M. Fuchs, G. A. Seregin, Partial regularity of the deformation gradient for some model problems in nonlinear twodimensional elasticity,
Algebra i Analiz, 1994, Volume 6, Issue 6, 128–153

Алгебра и анализ
Том 6 (1994), вып. 6

PARTIAL REGULARITY OF THE DEFORMATION GRADIENT FOR SOME MODEL PROBLEMS IN NONLINEAR TWODIMENSIONAL ELASTICITY

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Abstract. We consider the model problem of minimizing the functional $\int_\Omega \frac{1}{2}|\nabla u|^2 + h(\det \nabla u)\,dx$ where $u : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ and $h : \mathbb{R} \to [0,\infty]$ denotes a function which is convex and smooth on $(0,\infty)$, $\lim\limits_{t\downarrow 0} h(t) = +\infty$ and $h \equiv +\infty$ on $(-\infty, 0]$. In particular, we show that it is possible to introduce an approximation $\int_\Omega \frac{1}{2}|\nabla u|^2 + h_\delta(\det \nabla u)\,dx$ for the energy whose minimizers u_δ are of class C^1 on some open subset Ω_δ of Ω and converge strongly in $H^{1,2}(\Omega,\mathbb{R}^2)$ to a minimizer u of the original problem. Moreover, we have control on the measure of the exceptional set in the sense that $|\Omega - \Omega_\delta| \to 0$ as $\delta \to 0$.

§0. Introduction

In this article we consider the following two-dimensional variational problem which can be seen as a simple model for Neo–Hookean materials (compare, e.g., [B1, BOP, C]):

$$\begin{cases} \text{find} \quad u \in \overset{\circ}{H}{}^{1,2}(\Omega,\mathbb{R}^2) + u_0 \ (\Omega \quad \text{is a bounded domain in} \quad \mathbb{R}^2) \quad \text{such that,} \\ J(u) = \inf\{J(v) : v \in \overset{\circ}{H}{}^{1,2}(\Omega,\mathbb{R}^2) + u_0, J(v) < +\infty\} \end{cases} \tag{0.1}$$

where the energy J is defined as

$$J(w) := \int\limits_\Omega \frac{1}{2}|\nabla w|^2 + h(\det \nabla w)\,dx$$

and u_0 is a given mapping of class $H^{1,2}(\Omega,\mathbb{R}^2)$ representing the boundary values. The function $h : \mathbb{R} \to [0,+\infty]$ is required to have the following characteristic properties

$$h \quad \text{is of class} \quad C^2 \quad \text{and convex on} \quad (0,\infty), \tag{0.2}$$

$$\lim_{t\downarrow 0} h(t) = +\infty, \quad h \equiv +\infty \quad \text{on} \quad (-\infty, 0]. \tag{0.3}$$

Key words and phrases. Nonlinear elasticity, partial regularity, approximation.

Although the formulation of problem (0.1) looks quite simple it inherits some unsolved mathematical problems of nonlinear elasticity. Concerning existence of minimizers it is well known (see [B1, BM] that there is a solution of (0.1) if $J(u_1) < +\infty$ for some function u_1 in the space $\overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0$ which we will assume from now on. The first question which arises is: Are minimizers u weak solutions of the corresponding Euler-Lagrange equation

$$\operatorname{div} \frac{\partial W}{\partial F}(\nabla u) = 0 \quad \text{on} \quad \Omega, W(F) := \frac{1}{2}|F|^2 + h(\det F), \tag{0.4}$$

i.e., do the equilibrium equations hold? The answer is positive for sufficiently smooth functions u which in addition satisfy $\det \nabla u \geqslant \delta$ for some positive number δ. But—and this is the second problem—it is a priori unknown if minimizers have any differentiability properties. So the answers to both questions will be closely related.

The difficulties are caused by condition (0.3) imposed on h. If u denotes a minimizer then $\det(\nabla u(x) + t\nabla\varphi(x)) \leqslant 0$ on a set of positive measure can occur for $|t| << 1$ and $\varphi \in C_0^\infty(\Omega, \mathbb{R}^2)$, hence $J(u + t\varphi) = +\infty$ which makes it impossible to obtain (0.4) along standard lines. On the other hand—since we do not require h to have linear growth at infinity—it might also happen that $J(v) = +\infty$ for some $v \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0$ even if $\det \nabla v \geqslant \delta > 0$ a.e.

Returning to the question of regularity for minimizers u of problem (0.1) we can only state that u is an everywhere continuous function. This follows alone from the facts $\nabla u \in L^2(\Omega, \mathbb{R}^{2\times 2})$ and $\det \nabla u(x) > 0$ a.e. by quoting results of Vodopyanow-Goldstein [VG] or more recently Šverak [Š]. But we believe that the following conjecture is true.

Conjecture. *Let u denote a minimizer of* (0.1). *Then there exists a set Ω_0 whose complement in Ω is of vanishing measure such that if $x_0 \in \Omega_0$ and $\det \nabla u(x_0) > 0$ then ∇u is Hölder continuous in a neighborhood of x_0.*

In the above statement x_0 is assumed to be a Lebesgue point for ∇u. We are not able to prove this conjecture but known examples (see [BOP2]) and also some other results which will be described below confirm correctness of our conjecture.

In the present paper we introduce special approximations for problem (0.1). Instead of h we consider the functions

$$h_\delta(t) := \begin{cases} h(\delta) + h'(\delta)(t - \delta) & if \quad t \leqslant \delta, \\ h(t) \quad if \quad \delta \leqslant t \leqslant \delta^{-1}, \\ h(\frac{1}{\delta}) + h'(\frac{1}{\delta})(t - \frac{1}{\delta}) & if \quad t \geqslant \delta^{-1} \end{cases} \tag{0.5}$$

for $t \in \mathbb{R}$ and $0 < \delta < 1$ provided

$$h'(t) \to +\infty \quad \text{as} \quad t \to +\infty. \tag{0.6}$$

If (0.6) does not hold we just let

$$h_\delta(t) := \begin{cases} h(\delta) + h'(\delta)(t-\delta) & \textit{if} \quad t \leqslant \delta, \\ h(t) & \text{if} \quad t \geqslant \delta. \end{cases} \tag{0.7}$$

Definition (0.5) covers the case of functions h growing stronger than linear at infinity whereas (0.7) is appropriate for at most linear growth. Since both cases are similar we restrict ourselves to (0.5). Clearly h_δ is an approximation of h in the sense that

$$h_\delta(t) \leqslant h_\varepsilon(t) \leqslant h(t), \quad \lim_{\delta \downarrow 0} h_\delta(t) = h(t) \tag{0.8}$$

for all $t \in \mathbb{R}$ and $0 < \varepsilon \leqslant \delta < 1$. Instead of (0.1) we now look at the problem

$$\begin{cases} \text{find} \quad u_\delta \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0 \quad \text{such that,} \\ J_\delta(u_\delta) = \inf\{J_\delta(v) : v \in H^{1,2}(\Omega, \mathbb{R}^2) + u_0\} \end{cases} \tag{0.9}$$

where

$$J_\delta(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 + h_\delta(\det \nabla v)\, dx \tag{0.10}$$

is well defined on the space $H^{1,2}(\Omega, \mathbb{R}^2)$ due to the linear growth of h_δ at $\pm\infty$. Moreover, the existence of a minimizer u_δ follows from the general theory of quasiconvex integrals (see for example [D]). In Theorem 1.1 we show that u_δ is Hölder continuous on Ω. Since h''_δ is discontinuous it is by no means obvious if the (partial) $C^{1,\alpha}$-regularity results of Fusco–Hutchinson [FH3] and Giaquinta–Modica [GM] apply. In Theorem 1.2 and 1.3 we show that some version of the Conjecture is true: There is an open subset Ω_δ of Ω such that

$$|\Omega_\delta \Delta [\delta < \det \nabla u_\delta < \delta^{-1}]| = 0$$

and

$$\nabla u \in C^{0,\alpha}(\Omega_\delta) \quad \text{for all} \quad 0 < \alpha < 1.$$

Here Δ is the symmetric difference, $[\delta < \det \nabla u_\delta < \delta^{-1}]$ denotes the set $\{x \in \Omega : \delta < \det \nabla u_\delta(x) < \delta^{-1}\}$ and $|\cdot|$ is Lebesgue's measure. We prove these statements by a version of Caccioppoli's inequality obtained in [FH2] for the functional (compare also [FH1])

$$\int_\Omega |\nabla u|^2 + (\det \nabla u)^2\, dx.$$

It turns out that the arguments used in [FH2] immediately extend to functionals which have linear growth with respect to $\det \nabla u$.

In Theorem 1.4 we finally prove strong converge $u_\delta \to u$ in $H^{1,2}(\Omega, \mathbb{R}^2)$ to a solution u of (0.1) and we also show $|\Omega - \Omega_\delta| \to 0$ as $\delta \to 0$. This behaviour might serve as a first step towards a numerical computation of solutions for (0.1).

