

$$\mathcal{L}\psi = \psi S \text{ and } \partial_i(\psi) = \mathcal{B}_i(S)\psi, \text{ for all } i \geqslant 1. \tag{3.2}$$

Since $\mathcal{O}(S)$ is a free $LT_{\mathbb{N}}(R)$-module with generator ψ, the first equation $\mathcal{L}\psi = \psi S$ implies $\mathcal{L}h(S) = h(S)S$ and thus $\mathcal{L} = h(S)Sh(S)^{-1}$. By Lemma 2.1 this determines $h(S)$ uniquely. The same argument allows you to translate each $\partial_i(\psi) = \mathcal{B}_i(S)\psi$ into an identity in $LT_{\mathbb{N}}(R)$:

$$\partial_i(\psi) = \{\partial_i(h(S)) + h(S)S^i\}\psi_0 = \{\partial_i(h(S))h(S)^{-1} + \mathcal{L}^i\}\psi = \mathcal{B}_i(S)\psi.$$

Thus we get that $h(S)$ satisfies the Sato–Wilson equations (2.9) and $\mathcal{L}$ is a solution of the $k[S]$-hierarchy. The system (3.2) is called the *linearization of the $k[S]$-hierarchy* and ψ a *wave matrix of the $k[S]$-hierarchy*. Note that ψ_0 is the wave matrix corresponding to the trivial solution of the $k[S]$-hierarchy, $\mathcal{L} = S$.

In the case of the strict $k[S]$-hierarchy we start with a *$\mathcal{P}_-$-deformation* $\mathcal{M}$ of S together with the projections $\{\mathcal{C}_i(S) := \pi_{sut}(\mathcal{M}^i)\}$. Now we look for an oscillating $\mathbb{N}\times\mathbb{N}$-matrix $\varphi = \{k(S)\}\psi_0$ of type $\mathcal{P}_-$ that satisfies in $\mathcal{O}(S)$ the following set of equations:

$$\mathcal{M}\varphi = \varphi S \text{ and } \partial_i(\varphi) = \mathcal{C}_i(S)\varphi, \text{ for all } i \geqslant 1. \tag{3.3}$$

Also φ is a generator of $\mathcal{O}(S)$ and again we can translate the equations (3.3) into identities in $LT_{\mathbb{N}}(R)$. Thus the first equation $\mathcal{M}\varphi = \varphi S$ becomes $\mathcal{M} = k(S)Sk(S)^{-1}$ and the second set of equations in (3.3) yields the Sato–Wilson equations (2.14) of the strict $k[S]$-hierarchy. In particular, $\mathcal{M}$ is a solution of that hierarchy. The system (3.3) is called the *linearization of the strict $k[S]$-hierarchy* and a φ satisfying this system a *wave matrix of the strict $k[S]$-hierarchy*. Because the second set of equations in (3.3) is a different form of those in (2.14), the conditions in Remark 2.2 translate directly to a link between wave matrices of the hierarchies under consideration.

For both hierarchies, we use in the sequel a milder property that oscillating $\mathbb{N}\times\mathbb{N}$-matrices of a certain type have to satisfy, in order to become a wave matrix of that hierarchy. For a proof, see [1].

Proposition 3.1. *Let $\psi = \{h(S)\}\psi_0$ be an oscillating $\mathbb{N}\times\mathbb{N}$-matrix of type $\mathcal{U}_-$ in $\mathcal{O}(S)$ and $\mathcal{L} = h(S)Sh(S)^{-1}$ the corresponding potential solution of the $k[S]$-hierarchy. Similarly, let $\varphi = \{k(S)\}\psi_0$ be an oscillating $\mathbb{N}\times\mathbb{N}$-matrix of type $\mathcal{P}_-$ in $\mathcal{O}(S)$ with potential solution $\mathcal{M} = k(S)Sk(S)^{-1}$ of the strict version.*

(a) Assume there exists for each $i \geqslant 1$ an element $M_i \in \pi_{ut}(LT_{\mathbb{N}}(R))$ such that

$$\partial_i(\psi) = M_i\psi.$$

Then each $M_i = \pi_{ut}(\mathcal{L}^i)$ and ψ is a wave matrix for the $k[S]$-hierarchy.

(b) Suppose there exists for each $i \geqslant 1$ an element $N_i \in \pi_{sut}(LT_{\mathbb{N}}(R))$ such that

$$\partial_i(\varphi) = N_i\varphi.$$

Then each $N_i = \pi_{sut}(\mathcal{M}^i)$ and φ is a wave matrix for the strict $k[S]$-hierarchy.

Remark 3.1. Since you do not meet formal products of lower triangular $\mathbb{N}\times\mathbb{N}$-matrices and upper triangular $\mathbb{N}\times\mathbb{N}$-matrices in real life, the only way to construct wave matrices of both hierarchies is to give an analytic framework, where you can produce well-defined products of such matrices. This is done in the next section.

