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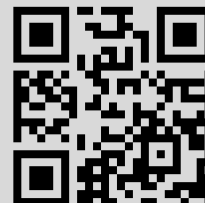
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## Holomorphic bundles and scalar difference operators: one-point constructions

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As in [1], we shall consider a non-singular algebraic curve  $\Gamma$  with a distinguished point  $P_0 = \infty$  and a local coordinate  $z = k^{-1}$ ,  $z(P_0) = 0$ . We denote by  $A = A(\Gamma, P_0)$  the ring of algebraic functions with a single pole at  $P_0$ . We prescribe the ‘inverse problem data’ as a sequence of points  $(\gamma_1, \dots, \gamma_{lg})$ , where  $l$  is the ‘rank’ and  $g$  is the genus of  $\Gamma$ , parameters  $\alpha_{sj}$ ,  $s = 1, \dots, lg$ ,  $j = 1, \dots, l-1$  and an  $l \times l$  matrix-valued function  $\chi_n^{(0)}(k)$  ( $n \in \mathbb{Z}$ ) with the only non-zero elements  $\chi^{(0)p, p+1} = 1$ ,  $p \leq l-1$ ,  $\chi^{(0)lq} = a_q$ ,  $q = 1, \dots, l$ , where  $a_q$  are polynomials in  $k$  depending on  $n$ .

**Theorem 1.** *For any vector  $\eta_0$  and generic data there exists a unique vector-valued ‘Baker-Akhiezer’ function  $\psi_n(P)$ ,  $P \in \Gamma$ , meromorphic on  $\Gamma \setminus P_0$  with poles of order one at the points  $\gamma_s$ , where the residues are connected by  $(\text{res}_{\gamma_s} \psi^{q+1}) = \alpha_{sq} (\text{res}_{\gamma_s} \psi^1)$ ,  $s = 1, \dots, lg$ ,  $q = 1, \dots, l-1$ . In a neighbourhood of  $\infty = P_0$  the vector-valued function  $\psi$  has the asymptotics  $\psi = [\eta_0 + \sum_{s \geq 1} \eta_{sn} k^{-s}] \Psi^{(0)}$ ,  $\Psi_x^{(0)} = \chi^{(0)} \Psi^{(0)}$  or  $\Psi_{n+1}^{(0)} = \chi_n^{(0)} \Psi_n^{(0)}$ ,  $\Psi^{(0)}$  being an  $l \times l$ -matrix.*

**Theorem 2.** *Suppose that the matrix  $\chi^{(0)}$  depends on  $k$  in such a way that only one of the functions  $a_{jn}(k)$  has the form  $a_j = k - v_{j, n+1}^{(0)}$  and all the remaining  $a_q$  are independent of  $k$  for  $q \neq j$ . Then for any function  $f(P) \in A(\Gamma, P_0)$  with a pole of order  $\tau$  there is a unique operator  $L_f$  of the form*

$$L_f = \sum_{-M}^{+N} u_{pn} T^p,$$

where  $N = \tau(l-j+1)$ ,  $M = \tau(j-1)$ ,  $T\psi_n = \psi_{n+1}$  and  $M+N = \tau l$ , such that the vector-valued Baker-Akhiezer function  $\psi$  constructed in Theorem 1 with  $\eta_0 = (\eta_0^q)$ ,  $\eta_0^q = \delta^{qj}$  satisfies the equation

$$L_f \psi = f \psi.$$

*Remark.* For  $j = 1$  this assertion is contained in [1] and [2] in the continuous case. Recall that  $(\alpha, \gamma)$  are the Tyurin parameters characterizing the framed holomorphic stable bundle  $\eta$ , where  $c_1(\det \eta) = lg$ . All the earlier constructions of commutative difference operators (of rank 1) required no fewer than two ‘infinite’ points on  $\Gamma$ . We note that symmetric operators  $M = N$  are possible only in the case of even rank  $l = 2j - 2$ .

Following the idea of [1], we consider the multiparameter vector-valued Baker-Akhiezer function. It is determined by the same data  $(\Gamma, P_0, \gamma_s, \alpha_{sj}, z = k^{-1})$  as in Theorem 1, but in addition for each new variable  $t_p$  a matrix  $M^{(0p)}$ ,  $p = 1, 2, \dots$  is given. The ‘input’ matrix  $\Psi_n^{(0)}$  is determined by the equations ( $t = (t_1, t_2, \dots)$ )

$$\Psi_{n+1}^{(0)} = \chi_n^{(0)} \Psi_n^{(0)}, \quad \Psi_{t_p}^{(0)} = M^{(0p)} \Psi^{(0)}, \quad p = 1, 2, \dots,$$

where the  $\chi^{(0)}$  are chosen as in Theorem 2.