We would like to remark that any minimizer u of (0.1) is a weak solution of

$$\operatorname{div}\big((\nabla u)^T \frac{\partial W}{\partial F}(\nabla u) - W(\nabla u)\mathbf{1}\big) = 0 \tag{0.11}$$

which can be seen by calculating the first variation with respect to reparametrisations of Ω (compare [FS, BOP]). Since for all smooth mappings $u : \Omega \to \mathbb{R}^2$ with positive determinant the identity

$$\operatorname{div}\big((\nabla u)^T \frac{\partial W}{\partial F}(\nabla u) - W(\nabla u)\mathbf{1}\big) = (\nabla u)^T \operatorname{div}\big(\frac{\partial W}{\partial F}(\nabla u)\big)$$

holds it follows that in this case (0.4) and (0.11) are equivalent. It is therefore of interest to study also the smoothness of weak solutions to (0.11) but here we are confronted with the same difficulties as in the case of minimizers. If we replace W in (0.11) by the approximation

$$\begin{aligned} \tilde{W}_\delta(F) &:= \frac{1}{2}|F|^2 + \tilde{h}_\delta(\det F), \quad 0 < \delta < 1, \\ \tilde{h}_\delta(t) &:= \begin{cases} +\infty, & t < 0, \\ h_\delta(t), & t \geqslant 0 \end{cases} \end{aligned}$$

then it was shown in [FS] that weak solutions $\tilde{u}_\delta$ with the property $\det \nabla \tilde{u}_\delta \geqslant 0$ are Hölder continuous on Ω for any exponent $0 < \alpha < 1$.

It can also be proved that the functions $\tilde{u}_\delta$ converge strongly towards a solutions of (0.11), i.e., we have a corresponding approximation procedure for stationary points (w.r.t. reparametrisations of Ω) of the functional J. But since the details are more technical than in the case of minimizers we limited our discussion to minimization problems.

§1. Notation and main results

We first fix our notation which will be frequently used throughout the paper. Let

$$\begin{aligned} \mathbb{M} &:= \text{space of all real } (2 \times 2)\text{-matrices } F = (F_{ij}), \\ \mathbb{M}_+ &:= \{F \in \mathbb{M} : \det F > 0\}, \\ \bar{\mathbb{M}}_+ &:= \{F \in \mathbb{M} : \det F \geqslant 0\}, \\ a \cdot b &:= a_i b_i, |a| := (a \cdot a)^{\frac{1}{2}} \quad \text{for} \quad a = (a_i), b = (b_i) \in \mathbb{R}^2, \\ F(a) &:= (F_{ik} a_k) \in \mathbb{R}^2, F^T := (F_{ji}), FH := (F_{ik} H_{kj}) \in \mathbb{M}, \\ F : H &:= tr(F^T H) = F_{ij} H_{ij}, |F| := (F : F)^{\frac{1}{2}}, \\ [F] &:= \sup_{|a|=1} |F(a)| \end{aligned}$$

for matrices $F = (F_{ij})$, $H = (H_{ij}) \in \mathbb{M}$ and vectors $a \in \mathbb{R}^2$.

We further assume that Ω denotes a bounded open region in $\mathbb{R}^2$. Next we formulate a few statements concerning regularity of minimizers of certain variational integrals whose energy densities have quadratic growth with respect to ∇u and linear growth in $\det \nabla u$. More precisely, we consider the following problem:

$$\begin{cases} \text{find} \quad u \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0 \quad \text{such that} \\ I(u) = \inf\{I(v) : v \in \overset{\circ}{H}{}^{1,2}(\Omega\mathbb{R}^2) + u_0\} \end{cases} \tag{1.1}$$

where $I(w) := \int_\Omega \frac{1}{2} |\nabla w|^2 + h(\det \nabla w)\, dx$, u_0 is a given function of Sobolev class $H^{1,2}(\Omega, \mathbb{R}^2)$ and $h : \mathbb{R} \to [0, \infty)$ denotes a convex function of class C^1 with the property

$$d_0 := \|h'\|_{L^\infty(\mathbb{R})} < +\infty. \tag{1.2}$$

Since the integrand is quasiconvex with appropriate growth rates standard arguments (see for example [D, B1, BM]) imply that problem (1.1) has at least one solution for which the following Euler–Lagrange equation can easily be obtained:

$$\int_\Omega (\nabla u + h'(\det \nabla u) \operatorname{Cof} \nabla u) : \nabla p\, dx = 0 \tag{1.3}$$

for any $p \in H^{1,2}(\Omega, \mathbb{R}^2)$. Our first result concerns the behaviour of weak solutions of (1.3).

Theorem 1.1. *Suppose that h is a function as described above satisfying* (1.2) *and let $u \in H^{1,2}(\Omega, \mathbb{R}^2)$ denote a weak solution of* (1.3)*. Then u is locally Hölder continuous on Ω.*

In order to obtain C^1-partial regularity for the approximations of problem (0.1) we impose the following additional condition on the function h

$$\begin{cases} h \quad \text{is of class} \quad C^2 \quad \text{on the whole line except for} \\ \text{a finite number of points} \quad t_1, \dots, t_s \quad \text{and moreover} \\ d := \|h''\|_{L^\infty(\mathbb{R})} < \infty. \end{cases} \tag{1.4}$$

We introduce the sets $E_0 := \{t_1, \dots, t_s\}$, $E_1 := \mathbb{R} - E_0$ and

$$\sum := \{x \in \Omega : \liminf_{r \downarrow 0} U(x,r) > 0\} \cup \{x \in \Omega : \nexists \lim_{r \downarrow 0} \fint_{B_r(x)} \nabla u\, dx\} \tag{1.5}$$

for functions $u \in H^{1,2}(\Omega, \mathbb{R}^2)$. Here $U(x,r)$ denotes the excess of ∇u with respect to the disc $B_r(x) \subset \Omega$, i.e.,

$$\begin{cases} U(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 dz, \\ (\nabla u)_{x,r} := \fint_{B_r(x)} \nabla u dz := |B_r(x)|^{-1} \int_{B_r(x)} \nabla u \, dz. \end{cases} \tag{1.6}$$

It is well known that ($|A| :=$ Lebesgue measure of the set A)

$$\left|\sum\right| = 0 \tag{1.7}$$

holds for any Sobolev function u of class $H^{1,2}(\Omega, \mathbb{R}^2)$, hence

$$\liminf_{r \downarrow 0} U(x,r) = 0 \tag{1.8}$$

and

$$\nabla u(x) := \lim_{r \downarrow 0} \fint_{B_r(x)} \nabla u \, dz \quad \text{exists} \tag{1.9}$$

is true for almost all points $x \in \Omega$.

Theorem 1.2. *Let* (1.2) *and* (1.4) *hold for the convex function* $h : \mathbb{R} \to [0, \infty)$. *Consider a minimizer* u *of problem* (1.1). *Then for any real number* $\nu \in (0,1)$ *and any point* $x_0 \in \Omega - \sum$ *satisfying*

$$\det \nabla u(x_0) \in E_1 \tag{1.10}$$

there is neighborhood V *of* x_0 *in* Ω *such that* $\nabla u \in C^{0,\nu}(V, \mathbb{M})$.

We now apply these results to the problems (0.1) and (0.9) which gives

Theorem 1.3. *Suppose that* $h : \mathbb{R} \to [0, \infty]$ *satisfies the hypotheses* (0.2) *and* (0.3) *and let* u^δ *denote a minimizer of the regularized problem* (0.9) *for some* $\delta \in (0.1)$. *Then* u^δ *is Hölder continuous on* Ω. *Moreover, there exists an open subset* Ω_δ *of* Ω *such that* $\nabla u^\delta \in C^{0,\nu}(\Omega_\delta, \mathbb{M})$ *for any* $0 < \nu < 1$. *In addition we have* $\delta < \det \nabla u^\delta(x) < \frac{1}{\delta}$ *for all points* $x \in \Omega_\delta$. *On* Ω_δ *the following Euler–Lagrange equation is satisfied:*

$$\int_{\Omega_\delta} (\nabla u^\delta + h'(\det \nabla u_\delta) \operatorname{Cof} \nabla u^\delta) : \nabla p \, dx = 0$$

for any $p \in H^{1,2}(\Omega_\delta, \mathbb{R}^2)$.

Concerning the size of Ω_δ we can prove

Theorem 1.4. *Let the assumptions of Theorem* 1.3 *hold and suppose* $J(u_1) < +\infty$ *for some function* $u_1 \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0$. *Then there is a sequence* $\{u^\delta\}_{0<\delta<1}$ *with the following properties:*

$$u^\delta \ \textit{ is a solution of problem } (0.9) \tag{1.11}$$

$$\begin{cases} u^\delta \to u \quad \textit{in} \quad H^{1,2}(\Omega, \mathbb{R}^2), \\ \int\limits_\Omega h_\delta(\det \nabla u^\delta)\, dx \to \int\limits_\Omega h(\det \nabla u)\, dx \quad \textit{as} \quad \delta \downarrow 0 \end{cases} \tag{1.12}$$

$$u \ \textit{ solves problem } (0.1). \tag{1.13}$$

Moreover, we have the estimate

$$|\Omega - \Omega_\delta| \leqslant \min\{h(\delta), h(1/\delta)\}^{-1} J(u) \tag{1.14}$$

so that $|\Omega - \Omega_\delta| \xrightarrow{\delta \downarrow 0} 0$.