4. The construction of solutions of both hierarchies

All the relevant $\mathbb{N}\times\mathbb{N}$-matrices that will be produced in the sequel correspond to bounded operators acting on a separable real or complex Hilbert space. Since $k=\mathbb{R}$ or $\mathbb{C}$, we have on k a norm $|\cdot|: k\to\mathbb{R}_{\geqslant 0}$. The Hilbert space we will work with is a space of $1\times\mathbb{N}$ matrices. Thereto we denote, for each $n\in\mathbb{N}$, the row vector with a 1 on the n-th entry and all other entries equal to zero by $\vec{e}(n)^T$, i. e.

$$\vec{e}(n)^T=(\cdots,0,1,0,\cdots).$$

Consider now the k-linear space of $1\times\mathbb{N}$ matrices

$$H=\{\sum_{n\in\mathbb{N}}a_n\vec{e}(n)^T=(a_0,a_1,a_2,\cdots)\mid a_n\in k,\sum_{n\in\mathbb{N}}|a_n|^2<\infty\},$$

which becomes a real or complex Hilbert space w.r.t. the inner product

$$(\sum_{n\in\mathbb{N}}a_n\vec{e}(n)^T\mid\sum_{n\in\mathbb{N}}b_n\vec{e}(n)^T):=\sum_{n\in\mathbb{N}}a_n\overline{b_n},$$

depending of $k=\mathbb{R}$ or $\mathbb{C}$. The elements $\{\vec{e}(n)^T\mid n\in\mathbb{N}\}$ form by definition a Hilbert basis in H. In the sequel we need the subspaces $\{H_i, i\in\mathbb{N}\}$ of H and their orthogonal complements $H_i^{\perp}$ given by

$$H_i=\{\sum_{n\leqslant i}a_n\vec{e}(n)^T\in H\}\text{ and }H_i^{\perp}=\{\sum_{n>i}a_n\vec{e}(n)^T\in H\}.$$

Any $b\in B(H)$, the space of all bounded k-linear maps from H to itself, can be defined w.r.t. the $\{\vec{e}(n)^T\}$ by right multiplication $M_{[b]}$ with an $\mathbb{N}\times\mathbb{N}$-matrix $[b]=(b_{ij})$ i. e.

$$b(\vec{e}(j)^T)=M_{[b]}(\vec{e}(j)^T)=\vec{e}(j)^T[b]=\sum_{i\in\mathbb{N}}b_{ji}\vec{e}(i)^T.$$

This choice implies for all b_1 and $b_2\in B(H)$ that $[b_1\circ b_2]=[b_2][b_1]$. The invertible transformations in $B(H)$ are denoted by $\mathrm{GL(H)}$ and its group of matrices by $[\mathrm{GL(H)}]$.

Two decompositions of $B(H)$ play a role in the sequel. The first splits a $b\in B(H)$ as $b=u_-(b)+p_+(b)$, with the corresponding matrices

$$[u_-(b)]=\begin{pmatrix}0&0&0&\dots\\ b_{1\,0}&0&0&\dots\\ b_{2\,0}&b_{2\,1}&0&\dots\\ \vdots&\vdots&\vdots&\ddots\end{pmatrix}\text{ and }[p_+(b)]=\begin{pmatrix}b_{0\,0}&b_{0\,1}&b_{0\,2}&\dots\\ 0&b_{1\,1}&b_{1\,2}&\dots\\ 0&0&b_{2\,2}&\dots\\ \vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

It gives rise to the decomposition of the Lie algebra $B(H)$ as $\mathcal{U}_-(H)\oplus\mathcal{P}_+(H)$, where

$$\mathcal{U}_-(H)=\{b\in B(H)\mid b=u_-(b)\}\text{ and }\mathcal{P}_+(H)=\{b\in B(H)\mid b=p_+(b)\}.\tag{4.1}$$

The second decomposition consists of splitting a $b \in B(H)$ as $b = p_-(b) + u_+(b)$, where the matrices of both components are given by

$$[p_-(b)] = \begin{pmatrix} b_{0\,0} & 0 & 0 & \dots \\ b_{1\,0} & b_{1\,1} & 0 & \dots \\ b_{2\,0} & b_{2\,1} & b_{2\,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } [u_+(b)] = \begin{pmatrix} 0 & b_{0\,1} & b_{0\,2} & \dots \\ 0 & 0 & b_{1\,2} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This leads to the decomposition $B(H) = \mathcal{P}_-(H) \oplus \mathcal{U}_+(H)$ of $B(H)$, where

$$\mathcal{P}_-(H) = \{b \in B(H) \mid b = p_-(b)\} \text{ and } \mathcal{U}_+(H) = \{b \in B(H) \mid b = u_+(b)\}. \tag{4.2}$$