**Theorem 3.** *For any  $l \geq 2$  one can choose matrices  $M^{(0p)}$ ,  $p = 1, 2, \dots$ , in such a way that the vector-valued Baker-Akhiezer function  $\psi$  determines a two-dimensionalized Toda lattice hierarchy of solutions of the inverse problem  $(\Gamma, P_0, z = k^{-1}, \gamma_s, \alpha_{sq})$ . (The solution determined by  $\psi$  will be called a solution of rank  $l$ .)*

**Example.** Let  $g = 1, l = 2, a_1 = -c_{n+1}^{(0)}, a_2 = k - v_{n+1}^{(0)}$  and let the data  $(\gamma_1, \gamma_2, \alpha_1, \alpha_2)$  and the function  $f(P) = \lambda = k^2$  be given on  $\Gamma$ . We use the Baker-Akhiezer vector  $\Psi_n$  to construct a matrix  $\widehat{\Psi}_n$  with rows  $\psi_n, \psi_{n+1}$ . We have  $\widehat{\Psi}_{n+1} = \chi_n \widehat{\Psi}_n$ , where  $\chi_n = (-c_{n+1}^{0,1}, k - v_{n+1}) + O(k^{-1})$ . The poles of  $\chi_n$  are at the points  $\gamma_{sn}$ , where  $\gamma_{s0} = \gamma_s, s = 1, 2$ . The zeros of  $(\det \chi_n)$  are at the points  $\gamma_{s,n+1}$ . Moreover,  $\alpha_{sn} \operatorname{res}_{\gamma_{sn}} \chi^{i1} = \operatorname{res}_{\gamma_{sn}} \chi^{i2}, i = 1, 2, \alpha_{s,n+1} = -\chi^{22}(\gamma_{s,n+1})$ . The quantity  $\gamma_{1n} + \gamma_{2n} = c$  is independent of  $n$ . The operators  $L_f$  can be computed effectively. For  $f = \lambda = -P(z)$  and  $c = 0$  we have the fourth-order symmetrizable operator

$$\begin{aligned} \lambda \psi_n &= L_\lambda \psi_n = [(L_2)^2 + u_n] \psi_n, & L_2 \psi_n &= \psi_{n+1} + v_n \psi_n + c_n \psi_{n-1}, \\ u_n &= -[\wp(\gamma_{n-1}) + \wp(\gamma_{n-2})] + b_{n-1} + b_{n-2}, & \wp(z) &= -\zeta'(z), \\ b_n &= 2\wp'(\gamma_n)[\wp(\gamma_{n+1} + \gamma_n) - \wp(\gamma_{n+1} - \gamma_n)][\wp'(\gamma_{n+1} + \gamma_n) - \wp'(\gamma_{n+1} - \gamma_n)]^{-1}, \\ c_n &= (\alpha_{1n} - \alpha_{2n})^{-1}[\zeta(\gamma_{n+1} - \gamma_n) - \zeta(\gamma_{n+1} + \gamma_n) + 2\zeta(\gamma_n)]. \end{aligned}$$

Here  $\gamma_n = \gamma_{1n}$  and  $v_n$  are arbitrary functions,  $\alpha_{1n}$  and  $\alpha_{2n}$  being determined by

$$\begin{aligned} \alpha_{1,n+1} &= -v_{n+1} + \zeta(\gamma_{n+1}) + \frac{\alpha_{1n}}{\alpha_{1n} - \alpha_{2n}} \zeta(\gamma_{n+1} - \gamma_n) + \frac{\alpha_{2n}}{\alpha_{1n} - \alpha_{2n}} \zeta(\gamma_{n+1} + \gamma_n), \\ \alpha_{2,n+1} &= -v_{n+1} - \zeta(\gamma_{n+1}) - \frac{\alpha_{1n}}{\alpha_{1n} - \alpha_{2n}} \zeta(\gamma_{n+1} - \gamma_n) - \frac{\alpha_{2n}}{\alpha_{1n} - \alpha_{2n}} \zeta(\gamma_{n+1} + \gamma_n). \end{aligned}$$

We consider the time dynamics with  $t = t_1, M^{(01)} = \chi_n^{(0)} + \operatorname{diag}(v_n^{(0)}, v_{n+1}^{(0)})$ . For the Baker-Akhiezer matrix  $\widehat{\Psi}_n(t)$  we have  $\widehat{\Psi}_{nt} = M_n \widehat{\Psi}_n$ , where  $M_n = \chi_n + \operatorname{diag}(v_n, v_{n+1}) + O(k^{-1})$ . From the compatibility of the variables  $n$  and  $t$  we obtain a non-linear system for  $(c_n(t), v_n(t))$ :

$$\begin{aligned} \dot{c}_{n+1} &= c_{n+1}(v_{n+1} - v_n), & \dot{v}_{n+1} &= c_{n+2} - c_{n+1} + \varkappa_{n+1} - \varkappa_n, \\ \chi_n^{22} &= k - v_n + \varkappa_n k^{-1} + O(k^{-2}). \end{aligned}$$

This system is a discretization of the so-called ‘Krichever-Novikov equation’ in [1]. The coefficient  $\varkappa_n$  can be computed explicitly using the Tyurin parameter dynamics. In a forthcoming note we shall consider two- (and more) point constructions of rank  $l - 1$ , in which case a number of new phenomena will appear.

### Bibliography

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