§2. Hölder continuity for the solutions of the regularized problems

In this section we give a short

Proof of Theorem 1.1. Of course we may quote [G], V. Theorem 3.1, by remarking that $h \geqslant 0$ combined with (1.2) make sure that the integrand has the required growth. Thus we deduce $\nabla u \in L^p_{\mathrm{loc}}(\Omega, \mathbb{M})$ for some $p > 2$ so that u is Hölder continuous.

On the other hand—using convexity of h—the desired claim can be proved very easily. Let $\varphi \in C^1_0(\Omega)$ denote a function with the properties

$$\operatorname{spt}\varphi \subset B_R(x_0) \subset\subset \Omega, \quad 0 \leqslant \varphi \leqslant 1,$$
$$\varphi = 1 \quad \text{on} \quad B_{R/2}(x_0), \quad |\nabla\varphi| \leqslant c_{21} R^{-1}$$

for some positive constant c_{21}. Inserting $p := \varphi^2(u - (u)_{x_0,R})$ into equation (1.3) we get

$$\int\limits_{B_R(x_0)} \varphi^2(\nabla u + (h'(\det \nabla u) - h'(0)) \operatorname{Cof} \nabla u) : \nabla u \, dx$$
$$= -2 \int\limits_{B_R(x_0} \varphi(\nabla u + h'(\det \nabla u) \operatorname{Cof} \nabla u) : (\nabla\varphi \otimes (u - (u)_{x_0,R})$$
$$- \int\limits_{B_R(x_0)} \varphi^2 h'(0) \operatorname{Cof} \nabla u : \nabla u \, dx.$$

Recall the identities

$$\operatorname{Cof} \nabla u : \nabla u = 2 \det \nabla u, \qquad \int\limits_{B_R(x_0)} \operatorname{Cof} \nabla u : \nabla \psi \, dx = 0, \quad \psi \in H^{1,2}(B_R(x_0), \mathbb{R}^2),$$

so that

$$-\int_{B_R(x_0)} \varphi^2 h'(0)\,\mathrm{Cof}\,\nabla u : \nabla u\,dx = 2\int_{B_R(x_0)} \varphi h'(0)\,\mathrm{Cof}\,\nabla u : (\nabla\varphi \otimes (u-(u)_{x_0,R}))\,dx.$$

Hence we arrive at

$$\int_{B_R(x_0)} \varphi^2\big(|\nabla u|^2 + 2\det\nabla u(h'(\det\nabla u) - h'(0))\big)\,dx$$

$$= -2\int_{B_R(x_0)} \varphi(\nabla u + [h'(\det\nabla u) - h'(0)]\,\mathrm{Cof}\,\nabla u) : (\nabla\varphi \otimes (u-(u)_{x_0,R}))\,dx.$$

By convexity we have

$$\det\nabla u(h'(\det\nabla u) - h'(0)) \geqslant 0.$$

so that the above equation turns into Caccioppoli's inequality

$$\int_{B_{R/2}(x_0)} |\nabla u|^2\,dx \leqslant c_{22}(d_0)R^{-2}\int_{B_R(x_0)} |u-(u)_{x_0,R}|^2\,dx$$

for a suitable constant c_{22} depending on the bound d_0 for h'. This completes the proof (see [G]).

§3. A decay estimate for the excess

In this section we suppose that all assumptions of Theorem 1.2 are fulfilled. The main step towards partial regularity is an excess decay estimate at points x_0 where this quantity is already sufficiently small. Unfortunately we can not rely on the approaches given for example in the papers [FH3] and [GM] because this would require h to be of class C^2 on the whole line which is not appropriate for our purposes. To overcome these difficulties we first observe the following lemma which is a straightforward extension of Lemma 2.1 in [FH2].

Lemma 3.1. *Suppose that we are given numbers $Q \geqslant 1$ and $\varepsilon > 0$ and a function $w \in H^{1,2}_{\mathrm{loc}}(B,\mathbb{R}^2)$ such that*

$$\int_{[w\neq\tilde w]} |\nabla w|^2\,dx \leqslant Q\int_{[w\neq\tilde w]} (|\nabla\tilde w|^2 + \varepsilon^2(\det\nabla\tilde w)^2)\,dx \tag{3.1}$$

holds for any function $\tilde\omega \in H^{1,2}_{\mathrm{loc}}(B,\mathbb{R}^2)$ which coincides with w outside of some compact subset of B. Then there is a constant c_{31} depending only on Q such that

$$\fint_{B_{R/2}} |\nabla w|^2\,dx \leqslant c_{31}\left(R^{-2}\fint_{B_R} |w-(w)_R|^2\,dx + \varepsilon^2\left(\fint_{B_R} |\nabla w|^2\,dx\right)^2\right) \tag{3.2}$$

holds for all $0 < R < 1$. Here we have used the notions $B := B_1$, $B_R := B_R(0)$, $(w)_R := (w)_{0,R}$ and $[w \neq \tilde w] := \{x \in B : w(x) \neq \tilde w(x)\}$.

Now we can state our

Main Lemma 3.2. *For any open set $E_* \subset\subset E_1$ and for arbitrary numbers $M \in (0,\infty)$, $t \in (0,\frac{1}{4})$ there exist constants $c_* = c_*(d,M)$, $\varepsilon_0 = \varepsilon_0(t,d,M,E_*)$, $R_0 = R_0(t,d,M,E_*)$ with the following properties: If*

$$0 < R < R_0, \quad B_R(x_0) \subset\subset \Omega \tag{3.3}$$

$$\left| \fint_{B_{tR}(x_0)} \nabla u \, dx \right|, \quad \left| \fint_{B_R(x_0)} \nabla u \, dx \right| < M, \tag{3.4}$$

$$\det \fint_{B_R(x_0)} \nabla u \, dx \in E_*, \tag{3.5}$$

$$U(x_0,R) \leqslant \varepsilon_0^2 \tag{3.6}$$

hold for some disc $B_R(x_0)$, then

$$U(x_0,tR) \leqslant c_* t^2 U(x_0,R). \tag{3.7}$$

Remark. The precise value of c_* can be found at the end of the proof of Lemma 3.2.

In order to establish the decay estimate (3.7) we recall a known result for linear elliptic systems with constant coefficients (see [G]).

Lemma 3.3. *Suppose that $A \in \mathbb{M}$ satisfies*

$$\det A \in E_1, \tag{3.8}$$

$$|A| \leqslant M \tag{3.9}$$

and let $v \in H^{1,2}(B,\mathbb{R}^2)$ denote a solution of

$$\int_B (\nabla v : \nabla w + h''(\det A) \operatorname{Cof} A : \nabla v \operatorname{Cof} A : \nabla w) \, dx = 0 \tag{3.10}$$

being valid for any $w \in \overset{\circ}{H}{}^{1,2}(B,\mathbb{R}^2)$. Then there is a constant $c_{**} = c_{**}(d,M)$ such that

$$V(t) \leqslant c_{**} t^2 V(1) \tag{3.11}$$

for all $t \in (0,\frac{1}{2})$. Here we have abbreviated $V(t) = \fint_{B_t} |\nabla v - (\nabla_v)_t|^2 \, dx$.

We now come to the

Proof of the Main Lemma. Suppose that the statement of the lemma is false; then there are numbers $t \in (0, \frac{1}{4})$, $M \in (0, \infty)$, a set $E_* \subset\subset E_1$ and sequences x_k, R_k, ε_k such that

$$B_{R_k}(x_k) \subset\subset \Omega, \quad R_k \to 0, \tag{3.12}$$

$$\left| \fint_{B_{tR_k}(x_k)} \nabla u \, dx \right|, \quad \left| \fint_{B_{R_k}(x_k)} \nabla u \, dx \right| < M, \tag{3.13}$$

$$\det \fint_{B_{R_k}(x_k)} \nabla u \in E_*, \tag{3.14}$$

$$U(x_k, R_k) = \varepsilon_k^2 \to 0, \tag{3.15}$$

$$U(x_k, tR_k) > c_* t^2 \varepsilon_k^2 \tag{3.16}$$

where c_* will be chosen at the end of the proof. We know (compare (1.3))

$$\int_{B_{R_k}(x_k)} (\nabla u + h'(\det \nabla u) \operatorname{Cof} \nabla u) : \nabla p \, dx = 0 \tag{3.17}$$

for all $p \in \overset{\circ}{H}{}^{1,2}(B_{R_k}(x_k), \mathbb{R}^2)$. Let $A_k := \fint_{B_{R_k}(x_k)} \nabla u \, dx$ and $\tilde{A}_k := \fint_{B_{tR_k}(x_k)} \nabla u \, dx$. We now consider the normalized sequence

$$v_k(y) := \varepsilon_k^{-1} R_k^{-1} (u(x_k + R_k y) - A_k(R_k y) - (u)_{x_k, R_k}), \quad y \in B. \tag{3.18}$$