Before we present the Lie algebra of the central group $G(\mathcal{S}_2)$ in this paper, we need some notations. We denote the space of Hilbert-Schmidt operators from H to H by $\mathcal{S}_2$. It is the two-sided ideal of compact operators in $B(H)$ such that for each $A \in \mathcal{S}_2$

$$||A||_2^2 := \text{trace}(A^*A) = \text{trace}(|A|^2) < \infty.$$

Here A^* denotes the adjoint of A. In terms of the matrix coefficients (a_{ij}) of $[A]$ the Hilbert-Schmidt condition is simply

$$\sum_{i,j \in \mathbb{N}} |a_{ij}|^2 < \infty. \tag{4.3}$$

The map $A \to ||A||_2$ defines the Hilbert-Schmidt norm on $\mathcal{S}_2$, with respect to which it is complete. Consider now the subspace $B(\mathcal{S}_2)$ of $B(H)$ defined by

$$B(\mathcal{S}_2) = \{b \in B(H) \mid u_-(b) \in \mathcal{S}_2\}.$$

Consider two elements b_1 and b_2 in $B(\mathcal{S}_2)$. Since $\mathcal{S}_2$ is a two-sided ideal in $B(H)$, all $\mathcal{U}_-(H)$-components of $u_-(b_1)p_+(b_2)$, $p_+(b_1)u_-(b_2)$ and $u_-(b_1)u_-(b_2)$ belong to $\mathcal{S}_2$. Hence $b_1b_2 \in B(\mathcal{S}_2)$ and thus $B(\mathcal{S}_2)$ is een algebra. We put on $B(\mathcal{S}_2)$ a different Banach structure than the one induced by the operator norm on $B(H)$, namely we take the Banach structure that is the direct sum of the Hilbert-Schmidt norm on $\mathcal{U}_-(\mathcal{S}_2)$ and the operator norm on $\mathcal{P}_+(H)$. The group $G(\mathcal{S}_2)$ will be the elements in $B(\mathcal{S}_2)$ that have an inverse in $B(\mathcal{S}_2)$ and is an open subset of $B(\mathcal{S}_2)$. Inside $B(\mathcal{S}_2)$ we have the two Lie subalgebras

$$\mathcal{U}_-(\mathcal{S}_2) = \{b \in B(\mathcal{S}_2) \mid b = u_-(b)\}$$

and $\mathcal{P}_+(H)$ and $B(\mathcal{S}_2)$ is equal to their direct sum. To both Lie subalgebras there corresponds a subgroup of $G(\mathcal{S}_2)$, respectively

$$U_-(\mathcal{S}_2) = \{b \in B(\mathcal{S}_2) \mid b = \text{Id} + u_-(b)\}$$

and

$$P_+(H) = \{p \in \mathcal{P}_+(H) \mid p \text{ invertible }, p^{-1} \in \mathcal{P}_+(H)\}.$$

The characterization (4.3) implies that, if we define for each $N \in \mathbb{N}$ the map $p_N : \mathcal{U}_-(\mathcal{S}_2) \to \mathcal{U}_-(\mathcal{S}_2)$ by taking the first $N+1$ rows of $p_N(u)$ equal to those of u and the remaining ones equal to zero, then we have for all $u \in \mathcal{U}_-(\mathcal{S}_2)$ that

$$\lim_{N \to \infty} p_N(u) = u. \tag{4.4}$$

The product $\Omega(\mathcal{S}_2) = U_-(\mathcal{S}_2)P_+(H)$ is called the *big cell* w.r.t. these subgroups and the splitting $\omega = \mathbf{u}_-(\omega)\mathbf{p}_+(\omega)$, with $\mathbf{u}_-(\omega) \in U_-(\mathcal{S}_2)$ and $\mathbf{p}_+(\omega) \in P_+(H)$, we call the $(\,U_-(\mathcal{S}_2),\, P_+(H)\,)$-splitting of $\Omega(\mathcal{S}_2)$ and $\mathbf{u}_-(\omega)$ is the unipotent component of this decomposition. The second decomposition of $B(H)$ induces also a different splitting of $B(\mathcal{S}_2)$ in two Lie subalgebras, namely as $\mathcal{P}_-(\mathcal{S}_2) \oplus \mathcal{U}_+(H)$, where

$$\mathcal{P}_-(\mathcal{S}_2) = \{b \in B(\mathcal{S}_2) \mid b = p_-(b)\}.$$

To each of these two Lie subalgebras we can associate subgroups of $G(\mathcal{S}_2)$. For $\mathcal{P}_-(\mathcal{S}_2)$ that will be