From (3.18) we deduce

$$(v_k)_1 = 0, \tag{3.19}$$

$$(\nabla v_k)_1 = 0, \tag{3.20}$$

$$\nabla v_k(y) = \varepsilon_k^{-1}(\nabla u(x_k + R_k y) - A_k) \tag{3.21}$$

and (3.15)-(3.17) turn into

$$V_k(1) := \fint_B |\nabla v_k|^2 \, dx = 1, \tag{3.22}$$

$$V_k(t) := \fint_{B_t} |\nabla v_k - (\nabla v_k)_t|^2 \, dx > c_* t^2, \tag{3.23}$$

$$\int_B [\nabla v_k + \varepsilon_k^{-1}(h'(\det(A_k + \varepsilon_k \nabla v_k)) - h'(\det A_k)) \operatorname{Cof}(A_k + \varepsilon_k \nabla v_k)] : \nabla w \, dx = 0,$$

$$w \in \overset{\circ}{H}{}^{1,2}(B, \mathbb{R}^2). \tag{3.24}$$

For (3.24) we again observe $\int_B \operatorname{Cof} \nabla p : \nabla \psi \, dx = 0$ for functions $p \in H^{1,2}(B, \mathbb{R}^2)$, $\psi \in \overset{\circ}{H}{}^{1,2}(B, \mathbb{R}^2)$. Using (3.13), (3.14), (3.19) and (3.22) we may assume (at least for a subsequence) that there exist $v \in H^{1,2}(B, \mathbb{R}^2)$, $A \in \mathbb{M}$ such that

$$v_k \to v \quad \text{weakly in} \quad H^{1,2}(B, \mathbb{R}^2), \tag{3.25}$$

$$v_k \to v \quad \text{strongly in} \quad L^2(B, \mathbb{R}^2), \tag{3.26}$$

$$A_k \to A \quad \text{in} \quad \mathbb{M}, \quad \det A \in \bar{E}_* \subset E_1, \quad |A| \leqslant M. \tag{3.27}$$

We claim that the limit v is a solution of the system (3.10). To this purpose we introduce the quantities

$$I_k^1(w) := \varepsilon_k^{-1} \int_B (h'(\det(A_k + \varepsilon_k \nabla v_k)) - h'(\det A_k)) \operatorname{Cof} A_k : \nabla w \, dx$$

and

$$I_k^2(w) := \int_B (h'(\det(A_k + \varepsilon_k \nabla v_k)) - h'(\det A_k)) \operatorname{Cof} \nabla v_k : \nabla w \, dx$$

where w is a fixed function in $C_0^1(B, \mathbb{R}^2)$. We calculate

$$I_k^1(w) = \int_B \int_0^1 h''(\det A_k + s(\det(A_k + \varepsilon_k \nabla v_k) - \det A_k)),$$
$$(\operatorname{Cof} A_k : \nabla v_k + \varepsilon_k \det \nabla v_k) \operatorname{Cof} A_k : \nabla w \, ds \, dx$$

(observe

$$\begin{aligned} \det(A_k + \varepsilon_k \nabla v_k) - \det A_k &= \frac{1}{2}[\operatorname{Cof}(A_k + \varepsilon_k \nabla v_k) : (A_k + \varepsilon_k \nabla v_k) - \operatorname{Cof} A_k : A_k] \\ &= \frac{1}{2}\varepsilon_k(\operatorname{Cof} \nabla v_k : A_k + \operatorname{Cof} A_k : \nabla v_k + \varepsilon_k \operatorname{Cof} \nabla v_k : \nabla v_k) \\ &= \frac{1}{2}\varepsilon_k(2 \operatorname{Cof} A_k : \nabla v_k + 2\varepsilon_k \cdot \det \nabla v_k)). \end{aligned}$$

Since

$$0 \leqslant h''(\det A_k + s(\det(A_k + \varepsilon_k \nabla v_k) - \det A_k)) \leqslant d$$

by assumption (1.4) and the fact that

$$h''(\det A_k + s(\det(A_k + \varepsilon_k \nabla v_k) - \det A_k)) \xrightarrow{k \to \infty} h''(\det A)$$

almost everywhere on $\Omega \times [0,1]$ (recall $\det A_k \in E_* \subset\subset E_1$) we deduce from dominated convergence (using (3.22)) that

$$\lim_{k\to\infty} I_k^1(w) = \lim_{k\to\infty} \int_B h''(A)(\operatorname{Cof} A_k : \nabla v_k + \varepsilon_k \det \nabla v_k) \operatorname{Cof} A_k : \nabla w \, dx$$

$$=_{(3.25)} \int_B h''(A)(\operatorname{Cof} A : \nabla v)(\operatorname{Cof} A : \nabla w) \, dx.$$

Using (1.2) and dominated convergence it is easy to see that

$$\lim_{k\to\infty} I_k^2(w) = 0.$$

Recalling (3.24) we arrive at equation (3.10). Inequality (3.11) implies

$$V(2t) \leqslant c_{**}(d,M) 4t^2 V(1) \leqslant_{(3.22)} c_{**}(d,M) 4t^2 \tag{3.28}$$

where $V(s) := \fint_{B_s} |\nabla v - (\nabla v)_s|^2 \, dx$.

In order to get the desired contradiction we introduce the functions

$$u_k(y) := v_k(y) - (\nabla v_k)_t y - (v_k)_t, \quad y \in B, \tag{3.29}$$

and choose $w \in H^{1,2}(B, \mathbb{R}^2)$ such that $w = u_k$ outside of some compact subset of B. Let

$$\tilde{w}(x) := \varepsilon_k R_k w(R_k^{-1}(x - x_k)) + \tilde{A}_k(x - x_k) + (u)_{x_k, tR_k}, x \in B_{R_k}(x_k).$$

Then—using definition (3.29)—we infer $\tilde{w} = u$ outside a compact set in $B_{R_k}(x_k)$ and since u is a minimizer we arrive at

$$\int_{B_{R_k}(x_k)} \frac{1}{2}|\nabla u|^2 + h(\det \nabla u) \, dx \leqslant \int_{B_{R_k}(x_k)} \frac{1}{2}|\nabla \tilde{w}|^2 + h(\det \nabla \tilde{w}) \, dx. \tag{3.30}$$

After change of variables and using the definitions introduced before (3.30) turns into the inequality

$$\int_B \frac{1}{2}|\tilde{A}_k + \varepsilon_k \nabla u_k|^2 \, dx + h(\det(\tilde{A}_k + \varepsilon_k \nabla u_k)) \, dx$$

$$\leqslant \int_B \frac{1}{2}|\tilde{A}_k + \varepsilon_k \nabla w|^2 \, dx + h(\det(\tilde{A}_k + \varepsilon_k \nabla w)) \, dx. \tag{3.31}$$

Observing

$$\int_B \nabla u_k\,dx = \int_B \nabla w\,dx,\ \int_B \det(\varepsilon_k \nabla u_k + \tilde{A}_k)\,dx = \int_B \det(\varepsilon_k \nabla w + \tilde{A}_k)\,dx$$

(recall $\int_B \det \nabla\varphi\,dx = \int_B \det \nabla\psi\,dx$ for functions $\varphi,\ \psi \in H^{1,2}(B,\mathbb{R}^2)$ such that $\mathrm{spt}(\varphi - \psi) \subset\subset B$) we may rewrite (3.31) in the following way:

$$\begin{aligned}\int_B \frac{1}{2}|\nabla u_k|^2 &+ \varepsilon_k^{-2}\big(h(\det(\tilde{A}_k + \varepsilon_k \nabla u_k)) - h(\det \tilde{A}_k)\\ &- h'(\det \tilde{A}_k)(\det(\tilde{A}_k + \varepsilon_k \nabla u_k) - \det \tilde{A}_k)\big)\,dx\\ \leqslant \int_B \frac{1}{2}|\nabla w|^2 &+ \varepsilon_k^{-2}\big(h(\det(\tilde{A}_k + \varepsilon_k \nabla w)) - h(\det \tilde{A}_k)\\ &- h'(\det \tilde{A}_k)(\det(\tilde{A}_k + \varepsilon_k \nabla w) - \det \tilde{A}_k)\big)\,dx. \qquad (3.32)\end{aligned}$$

By convexity of h the left hand-side of (3.32) is bounded from below by the quantity $\int_{[u_k \neq w]} \frac{1}{2}|\nabla u_k|^2\,dx$. In order to get an upper bound for the right hand-side we apply the calculus formula

$$h(y) - h(x) - h'(x)(y - x) = \int_0^1 h''(x + s(y - x))(1 - s)ds(y - x)^2$$

and infer from (3.32)

$$\begin{aligned}\int_{[u_k \neq w]} \frac{1}{2}|\nabla u_k|^2\,dx \leqslant \int_{[u_k \neq w]} \Bigg(\frac{1}{2}|\nabla w|^2 + \varepsilon_k^{-2} \int_0^1 h''(\det \tilde{A}_k + s[\det(\tilde{A}_k + \varepsilon_k \nabla w)\\ - \det \tilde{A}_k])(1 - s)ds(\det(\tilde{A}_k + \varepsilon_k \nabla w) - \det \tilde{A}_k)^2\Bigg)\,dx. \qquad (3.33)\end{aligned}$$

For arbitrary $A, B \in \mathbb{M}$ we know

$$\begin{aligned}\det(A + B) - \det A &= \int_0^1 \frac{d}{dt}\det(A + tB)\,dt = \int_0^1 \mathrm{Cof}(A + tB) : B\,dt\\ &= \mathrm{Cof}\,A : B + \frac{1}{2}\mathrm{Cof}\,B : B = \mathrm{Cof}\,A : B + \det B,\end{aligned}$$

hence by (1.4) and (3.33)

$$\begin{aligned}\int_{[u_k\neq w]} |\nabla u_k|^2\,dx &\leqslant \int_{[u_k\neq w]} |\nabla w|^2 + \varepsilon_k^{-2} d(\operatorname{Cof}\tilde{A}_k : \nabla w\varepsilon_k + \varepsilon_k^2 \det \nabla w)^2\,dx \\ &= \int_{[u_k\neq w]} |\nabla w|^2 + d(\operatorname{Cof}\tilde{A}_k : \nabla w + \varepsilon_k \det\nabla w)^2\,dx \\ &\leqslant \int_{[u_k\neq w]} |\nabla w|^2 + 2d(M^2|\nabla w|^2 + \varepsilon_k^2(\det\nabla w)^2)\,dx\end{aligned}$$

where we have used $|\operatorname{Cof}\tilde{A}_k| = |\tilde{A}_k| < M$. Thus we arrive at

$$\int_{[u_k\neq w]} |\nabla u_k|^2\,dx \leqslant Q(d,M) \int_{[u_k\neq w]} (|\nabla w|^2 + \varepsilon_k^2(\det\nabla w)^2)\,dx$$

for a suitable constant $Q(d,M)$.

From inequality (3.2) we now deduce

$$⨍_{B_t} |\nabla u_k|^2\,dx \leqslant c_{32}(d,M)((2t)^{-2} ⨍_{B_{2t}} |u_k - (u_k)_{2t}|^2\,dx + \varepsilon_k^2 (⨍_{B_{2t}} |\nabla u_k|^2\,dx)^2). \tag{3.34}$$

Recalling definition (3.29) estimate (3.34) implies

$$V_k(t) \leqslant c_{32}(d,M)((2t)^{-2} ⨍_{B_{2t}} |v_k(x) - (\nabla v_k)_t(x) - (v_k)_{2t}|^2\,dx + \varepsilon_k^2 ⨍_{B_{2t}} |\nabla v_k - (\nabla v_k)_t|^2\,dx)$$

and from (3.25), (3.26) we infer (using Poincaré's inequality)

$$\begin{aligned}\limsup_{k\to\infty} V_k(t) &\leqslant c_{32}(d,M)(2t)^{-2} ⨍_{B_{2t}} |v(x) - (\nabla v)_t(x) - (v)_{2t}|^2\,dx \\ &\leqslant c_{33}(d,M)\Big[(2t)^{-2} ⨍_{B_{2t}} |v(x) - (\nabla v)_{2t}(x) - (v)_{2t}|^2\,dx \\ &\qquad + (2t)^{-2} ⨍_{B_{2t}} |(\nabla v)_{2t}(x) - (\nabla v)_t(x)|^2\,dx\Big] \\ &\leqslant c_{34}(d,M)V(2t).\end{aligned}$$

By (3.28) we have $V(2t) \leqslant c_{**}(d,M)4t^2$, hence

$$\limsup_{k\to\infty} V_k(t) \leqslant c_{34}(d,M)c_{**}(d,M)4t^2 = c_{35}(d,M)t^2.$$

If we choose $c_* := 2 \cdot c_{35}(d, M)$ the last inequality contradicts (3.23) which proves the Main Lemma. •

For completeness we give an alternative proof of the Main Lemma which does not make use of Lemma 3.1. Going back to inequality (3.28) it is clear that the desired contradiction can also be obtained by showing

$$\nabla v_k \to \nabla v \quad \text{strongly in} \quad L^2_{\text{loc}}(B, \mathbb{M}). \tag{3.35}$$

To this purpose we first observe that v_k minimizes the functional

$$\varphi \mapsto \int_B \frac{1}{2}|\nabla\varphi|^2 + \varepsilon_k^{-2} h(\det[A_k + \varepsilon_k \nabla\varphi])\, dx$$

in the class $H^{1,2}(B, \mathbb{R}^2)$ with respect to its boundary values. Letting

$$H_k(P) := \varepsilon_k^{-2}(h(\det[A_k + \varepsilon_k P]) - h(\det A_k) - h'(\det A_k)[\det(A_k + \varepsilon_k P) - \det A_k])$$

we see

$$I_k^r(v_k) := \int_{B_r} \frac{1}{2}|\nabla v_k|^2 + H_k(\nabla v_k)\, dx \leqslant I_k^r(w) \tag{3.36}$$

for any $0 < r < 1$ and all $w \in H^{1,2}(B_r, \mathbb{R}^2)$ such that $v_k = w$ on ∂B_r. We claim

$$\frac{1}{2}|P|^2 \leqslant \frac{1}{2}|P|^2 + H_k(P) \leqslant c_1|P|^2, \quad P \in \mathbb{M} \tag{3.37}$$

for some positive constant c_1 independent of k. The first inequality follows from convexity of H_k. For the second one we observe

$$\det A_k \in E_* \subset\subset E_1 = \mathbb{R} - \{t_1, \dots, t_s\}$$

for all k, hence $\det A \in E_1$ and we can calculate $\sigma > 0$ such that $\det P \in E_1$ for all $P \in \mathbb{M}$ such that $|P - A| \leqslant \sigma$.

Now suppose that $P \in \mathbb{M}$ is given.

Case 1: $\varepsilon_k|P| < \sigma$

$$\begin{aligned}
H_k(P) &= \varepsilon_k^{-2} \int_0^1 h''(\det A_k + s[\det(A_k + \varepsilon_k P) - \det A_k])(1 - s)ds \\
&\quad \times (\det(A_k + \varepsilon_k P) - \det A_k)^2 \\
&\leqslant_{(1.4)} \varepsilon_k^{-2} d(\operatorname{Cof} A_k : P\varepsilon_k + \varepsilon_k^2 \det P)^2 \\
&= d(\operatorname{Cof} A_k : P + \varepsilon_k \det P)^2 \\
&\leqslant c_2((|A| + \sigma)|P| + \varepsilon_k|P|^2)^2 \\
&\leqslant c_2((|A| + \sigma)|P| + \sigma|P|)^2 =: c_3|P|^2.
\end{aligned}$$

Here we have used $|\operatorname{Cof} A_k| = |A_k| \leqslant \sigma + |A|$ for k large enough.

Case 2: $\varepsilon_k |P| > \sigma$.

$$\begin{aligned} H_k(P) &\leqslant c_4 \varepsilon_k^{-2} (1 + |\det(A_k + \varepsilon_k P)| + |\det A_k| \\ &\qquad + \sup|h'| |\det(A_k + \varepsilon_k P) - \det A_k|) \\ &\leqslant c_5 \varepsilon_k^{-2} (1 + \varepsilon_k |P| + \varepsilon_k^2 |\det P|) \\ &\leqslant c_5 (\sigma^{-2} |P|^2 + \varepsilon_k^{-1} |P| + |\det P|) \\ &\leqslant c_5 (\sigma^{-2} |P|^2 + \sigma^{-1} |P|^2 + |\det P|) \\ &\leqslant c_6 |P|^2. \end{aligned}$$

This implies (3.37) for large k with c_1 being independent of k.

We proceed as in the paper [EG]. Defining the measures

$$\mu_k(M) := \int_M |\nabla v_k|^2 + |\nabla v|^2 \, dx, M \subset B,$$

we may assume $\mu_k \to \mu$ weakly for some measure μ.