$$P_-(\mathcal{S}_2) = \{p \in \mathcal{P}_-(\mathcal{S}_2) \mid p \text{ invertible }, p^{-1} \in \mathcal{P}_-(\mathcal{S}_2)\}$$

and for $\mathcal{U}_+(H)$ that is $U_+(H) = \{p \in P_+(H), p = \mathrm{Id} + u_+(p)\}$. Let $D(H)$ denote the invertible diagonal transformations in $B(\mathcal{S}_2)$. Then $P_+(H) = D(H)U_+(H)$ and $P_-(\mathcal{S}_2) = U_-(\mathcal{S}_2)D(H)$ and thus $\Omega(\mathcal{S}_2)$ also equals $P_-(\mathcal{S}_2)U_+(H)$. We call the splitting $\omega = \mathbf{p}_-(\omega)\mathbf{u}_+(\omega)$, with $\mathbf{p}_-(\omega) \in P_-(\mathcal{S}_2)$ and $\mathbf{u}_+(\omega) \in U_+(H)$, the $(\,P_-(\mathcal{S}_2),\, U_+(H)\,)$-splitting of $\Omega(\mathcal{S}_2)$ and $\mathbf{p}_-(\omega)$ is called the parabolic component of this decomposition.

The next step will be the introduction of the group of commuting flows that is relevant to both hierarchies. This requires some notations. For $r > 0$, let $D_0(r)$ be the closed disc around the origin in the complex plane with radius r. As in [4], the space $\mathcal{O}(D_0(r))$ of holomorphic functions on $D_0(r)$, consists of the direct limit of the $\mathcal{O}(U)$ with U an open subset containing $D_0(r)$. On it we put the topology of uniform convergence and we consider the closed subspace $\mathcal{O}_0(D_0(r))$ of all $f \in \mathcal{O}(D_0(r))$ such that $f(0) = 0$. Now we can specify the group Γ of commuting flows in GL(H) we will work with. The matrices corresponding to the transformations in Γ are the image of the continuous map from $\mathcal{O}_0(D_0(1))$ to [GL(H)] built up from the substitution $z = S$ in an $f \in \mathcal{O}_0(D_0(1))$ and the exponential map. In concrete terms, the group of matrices of Γ can be described as follows:

$$[\Gamma] = \left\{ [\gamma] = \exp(\sum_{i=1}^{\infty} t_i S^i) \mid \begin{array}{c} \sum_{i=1}^{\infty} |t_i|(1+\varepsilon)^i < \infty \\ \text{for some } \varepsilon > 0 \end{array} \right\}. \tag{4.5}$$

Since the matrices in $[\Gamma]$ are upper triangular, Γ is a subgroup of $P_+(H)$ and hence of $G(\mathcal{S}_2)$. Therefore Γ acts by left translations on $G(\mathcal{S}_2)$. We need the following result w.r.t. this action:

Proposition 4.1. *The action of* Γ *on* $G(\mathcal{S}_2)$ *satisfies: for each element* $g \in G(\mathcal{S}_2)$ *there exists* $\gamma \in \Gamma$ *such that* $\gamma^{-1}g$ *belongs to the big cell* $\Omega(\mathcal{S}_2)$ *w.r.t.* $U_-(\mathcal{S}_2)$ *and* $P_+(H)$*. The set* $\mathcal{O}_0(D_0(1))(g)$ *of all* $f \in \mathcal{O}_0(D_0(1))$ *such that* $[\gamma] = \exp(f(S))$ *satisfies this condition is a non-zero open part of* $\mathcal{O}_0(D_0(1))$ *and let* $\Gamma(g)$ *correspond to its image in* Γ.

Proof. Because of relation (4.4) it suffices to prove the statement for the elements of $G(\mathcal{S}_2)$ that decompose for some $N \in \mathbb{N}$ w.r.t the splitting $H = H_N \oplus H_N^{\perp}$ as

$$[g] = (g_{i,j}) = \begin{pmatrix} g(0,0) & g(0,1) \\ 0 & g(1,1) \end{pmatrix}, \text{ with } g(0,0) = \begin{pmatrix} g_{0,0} & \cdots & g_{0,N} \\ \vdots & \ddots & \vdots \\ g_{N,0} & \cdots & g_{N,N} \end{pmatrix} \in \mathrm{GL}_{N+1}(k)$$

$$\text{and } g(1,1) = \begin{pmatrix} g_{N+1,N+1} & \cdots & \cdots & g_{N,N+k} & \cdots \\ 0 & \ddots & \dots & \vdots & \dots \\ \vdots & \ddots & \ddots & \vdots & \dots \\ 0 & \dots & 0 & g_{N+k,N+k} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \text{ invertible,}$$

since elements of this form are dense in $G(\mathcal{S}_2)$. The matrix of each element $\gamma \in \Gamma$ we split w.r.t. the same decomposition of H:

$$[\gamma] = \begin{pmatrix} \gamma_N(0,0) & \gamma(0,1) \\ 0 & \gamma(1,1) \end{pmatrix}, \text{ with } \gamma_N(0,0) = \begin{pmatrix} 1 & p_1 & \dots & p_N \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then the matrix of the operator $\gamma^{-1}g$ has the form

$$[g][\gamma]^{-1} = \begin{pmatrix} g(0,0)\gamma_N(0,0)^{-1} & x \\ 0 & g(1,1)\gamma(1,1)^{-1} \end{pmatrix},$$

where $x = -g(0,0)\gamma_N(0,0)^{-1}\gamma(0,1)\gamma(1,1)^{-1} + g(0,1)\gamma(1,1)^{-1}$. Hence, if we find a vector $\vec{t_N} = \{t_1, \cdots, t_N\}$ such that $g(0,0)\gamma_N(0,0)^{-1} = p_0(\vec{t_N})u_0(\vec{t_N})$ with $p_0(\vec{t_N})$ an invertible upper triangular $N+1 \times N+1$-matrix and $u_0(\vec{t_N})$ a unipotent lower triangular matrix of the same size, then we have

$$[g][\gamma]^{-1} = \begin{pmatrix} p_0(\vec{t_N}) & x \\ 0 & g(1,1)\gamma(1,1)^{-1} \end{pmatrix} \begin{pmatrix} u_0(\vec{t_N}) & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix}$$

and this is the decomposition we are looking for. Clearly, in order that we have the desired splitting of $[g][\gamma]^{-1}$ the condition

$$g(0,0)\gamma(0,0)^{-1} = p_0(\vec{t_N})u_0(\vec{t_N})$$

for the vector $\vec{t_N}$ is also necessary and by taking the inverse, one sees that it is equivalent to finding a $\vec{t_N}$ for $\gamma(0,0)h$, where $h = g(0,0)^{-1} = (h_{i,j})$, such that $\gamma(0,0)h = u(\vec{t_N})p(\vec{t_N})$ with $p(\vec{t_N})$ an invertible upper triangular $N+1 \times N+1$-matrix and $u(\vec{t_N})$ a unipotent lower triangular one. We will show by induction on N that there are N nonzero polynomials $\{q_1, \cdots, q_N\}$ such that for all $\vec{t_N}$ in the complement of the union of the zero-sets of all the $\{q_i\}$ one has the decomposition $\gamma_N(0,0)h = u(\vec{t_N})p(\vec{t_N})$. For $N = 0$, the matrix $[g]$ is upper triangular and the desired decomposition holds for all $\vec{t_N}$. Now we take $N \geqslant 1$, we split off the first row of $\gamma_N(0,0)$ as follows:

$$\gamma_N(0,0)h = \begin{pmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \gamma_{N-1}(0,0) \end{pmatrix} \begin{pmatrix} 1 & p_1 \cdots p_N \\ 0 & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} h_{0,0} & \cdots & h_{0,N} \\ \vdots & \ddots & \vdots \\ h_{N,0} & \cdots & h_{N,N} \end{pmatrix} \tag{4.6}$$

and we focus for the moment on the product of the last two matrices. Since the first column of h is nonzero, the polynomial $q_N := h_{0,0} + \sum_{k=1}^{N} h_{k,0}p_k$ is nonzero. Now we work on the complement of the zero-set of q_N, so q_N is invertible. Define for all $i, 0 \leqslant i \leqslant N$, the polynomials

$\hat h_{0,i} = h_{0,i} + \sum_{k=1}^{N} h_{k,i}p_k$. Note that $\hat h_{0,0} = q_N$. Then the product of the last two matrices in (4.6) is equal to

$$\begin{pmatrix} \hat h_{0,0} & \cdots & \cdots & \hat h_{0,N} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N,0} & h_{N,1} & \cdots & h_{N,N} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \tilde h_{1,0} & 1 & 0 & \cdots & \cdots \\ \vdots & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ \tilde h_{N,0} & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat h_{0,0} & \cdots & \cdots & \hat h_{0,N} \\ 0 & \hat h_{1,1} & \cdots & \hat h_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat h_{N,1} & \cdots & \hat h_{N,N} \end{pmatrix},$$

where each $\tilde h_{k,0} = h_{k,0}q_N^{-1}$ and all $\hat h_{ik}$ with $i \geqslant 1$ and $k \geqslant 1$ are defined by $\hat h_{ik} = h_{ik} - \tilde h_{i0}\hat h_{0k}$. Next we push the top row of the right matrix to the right

$$\begin{pmatrix} \hat h_{0,0} & \cdots & \cdots & \hat h_{0,N} \\ 0 & \hat h_{1,1} & \cdots & \hat h_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat h_{N,1} & \cdots & \hat h_{N,N} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \hat h_{1,1} & \cdots & \hat h_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat h_{N,1} & \cdots & \hat h_{N,N} \end{pmatrix} \begin{pmatrix} \hat h_{0,0} \cdots \hat h_{0,N} \\ 0 \quad \mathrm{Id}_N \end{pmatrix}.$$