Fix a radius $0 < r < 1$ such that $\mu(\partial B_r) = 0$ (which holds for almost all r). For $0 < s < r$ choose $\eta \in C_0^1(B, [0,1])$ such that $\eta = 1$ on B_s, $\eta = 0$ on $B - B_r$. Then

$$\begin{aligned} I_k^r(v_k) - I_k^r(v) &= I_k^r(v_k) - I_k^r((v_k + \eta(v - v_k)) + I_k^r(v_k + \eta(v - v_k)) - I_k^r(v) \\ &\leqslant_{(3.36)} I_k^r(v_k + \eta(v - v_k)) - I_k^r(v) \\ &\leqslant_{(3.37)} c_7 \int_{B_r - B_s} |\nabla v_k|^2 + |\nabla v|^2 + |\nabla \eta|^2 |v - v_k|^2 \, dx. \end{aligned}$$

Consequently strong convergence $v_k \to v$ in $L^2(B, \mathbb{R}^2)$ ensures that

$$\limsup_{k \to \infty} (I_k^r(v_k) - I_k^r(v)) \leqslant c_7 \mu(\overline{B_r - B_s})$$

being valid for any $s < r$. Taking the limit $s \nearrow r$ we deduce

$$\limsup_{k \to \infty} (I_k^r(v_k) - I_k^r(v)) \leqslant 0. \tag{3.38}$$

Unfortunately the hypothesis (H2) of [EG] is violated so that we have to exploit the special structure of our integrand.

Selecting $0 < r < 1$ as before we calculate (using convexity of h)

$$\begin{aligned} I_k^r(v_k) - I_k^r(v) &= \int_{B_r} \frac{1}{2}(|\nabla v_k|^2 - |\nabla v|^2) + \varepsilon_k^{-2}\{h(\det[A_k + \varepsilon_k \nabla v_k]) - h(\det[A_k + \varepsilon_k \nabla v]) \\ &\qquad - h'(\det A_k)(\det[A_k + \varepsilon_k \nabla v_k] - \det[A_k + \varepsilon_k \nabla v])\}\, dx \\ &\geqslant \int_{B_r} \frac{1}{2}(|\nabla v_k|^2 - |\nabla v|^2) + \varepsilon_k^{-2}\{h'(\det[A_k + \varepsilon_k \nabla v]) - h'(\det A_k)\} \\ &\qquad \times (\det(A_k + \varepsilon_k \nabla v_k) - \det(A_k + \varepsilon_k \nabla v))\, dx \\ &= \int_{B_r} \frac{1}{2}(|\nabla v_k|^2 - |\nabla v|^2) + \varepsilon_k^{-1}\{h'(\det[A_k + \varepsilon_k \nabla v]) - h'(\det A_k)\} \\ &\qquad \times (\operatorname{Cof} A_k : (\nabla v_k - \nabla v) + \varepsilon_k[\det \nabla v_k - \det \nabla v])\, dx \\ &=: \int_{B_r} \frac{1}{2}(|\nabla v_k|^2 - |\nabla v|^2)\, dx + R_k. \end{aligned}$$

We already know $v \in C^\infty(B_r; \mathbb{R}^2)$, hence

$$h''(\det A_k + s[\det(A_k + \varepsilon_k \nabla v(x)) - \det A_k]) \to h''(\det A) \quad \text{for all } x \in B_r,\ 0 \leqslant s \leqslant 1,$$

so that (using boundedness of h'')

$$\int_0^1 h''(\det A_k + s[\det(A_k + \varepsilon_k \nabla v) - \det A_k])ds \to h''(\det A) \quad \text{strongly in} \quad L^2(B_r).$$

Writing

$$\begin{aligned} R_k &= \varepsilon_k^{-1} \int_{B_r} \int_0^1 h''(\det A_k + s[\det(A_k + \varepsilon_k \nabla v) - \det A_k])ds \\ &\qquad \times (\det(A_k + \varepsilon_k \nabla v) - \det A_k) \\ &\qquad \times (\operatorname{Cof} A_k : (\nabla v_k - \nabla v) + \varepsilon_k[\det \nabla v_k - \det \nabla v])\, dx \\ &= \int_{B_r} \int_0^1 h''(\dots)ds(\operatorname{Cof} A_k : \nabla v + \varepsilon_k \det \nabla v) \\ &\qquad \times (\operatorname{Cof} A_k : (\nabla v_k - \nabla v) + \varepsilon_k[\det \nabla v_k - \det \nabla v])\, dx \end{aligned}$$

we arrive at $\lim_{k\to\infty} R_k = 0$ and in conclusion

$$\limsup_{k\to\infty}(I_k^r(v_k) - I_k^r(v)) \geqslant \frac{1}{2} \limsup_{k\to\infty} \int_{B_r} |\nabla v_k|^2 - |\nabla v|^2\, dx.$$

Recalling (3.38) the proof of (3.35) is complete.

Remark. With the help of the above arguments the Main Lemma and also Theorem 1.2 can be extended to functionals of the form

$$\int_\Omega |\nabla u|^2 + |\nabla u|^p + h(\det \nabla u)\, dx$$

with $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, $p \geqslant n$ and h satisfying (1.2), (1.4).

§4. Proof of Theorem 1.2

The proof of Theorem 1.2 combines the Main Lemma with a suitable iteration technique. Due to the fact that during this procedure we have to take care of various quantities we found it easier to proceed in several steps. First we have the following inequalities

$$\Big| \fint_{B_{t^k R}(x_0)} \nabla u\, dx \Big| \leqslant \Big| \fint_{B_R(x_0)} \nabla u\, dx \Big| + \frac{1}{t} \sum_{i=0}^{k-1} U^{\frac{1}{2}}(x_0, t^i R) \tag{4.1}$$

$$\left\{ \begin{aligned} &\Big| \det \fint_{B_{t^k R}(x_0)} \nabla u\, dx - \det \fint_{B_R(x_0)} \nabla u\, dx \Big| \\ &\leqslant t^{-1} \Big| \fint_{B_R(x_0)} \nabla u\, dx \Big| \sum_{i=0}^{k-1} U^{\frac{1}{2}}(x_0, t^i R) + t^{-2} \Big(\sum_{i=0}^{k-1} U^{\frac{1}{2}}(x_0, t^i R) \Big)^2 \\ &+ (2t^2)^{-1} \sum_{i=0}^{k-1} U(x_0, t^i R) \end{aligned} \right. \tag{4.2}$$

being valid for all $u \in H^{1,2}(B_R(x_0), \mathbb{R}^2)$, $0 < t < 1$ and $k \in \mathbb{N}$.

Lemma 4.1. *Suppose that the assumptions of Theorem* 1.2 *hold. Let numbers* $0 < \nu < 1$, $0 < M_1 < M < +\infty$ *and open sets* $E_{*1} \subset\subset E_* \subset\subset E_1$ *be given and define* $\Delta := \operatorname{dist}(E_{*1}, \partial E_*) > 0$. *Let* $t \in (0, \frac{1}{4})$ *satisfy the inequality*

$$c_*(d, M) t^{2-2\nu} \leqslant 1. \tag{4.3}$$

Suppose further that for some disc $B_R(x_0) \subset \Omega$ *the following conditions hold:*

$$B_R(x_0) \subset\subset \Omega, \quad 0 < R < R_0(t, d, M, E_*), \tag{4.4}$$

$$\Big| \fint_{B_R(x_0)} \nabla u\, dx \Big|, \Big| \fint_{B_{tR}(x_0)} \nabla u\, dx \Big| < M_1, \tag{4.5}$$

$$\det \fint_{B_R(x_0)} \nabla u\, dx \in E_{*1}, \tag{4.6}$$

$$U(x_0, R) < \varepsilon_1^2 \tag{4.7}$$

with $\varepsilon_1 := \frac{1}{2}\min\{1, \varepsilon_0(t,d,M,E_*), \Delta\big(\frac{1}{2}M_1\frac{1}{1-t^\nu}+\frac{1}{t^2}\frac{1}{(1-t^\nu)^2}+\frac{1}{2t^2}\frac{1}{1-t^{2\nu}}\big)^{-1}, \frac{t}{2}(1-t^\nu)(M-M_1)\}$*,* R_0*,* ε_0 *taken from the Main Lemma. Then we have*

$$U(x_0, t^k R) \leqslant t^{2\nu k} U(x_0, R) \tag{4.8}$$

for all $k \in \mathbb{N}$*.*

Proof. We proceed by induction. From (4.4)–(4.7) it follows that (3.3)–(3.6) hold, hence by (3.7), (4.3)

$$U(x_0, tR) \leqslant c_* t^{2-2\nu} t^{2\nu} U(x_0, R) \leqslant t^{2\nu} U(x_0, R).$$

Next suppose

$$U(x_0, t^s R) \leqslant t^{2\nu s} U(x_0, R)$$

for $s = 1, \ldots, k$. We have to show

$$\Bigg| \fint_{B_{t^k R}(x_0)} \nabla u \, dx \Bigg|, \Bigg| \fint_{B_{t^{k+1} R}(x_0)} \nabla u \, dx \Bigg| < M, \tag{4.9}$$

$$\det \fint_{B_{t^k R}(x_0)} \nabla u \, dx \in E_*, \tag{4.10}$$

$$U(x_0, t^k R) < \varepsilon_0^2. \tag{4.11}$$

In this case the Main Lemma implies our assertion (4.8) for $k+1$:

$$U(x_0, t^{k+1} R) \leqslant c_* t^2 U(x_0, t^k R) \leqslant t^{2\nu} U(x_0, t^k R) \leqslant t^{2\nu(k+1)} U(x_0, R).$$