The matrix $\begin{pmatrix} \hat h_{0,0} \cdots \hat h_{0,N} \\ 0 \quad \mathrm{Id}_N \end{pmatrix}$ at the right has determinant $\hat h_{0,0} = q_N \neq 0$ and will be part of $p(\vec t_N)$. Next we move the matrix $\begin{pmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \gamma_{N-1}(0,0) \end{pmatrix}$ in the product (4.6) to the right by using

$$\begin{pmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \gamma_{N-1}(0,0) \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \tilde h_{1,0} & 1 & 0 & \cdots & \cdots \\ \vdots & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ \tilde h_{N,0} & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \vec z & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \gamma_{N-1}(0,0) \end{pmatrix},$$

where $\vec z$ is the column of length N equal to $\gamma_{N-1}(0,0)\vec{\tilde h}$ with $(\vec{\tilde h})^T = (\tilde h_{1,0}, \cdots, \tilde h_{N,0})$. The matrix $\begin{pmatrix} 1 & 0 \\ \vec z & \mathrm{Id}_N \end{pmatrix}$ will be part of $u(\vec t_N)$. Thus we have reduced the case to the product

$$\begin{pmatrix} 1 & 0\cdots\cdots 0 \\ 0 & \gamma_{N-1}(0,0) \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \hat h_{1,1} & \cdots & \hat h_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat h_{N,1} & \cdots & \hat h_{N,N} \end{pmatrix},$$

where the matrix at the right has determinant $q_N^{-1} \neq 0$. The induction hypothesis gives us then the nonzero polynomials $\{q_1, \cdots, q_N\}$ so that on the complement of all their zeros we have the desired decomposition. This proves the claim in the proposition. □

Now we will construct for each $g \in G(\mathcal{S}_2)$ a solution of the $k[S]$-hierarchy and the strict $k[S]$-hierarchy. The appropriate setting in both cases is the algebra

$$R(g) = C^\infty(\mathcal{O}_0(D_0(1))(g), k), \tag{4.7}$$

with the derivations $\partial_i = \frac{\partial}{\partial t_i}, i \geqslant 1$. We start with the construction of the solutions of the $k[S]$-hierarchy. Then we have by definition for all $\gamma \in \Gamma(g)$ that

$$\gamma^{-1} g = \mathbf{u}_-(g,\gamma)\mathbf{p}_+(g,\gamma)^{-1} \tag{4.8}$$

and thus on the matrix level

$$[g][\gamma]^{-1} = [\mathbf{p}_+(g,\gamma)]^{-1}[\mathbf{u}_-(g,\gamma)].$$

Note that all matrix coefficients of $[\mathbf{u}_-(g,\gamma)]$ and $[\mathbf{p}_+(g,\gamma)]$ belong to $R(g)$, since the map $(\mathbf{u}_-, \mathbf{p}_+) \to \mathbf{u}_- \mathbf{p}_+^{-1}$ is a diffeomorphism between $U_-(\mathcal{S}_2) \times P_+(H)$ and $\Omega(\mathcal{S}_2)$. The equation (4.8) leads to the following identity

$$\Psi(g) := [\mathbf{u}_-(g,\gamma])[\gamma] = [\mathbf{p}_+(g,\gamma)][g]. \tag{4.9}$$

Clearly $\Psi(g)$ is an oscillating matrix in $\mathcal{O}(S)$ for which the products between the different factors are real. To show that $\Psi(g)$ is a wave matrix for the $k[S]$ hierarchy it suffices to prove the property in Proposition 3.1. Thereto we compute for all $i \geqslant 1$, the matrix $\partial_i(\Psi(g))\Psi(g)^{-1}$ using both the left and the right hand side of expression (4.9). We start with the right hand side. Since for all $i \geqslant 1$, $\partial([g]) = 0$, we get

$$\partial_i(\Psi(g))\Psi(g)^{-1} = \partial_i([\mathbf{p}_+(g,\gamma)])[\mathbf{p}_+(g,\gamma)]^{-1}.$$