We now check the hypotheses (4.9)–(4.11): From (4.7) and our assumption we infer

$$U(x_0, t^k R) \leqslant t^{2\nu k} U(x_0, R) < t^{2\nu k} \varepsilon_0^2 < \varepsilon_0^2. \tag{4.12}$$

By (4.2) and (4.12) we have

$$\begin{aligned}
&\Bigg| \det \fint_{B_{t^k R}(x_0)} \nabla u \, dx - \det \fint_{B_R(x_0)} \nabla u \, dx \Bigg| \\
&\leqslant t^{-1} M_1 \sum_{i=0}^{k-1} t^{\nu i} \varepsilon_1 + t^{-2} \Big(\sum_{i=0}^{k-1} t^{\nu i} \varepsilon_1 \Big)^2 + (2t^2)^{-1} \sum_{i=0}^{k-1} t^{2\nu i} \varepsilon_1^2 \\
&= M_1 t^{-1} \varepsilon_1 (1 - t^{\nu k})/(1 - t^\nu) + t^{-2} \varepsilon_1^2 (1 - t^{\nu k})^2/(1 - t^\nu)^2 \\
&\quad + \varepsilon_1^2 (2t^2)^{-1} (1 - t^{2\nu k})/(1 - t^{2\nu}) \\
&< \varepsilon_1 (t^{-1} M_1 (1 - t^\nu)^{-1} + t^{-2} (1 - t^\nu)^{-2} + (2t^2)^{-1} (1 - t^{2\nu})^{-1}) < \Delta
\end{aligned}$$

so that (4.10) is proved.

Finally we observe

$$\left| \fint_{B_{t^{k+1}R}(x_0)} \nabla u \, dx \right|, \left| \fint_{B_{t^kR}(x_0)} \nabla u \, dx \right| \leqslant_{(4.1)} \left| \fint_{B_R(x_0)} \nabla u \, dx \right| + t^{-1} \sum_{i=0}^{k} U^{\frac{1}{2}}(x_0, t^i R)$$

$$\leqslant_{(4.5)} M_1 + t^{-1} \sum_{i=0}^{k} t^{i\nu} U^{\frac{1}{2}}(x_0, R)$$

$$\leqslant_{(4.7)} M_1 + t^{-1}(1 - t^{\nu})^{-1} \varepsilon_1$$

$$\leqslant M_1 + \frac{1}{4}(M - M_1) < M.$$

This completes the proof of Lemma 4.1 •

The statement of Lemma 4.1 can be written in the following form.

Lemma 4.2. *Suppose that all assumptions of Lemma* (4.1) *are satisfied. Then we have*

$$U(x_0, p) \leqslant 4 \cdot t^{-2-2\nu} U(x_0, R) \Big(\frac{p}{R} \Big)^{2\nu} \tag{4.13}$$

for all $0 < p \leqslant R$.

Proof of Theorem 1.2. We select a point $x_0 \in \Omega$ with (1.8), (1.9), i.e.,

$$\liminf_{r \downarrow 0} U(x_0, r) = 0, \quad \nabla u(x_0) := \lim_{r \downarrow 0} \fint_{B_r(x_0)} \nabla u \, dx \quad \text{exists}$$

and the additional property $a := \det \nabla u(x_0) \in E_1$. Let

$$M_1 := |\nabla u(x_0)| + 1, \quad M := M_1 + 1$$

and define E_{*1}, E_* as small open intervals with center a such that $E_{*1} \subset\subset E_* \subset\subset E_1$ holds.

Next the quantities R_0, ε_1 are defined as in Lemma 4.1 where we choose $t := c_*(d, M)^{-1/(2-2\nu)}$. Then we can find a radius $R < R_0$ such that (4.5)–(4.7) hold. For points y close to x_0 we also have

$$\left| \fint_{B_{R(y)}} \nabla u \, dx \right|, \left| \fint_{B_{tR(y)}} \nabla u \, dx \right| < M_1, \qquad \det \fint_{B_{R(y)}} \nabla u \, dx \in E_{*1}, \qquad U(y, R) < \varepsilon_1^2,$$

thus the decay estimate (4.13) is established for points y in a neighborhood of x_0 and we have shown $\nabla u \in C^{0,\nu}$ near x_0. •

Theorem 1.3 is a direct consequence of Theorem 1.2: For δ fixed we consider the set

$$\Omega_\delta^1 := \Big\{ x \in \Omega : \liminf_{r\downarrow 0} \fint_{B_r(x)} |\nabla u_\delta - (\nabla u_\delta)_r|^2 dz = 0 \text{ and}$$

$$\nabla u_\delta(x) := \lim_{r\downarrow 0} \fint_{B_r(x)} \nabla u_\delta dz \text{ exists} \Big\}.$$

Let $\Omega_\delta := \{ x \in \Omega_\delta^1 : \det \nabla u_\delta(x) \in (\delta, \delta^{-1}) \}$. Since h_δ is of class C^2 on $\mathbb{R} - \{\delta, \delta^{-1}\} =: E_1$ we deduce from Theorem 1.2 that Ω_δ is open and $\nabla u_\delta \in C^{0,\nu}(\Omega_\delta, \mathbb{M})$ for any $\nu \in (0,1)$.

§5. Approximation

In this section we discuss Theorem 1.4 but in a more general setting. Let Ω now denote a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and fix some real number $p \geqslant n$. We introduce the energy

$$F(u) := \int_\Omega |\nabla u|^p + h(\det \nabla u)\, dx$$

on the space $\overset{\circ}{H}{}^{1,p}(\Omega, \mathbb{R}^n) + u_0$ where now $u_0 \in H^{1,p}(\Omega, \mathbb{R}^n)$ is fixed and $h : \mathbb{R} \to [0, \infty]$ denotes a function such that $\lim_{t\downarrow 0} h(t) = +\infty$ and $h \equiv +\infty$ on $(-\infty, 0]$. Moreover, we require h to be convex and of class C^2 on $(0, \infty)$. Suppose $F(u_1) < +\infty$ for some $u_1 \in H^{1,p}(\Omega, \mathbb{R}^n) + u_0$ and define h_δ according to (0.5), (0.7). Then the variational problem

$$F_\delta(u) := \int_\Omega |\nabla u|^p + h_\delta(\det \nabla u)\, dx \to \min \quad \text{in} \quad \overset{\circ}{H}{}^{1,p}(\Omega, \mathbb{R}^n) + u_0$$

admits a solution u_δ which can be easily deduced from quasiconvexity and the growth rates of the integrand. We next check the appropriate versions of (1.11)–(1.14). The minimality of u_δ implies

$$F_\delta(u_\delta) \leqslant F_\delta(u_1) \xrightarrow{\delta\downarrow 0} F(u_1) < +\infty$$

so that $\|u_\delta\|_{H^{1,p}(\Omega, \mathbb{R}^n)}$ is bounded independent of δ. (Observe $\int_\Omega h_\delta(\det \nabla u_1)\, dx \to \int_\Omega h(\det \nabla u_1)\, dx$ by monotone convergence)). After passing to a subsequence we find $u \in \overset{\circ}{H}{}^{1,p}(\Omega, \mathbb{R}^n) + u_0$ such that

$$u_\delta \to u \quad \text{weakly in} \quad H^{1,p}(\Omega, \mathbb{R}^n).$$

For any ε we have by weak lower semicontinuity

$$F_\varepsilon(u) \leqslant \liminf_{\delta\downarrow 0} F_\varepsilon(u_\delta) \leqslant \liminf_{\delta\downarrow 0} F_\delta(u_\delta), \tag{5.1}$$

hence

$$\sup_{\varepsilon} \int_{\Omega} h_{\varepsilon}(\det \nabla u)\, dx < \infty.$$

Again by monotone convergence we deduce

$$\int_{\Omega} h(\det \nabla u)\, dx < \infty$$

which means $\nabla u \in \mathbb{M}_{+}$ a.e. and $F(u) < \infty$. Recalling (5.1) we get

$$F_{\varepsilon}(u) \leqslant \liminf_{\delta \downarrow 0} F_{\delta}(u_{\delta}) \leqslant \limsup_{\delta \downarrow 0} F_{\delta}(u_{\delta}) \leqslant \limsup_{\delta \downarrow 0} F_{\delta}(u) = F(u),$$

thus

$$F(u) = \lim_{\delta \downarrow 0} F_{\delta}(u_{\delta}) = \lim_{\delta \downarrow 0} F_{\delta}(u). \tag{5.2}$$

Now let $v \in \overset{\circ}{H}{}^{1,p}(\Omega, \mathbb{R}^n) + u_0$ denote an arbitrary function. Then

$$F_{\delta}(u_{\delta}) \leqslant F_{\delta}(v) \xrightarrow{\delta \downarrow 0} F(v)$$

and by (5.2) we arrive at $F(u) \leqslant F(v)$, i.e., u solves problem (0.1) in the modified form. We claim

$$\int_{\Omega} |\nabla u_{\delta}|^p\, dx \xrightarrow{\delta \downarrow 0} \int_{\Omega} |\nabla u|^p\, dx. \tag{5.3}$$