Now the matrix $\partial_i([\mathbf{p}_+(g,\gamma)])[\mathbf{p}_+(g,\gamma)]^{-1}$ is of the form $\sum_{r \geqslant 0} d_r S^r$ with all $d_r \in \mathcal{D}_{\mathbb{N}}(R)$. Next we use the left hand side of (4.9) to compute $\partial_i(\Psi(g))\Psi(g)^{-1}$. This yields

$$\begin{aligned}\partial_i(\Psi(g))\Psi(g)^{-1} &= \partial_i([\mathbf{u}_-(g,\gamma)])[\mathbf{u}_-(g,\gamma)]^{-1} + [\mathbf{u}_-(g,\gamma)]\partial_i([\gamma])[\gamma]^{-1}[u_-(g,\gamma)]^{-1}\\ &= \partial_i([\mathbf{u}_-(g,\gamma)])[u_-(g,\gamma)]^{-1} + [\mathbf{u}_-(g,\gamma)]S^i[\mathbf{u}_-(g,\gamma)]^{-1}.\end{aligned}$$

In this formula expression $\partial_i([\mathbf{u}_-(g,\gamma)])[\mathbf{u}_-(g,\gamma)]^{-1}$ possesses only negative diagonals and $[\mathbf{u}_-(g,\gamma)]S^i[\mathbf{u}_-(g,\gamma)]^{-1}$ has the form

$$\sum_{r=0}^{i} v_r S^r + \sum_{r<0} (S^T)^{-r} v_r,$$

with all $v_r \in \mathcal{D}_{\mathbb{N}}(R)$ and $v_i = \mathrm{Id}$. Combining this with the expression found for the right hand side gives for all $i \geqslant 1$

$$\partial_i(\Psi(g)) = (\sum_{r=0}^{i} v_r S^r)\Psi(g) = \mathcal{B}_{i,\Psi(g)}\Psi(g).$$

Thus $\Psi(g)$ satisfies the conditions in part (a) of Proposition 3.1 and hence it is a wave matrix of the $k[S]$-hierarchy. In other words, $\Psi(g)$ is a solution of the linearization of the $k[S]$-hierarchy. The corresponding solution $\mathcal{L}_g$ of the $k[S]$-hierarchy is

$$\mathcal{L}_g = [u_-(g,\gamma)]S[u_-(g,\gamma)]^{-1}. \tag{4.10}$$

Note that, since the factor $p_+(g,\gamma)^{-1}$ plays no role at the construction of $\mathcal{L}_g$, multiplying g from the right with an element of $P_+(H)$ does not affect the solution $\mathcal{L}_g$.

Secondly, we present for a $g \in G(\mathcal{S}_2)$ the construction of the solution of the strict $k[S]$-hierarchy. We proceed similarly, but now we use the $(\,P_-(\mathcal{S}_2), U_+(H)\,)$-splitting of $\Omega(\mathcal{S}_2)$. By definition we have for all $\gamma \in \Gamma(g)$ that

$$\gamma^{-1} g = \mathbf{p}_-(g,\gamma)\mathbf{u}_+(g,\gamma)^{-1} \tag{4.11}$$

and thus on the matrix level

$$[g][\gamma]^{-1} = [\mathbf{u}_+(g,\gamma)]^{-1}[\mathbf{p}_-(g,\gamma)].$$

Note that all matrix coefficients of $[\mathbf{p}_-(g,\gamma)]$ and $[\mathbf{u}_+(g,\gamma)]$ belong to $R(g)$, since the map $(\mathbf{p}_-,\mathbf{u}_+) \to \mathbf{p}_-\mathbf{u}_+^{-1}$ is a diffeomorphism between $P_-(\mathcal{S}_2) \times U_+(H)$ and $\Omega(\mathcal{S}_2)$. The equation (4.11) leads to the following identity

$$\Phi(g) := [\mathbf{p}_-(g,\gamma][\gamma] = [\mathbf{u}_+(g,\gamma)][g]. \tag{4.12}$$

Clearly $\Phi(g)$ is an oscillating matrix in $\mathcal{O}(S)$ for which the products between the different factors are real. The idea is again to show that $\Phi(g)$ is a wave matrix for the strict $k[S]$-hierarchy and that is done by proving property (b) in Proposition 3.1. Thereto we compute for all $i \geqslant 1$, the matrix $\partial_i(\Phi(g))\Phi(g)^{-1}$ using both the left and the right hand side of expression (4.12). We start with the right hand side. Again all the $\partial_i([g])$ are zero, hence

$$\partial_i(\Phi(g))\Phi(g)^{-1} = \partial_i([\mathbf{u}_+(g,\gamma)])[\mathbf{u}_+(g,\gamma)]^{-1}.$$

The matrix $\partial_i([u_+(g,\gamma)])[u_+(g,\gamma)]^{-1}$ has only strict positive diagonals. Thus $\partial_i(\Phi(g))\Phi(g)^{-1}$ is equal to a matrix of the form $\sum_{r\geqslant 1} u_r S^r$ with all $u_r \in \mathcal{D}_{\mathbb{N}}(R)$. Next we use the left hand side of (4.12) to compute $\partial_i(\Phi(g))\Phi(g)^{-1}$ once more. This yields

$$\begin{aligned}\partial_i(\Phi(g))\Phi(g)^{-1} &= \partial_i([\mathbf{p}_-(g,\gamma)])[\mathbf{p}_-(g,\gamma)]^{-1} + [\mathbf{p}_-(g,\gamma)]\partial_i([\gamma])[\gamma]^{-1}[\mathbf{p}_-(g,\gamma)]^{-1}\\ &= \partial_i([\mathbf{p}_-(g,\gamma)])[\mathbf{p}_-(g,\gamma)]^{-1} + [\mathbf{p}_-(g,\gamma)]S^i[\mathbf{p}_-(g,\gamma)]^{-1}.\end{aligned}$$

In this formula expression $\partial_i([\mathbf{p}_-(g,\gamma)])[\mathbf{p}_-(g,\gamma)]^{-1}$ does not possess any strict positive diagonals and the matrix $[\mathbf{p}_-(g,\gamma)]S^i[\mathbf{p}_-(g,\gamma)]^{-1}$ has the form

$$\sum_{r=0}^{i} v_r S^r + \sum_{r<0} (S^T)^{-r} v_r,$$

with all $v_r \in \mathcal{D}_{\mathbb{N}}(R)$ and $v_i \in \mathcal{D}_{\mathbb{N}}(R)^*$. Combining this with the first expression found yields for all $i \geqslant 1$

$$\partial_i(\Phi(g)) = (\sum_{r=1}^{i} v_r S^r)\Phi(g) = \mathcal{C}_{i,\Phi(g)}\Phi(g).$$

Thus $\Phi(g)$ satisfies the conditions in part (b) of Proposition 3.1 and hence is a wave matrix of the strict $k[S]$-hierarchy. The corresponding solution $\mathcal{M}_g$ of the strict $k[S]$-hierarchy is

$$\mathcal{M}_g = [\mathbf{p}_-(g,\gamma)]S[\mathbf{p}_-(g,\gamma)]^{-1}. \tag{4.13}$$

Also here the factor $\mathbf{u}_+(g,\gamma)^{-1}$ plays no role at the construction of $\mathcal{M}_g$. Hence, multiplying g from the right with an element of $U_+(H)$ does not affect the solution $\mathcal{M}_g$. For completeness we resume the foregoing results in a

Theorem 4.1. *Let g be an element in the group $G(\mathcal{S}_2)$.*

(a) For any γ in $\Gamma(g)$ let $\mathbf{u}_-(g,\gamma)$ be the unipotent component of $\gamma^{-1}g$ in the $(U_-(\mathcal{S}_2), P_+(H))$-splitting of $\Omega(\mathcal{S}_2)$. Let the oscillating matrix $\Psi(g) \in \mathcal{O}(S)$ be defined by formula (4.9). Then $\Psi(g)$ satisfies the linearization of the $k[S]$-hierarchy w.r.t. the matrix $\mathcal{L}_g$ defined by formula (4.10). The solution $\mathcal{L}_g$ of the $k[S]$-hierarchy satisfies for all $g \in G(\mathcal{S}_2)$ and all $p \in P_+(H)$ that $\mathcal{L}_g = \mathcal{L}_{gp}$.

(b) For any γ in $\Gamma(g)$ let $\mathbf{p}_-(g,\gamma)$ be the parabolic component of $\gamma^{-1}g$ in the $(P_-(\mathcal{S}_2), U_+(H))$-splitting of $\Omega(\mathcal{S}_2)$. Let the oscillating matrix $\Phi(g) \in \mathcal{O}(S)$ be defined by formula (4.12). Then $\Phi(g)$ satisfies the linearization of the $k[S]$-hierarchy w.r.t. the matrix $\mathcal{M}_g$ defined by formula (4.13). The solution $\mathcal{M}_g$ of the $k[S]$-hierarchy satisfies for all $g \in G(\mathcal{S}_2)$ and all $u \in U_+(H)$ that $\mathcal{M}_g = \mathcal{M}_{gu}$.

Remark 4.1. The manifolds $G(\mathcal{S}_2)/P_+(H)$ and $G(\mathcal{S}_2)/U_+(H)$ are the analogues for the $k[S]$-hierarchy and its strict version of the Grassmann manifold Gr(H) and its cover, the flag variety $\mathcal{F}_1$, used in [5] resp. [6] to construct solutions of KP resp. strict KP.

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There is no conflict of interests.

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Received 17.06.2021
Reviewed 30.08.2021
Accepted for press 10.09.2021

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Поступила в редакцию 17.06.2021 г.
Поступила после рецензирования 30.08.2021 г.
Принята к публикации 10.09.2021 г.