Since $L^p(\Omega, \mathbb{M})$ is a uniformly convex space (5.3) implies strong convergence $u_{\delta} \xrightarrow{\delta \downarrow 0} u$ in $H^{1,p}(\Omega, \mathbb{R}^n)$. In order to prove (5.3) we write

$$\int_{\Omega} |\nabla u_{\delta}|^p\, dx = F_{\delta}(u_{\delta}) - F(u) + \int_{\Omega} |\nabla u|^p\, dx + \int_{\Omega} h(\det \nabla u)\, dx - \int_{\Omega} h_{\delta}(\det \nabla u_{\delta})\, dx.$$

Thus

$$\limsup_{\delta \downarrow 0} \int_{\Omega} |\nabla u_{\delta}|^p\, dx \leqslant \int_{\Omega} |\nabla u|^p\, dx$$

follows from (5.2) and

$$\limsup_{\delta \downarrow 0} \Big(\int_{\Omega} h(\det \nabla u)\, dx - \int_{\Omega} h_{\delta}(\det \nabla u_{\delta})\, dx \Big) \leqslant 0$$

$$\iff \int_{\Omega} h(\det \nabla u)\, dx \leqslant \liminf_{\delta \downarrow 0} \int_{\Omega} h_{\delta}(\det \nabla u_{\delta})\, dx.$$

The last inequality is in turn a consequence of

$$\int_\Omega h_\varepsilon(\det \nabla u)\,dx \leqslant \liminf_{\delta\downarrow 0} \int_\Omega h_\varepsilon(\det \nabla u_\delta)\,dx$$

$\psi \mapsto \int_\Omega h_\varepsilon(\det \nabla\psi)\,dx$ is a quasiconvex variational integral),

$$\liminf_{\delta\downarrow 0} \int_\Omega h_\varepsilon(\det \nabla u_\delta)\,dx \leqslant \liminf_{\delta\downarrow 0} \int_\Omega h_\delta(\det \nabla u_\delta)\,dx$$

(observe $h_\varepsilon(t) \leqslant h_\delta(t)$ for $\varepsilon \geqslant \delta$) and

$$\int_\Omega h_\varepsilon(\det \nabla u)\,dx \xrightarrow{\varepsilon\downarrow 0} \int_\Omega h(\det \nabla u)\,dx.$$

It remains to prove (1.14). We have

$$\int_\Omega h_\delta(\det \nabla u_\delta)\,dx \leqslant F_\delta(u_\delta) \leqslant F_\delta(u) \leqslant F(u),$$

i.e.,

$$\int_{[\det \nabla u_\delta \leqslant \delta]} h_\delta(\det \nabla u_\delta)\,dx + \int_{[\det \nabla u_\delta \geqslant \delta^{-1}]} h_\delta(\det \nabla u_\delta)\,dx \leqslant F(u)$$

and the statement follows from the definition of h_δ. •

§6. Final remarks

For completeness we indicate how to obtain a sequence of more regular approximations $\{u_\delta\}$ for problem (0.1) in the case that h satisfies (0.2), (0.3) and in addition

$$h \in C^3(0,\infty), \quad h^{(3)} \leqslant 0 \quad \text{on} \quad (0,\infty). \tag{6.1}$$

For simplicity we also assume

$$\limsup_{t\to\infty} h'(t) < \infty, \tag{6.2}$$

the case $h'(t) \to \infty$ as $t \to \infty$ can be treated in a similar way. For $\delta > 0$ we let

$$g_\delta(t) := h(\delta) + h'(\delta)(t-\delta) + \frac{1}{2}h''(\delta)(t-\delta)^2, \quad t \in \mathbb{R}.$$

Observing

$$h(t) = g_\delta(t) + \frac{1}{3!}h^{(3)}(\tau)(t-\delta)^3$$

for some $\tau \in (t, \delta)$ we see $h(t) \geqslant g_\delta(t)$ on $(-\infty, \delta]$.

The functions

$$H_\delta(t) := \begin{cases} h(t), & t \geqslant \delta, \\ g_\delta(t), & t \leqslant \delta \end{cases}$$

are of class $C^2(\mathbb{R})$, non-negative and convex and satisfy

$$\varepsilon < \delta \Rightarrow H_\delta(t) \leqslant H_\varepsilon(t), \quad \lim_{\varepsilon \downarrow 0} H_\varepsilon(t) = h(t) \quad \text{on} \quad \mathbb{R}. \tag{6.3}$$

In order to prove (6.3) it remains to show

$$H_\delta(t) \leqslant H_\varepsilon(t) \quad \text{on} \quad [\varepsilon, \delta],$$

i.e.,

$$g_\delta(t) \leqslant g_\varepsilon(t), \quad t \in [\varepsilon, \delta]. \tag{6.4}$$

Fix some $t \in \mathbb{R}$ and consider the function $\varepsilon \mapsto g_\varepsilon(t)$. We have $\frac{d}{d\varepsilon} g_\varepsilon(t) = \frac{1}{2} h^{(3)}(\delta)(t-\delta)^2 \leqslant 0$ on account of (6.1), which means $g_\varepsilon(t) \geqslant g_\delta(t)$ for $\delta > \varepsilon$, thus (6.4) holds.

We assume that

$$J(u_1) = \int_\Omega \frac{1}{2}|\nabla u_1|^2 + h(\det \nabla u_1)\, dx < \infty$$

holds for some function $u_1 \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0$ where Ω and u_0 are as in section 1. Instead of (0.9) we then consider the problem

$$\begin{cases} \text{find} \quad u_\delta \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0 \quad \text{such that} \\ J_\delta^*(u_\delta) = \inf\{J_\delta^*(v) : v \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0\} \end{cases} \tag{6.5}$$

where

$$J_\delta^*(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 + H_\delta(\det \nabla v)\, dx.$$

Note that H_δ can grow in certain directions like $|\nabla u|^4$ so that $J_\delta^*(v)$ may be infinite for certain $v \in \overset{\circ}{H}{}^{1,2}(\Omega, \mathbb{R}^2) + u_0$. But from $0 \leqslant J_\delta^*(u_1) < \infty$ (recall (6.3)) we deduce by quoting Theorem 3.1 in [FH1].

Lemma 6.1. *Problem* (6.5) *admits a solution* u_δ *which is of class* $C^{1,\alpha}(\Omega_\delta)$ *for any* $0 < \alpha < 1$ *on some open subset* Ω_δ *of* Ω *such that* $|\Omega - \Omega_\delta| = 0$.

The second statement of Lemma 6.1 is just the main theorem in the paper [FH1] of Fusco–Hutchinson.

Now we can proceed similar to the arguments used in section 5 with the following results.

Lemma 6.2. *After passing to a suitable subsequence we have*
i) $u_\delta \to u$ *strongly in* $H^{1,2}(\Omega, \mathbb{R}^2)$;
ii) $J^*_\delta(u_\delta) \to J(u)$;
iii) u *is a solution of problem* (0.1);
iv) $|[\det \nabla u_\delta \leqslant \delta]| \to 0$ *as* $\delta \downarrow 0$.

For example it is easy to see

$$\sup_{\delta>0}(\|\nabla u_\delta\|_{L^2} + \|\det \nabla u_\delta\|_{L^2}) < \infty$$

(provided $h''(\delta) > 0$) so that (for a subsequence)

$$u_\delta \rightharpoonup: u \quad \text{in} \quad H^{1,2}(\Omega, \mathbb{R}^2), \qquad \det \nabla u_\delta \rightharpoonup d \quad \text{in} \quad L^2(\Omega).$$

But $\det \nabla u_\delta \rightharpoonup \det \nabla u$ in the sense of distributions so that $d = \det \nabla u$.

From convexity of H_ε and weak convergence $\det \nabla u_\delta \rightharpoonup \det \nabla u$ in $L^2(\Omega)$ we deduce by quoting De Giorgi's theorem on lower semicontinuity

$$\int_\Omega H_\varepsilon(\det \nabla u)\, dx \leqslant \liminf_{\delta\downarrow 0} \int_\Omega H_\varepsilon(\det \nabla u_\delta)\, dx \leqslant_{(6.3)} \liminf_{\delta\downarrow 0} \int_\Omega H_\delta(\det \nabla u_\delta)\, dx < \infty$$

so that $\int_\Omega H_\varepsilon(\det \nabla u)\, dx$ is bounded independent of ε. This implies $\int_\Omega h(\det \nabla u)\, dx < \infty$ for the limit function.

We leave it to the reader to check the details of the proof of Lemma 6.1.

Typical examples satisfying (6.1) and (6.2) are given by the functions

$$h(t) = \begin{cases} 1/t, & t > 0, \\ +\infty, & t \leqslant 0, \end{cases} \qquad h(t) = \begin{cases} -\log(\frac{1}{t} + 1), & t > 0, \\ +\infty, & t \leqslant 0. \end{cases}$$

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Поступило 25 мая 1994 г.

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