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THE MONODROMY GROUP OF A CONFIGURATION OF LINES

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Abstract. A configuration of skew lines is an unordered collection of lines in general position in a real affine or projective three-dimensional space. Such configurations give rise to topological problems related to real algebraic geometry. In this paper, the notion of a monodromy group, which is a rigid isotopy invariant of such configurations, is introduced, and some of its properties are studied. It is shown that in two important cases, the monodromy group determines the configuration up to rigid isotopy and mirror image.

§1. Introduction

An *ordered* (respectively, *unordered*) configuration of lines is an ordered (respectively, unordered) collection of lines in a three-dimensional space. A configuration is *nonsingular* if the lines are in general position, otherwise it is *singular*. Throughout this paper, the three-dimensional space is the real projective space $\mathbb{R}P^3$; lines are in general position in $\mathbb{R}P^3$ if they are pairwise disjoint. Two configurations are *isotopic* if one configuration can be deformed into the other by an ambient isotopy of $\mathbb{R}P^3$. Two configuratons are *rigidly isotopic* if there exists an isotopy under which the lines remain lines in general position and such that one configuration is deformed into the other.

In the paper [Vi], O. Viro posed the problem of classification, up to rigid isotopy, of nonsingular configurations of n lines, for any fixed integer n. This problem is difficult, even for small values of n. The classification is known for $n \leq 6$ (see [Vi, Ma1, Ma2]). Recently, some progress has been made and the classification for $n = 7$ has been announced by V. Mazurovskiĭ; cf. [MP]. One can similarly pose the problem of classification of nonsingular configurations of affine lines in $\mathbb{R}^3$ (lines in general position in $\mathbb{R}^3$ are pairwise skew lines). A dimension argument shows that this problem is equivalent to the one mentioned above.

Key words and phrases. Three-dimensional space, configuration of lines, monodromy group.

In this note, we wish to address a related question. Let $\mathcal{C} = \{L_1, \dots, L_n\}$ be a nonsingular configuration of n skew lines ; the question consists in determining all permutations of lines of $\mathcal{C}$ that can be realized by rigid isotopies. Such permutations clearly form a subgroup of the symmetric group S_n on n elements, which we call the (combinatorial) *monodromy group* of the configuration $\mathcal{C}$. Obviously, this is a rigid isotopy invariant of configurations. Some of its properties are studied in this paper. The main result is the following: if the monodromy group of a nonsingular configuration of N lines is the full symmetric group S_N or $S_n \times S_p$ with $n+p=N$, then the configuration is completely determined up to rigid isotopy and mirror image.

§2. Definitions and notation

2.1. Preliminaries. First, we describe the basic constructions of O. Viro (see [Vi, §2 and §4]). We endow the real projective space $\mathbb{R}P^3$ with its canonical orientation. If $L_1,\ L_2$ are two disjoint unoriented lines in $\mathbb{R}P^3$, and if $L_1^*,\ L_2^*$ denote the same lines endowed with some orientation, we can define the linking number $\mathrm{lk}(L_1^*,\ L_2^*) \in \{-1, 1\}$. It is independent of the order of the pair $(L_1^*,\ L_2^*)$. Let $L_1,\ L_2,\ L_3$ be three disjoint unoriented lines in $\mathbb{R}P^3$, and let $L_1^*,\ L_2^*,\ L_3^*$ denote the same lines endowed with some orientation. Then the product $\mathrm{lk}(L_1^*,\ L_2^*)\cdot\mathrm{lk}(L_2^*,\ L_3^*)\cdot\mathrm{lk}(L_3^*,\ L_1^*)$, denoted by $\mathrm{lk}(L_1,\ L_2,\ L_3)$, does not depend on the choice of the orientation of the lines and is preserved under the rigid isotopies of $\mathcal{C} = \{L_1, L_2, L_3\}$.

Let $\mathcal{C}$ be a nonsingular configuration of any number of lines in $\mathbb{R}P^3$. Two lines $L_1,\ L_2 \in \mathcal{C}$ are *adjacent* (*contiguous* in [Ma1]) if, by means of a rigid isotopy, we can put them on one side of some quadric and the remaining lines in $\mathcal{C}$ on the other side. Two lines $L_1,\ L_2 \in \mathcal{C}$ are *homologous* if $\mathrm{lk}(L_1, L_2, C) = \mathrm{lk}(L_1, L_2, D)$ for any two lines $C, D \in \mathcal{C}$. This is equivalent to saying that L_1 and L_2 represent the same homological class in $H_1(\mathbb{R}P^3 \setminus (\mathcal{C} \setminus (L_1 \cup L_2)))$. A pair of homologous lines $L_1,\ L_2 \in \mathcal{C}$ such that $\mathrm{lk}(L_1, L_2, C) = \epsilon$ for any line $C \in \mathcal{C} \setminus \{L_1, L_2\}$ is called an *ϵ-pair.* Clearly, the properties of being homologous or adjacent define equivalence relations on the set of lines; two lines that are adjacent are homologous; any two lines homologous to homologous lines forming an ϵ-pair also form an ϵ-pair. A configuration consisting of exactly one class of homologous lines is said to be *homologically trivial.* A class of adjacent lines of a configuration is called an *ϵ-class* if it contains at least two elements and any two lines in it constitute an ϵ-pair. By definition, we can take the lines of each class of adjacent lines on a quadric as linear generators; in this way we obtain one quadric for each adjacency class, and we can make all quadrics bound disjoint regions in $\mathbb{R}P^3$.

Consider a subconfiguration of $\mathcal{C}$ that contains exactly one line from each adjacency class in $\mathcal{C}$; up to rigid isotopy, such a subconfiguration does not depend on the representatives chosen and is called the *derived configuration* of C. A configuration is said to be *simple* if it coincides with its derived configuration. If some multiple derived configuration of C coincides with a single line, then the configuration C is said to be *completely decomposable.* Here is Viro's notation for completely decomposable configurations. A configuration of p linear generators of a quadric that form ϵ-pairs is denoted $\langle \epsilon p \rangle$ and will be called a $Hopf$ configuration. The symbol $\langle +A_1, \dots, A_r \rangle$ (respectively, $\langle -A_1, \dots, A_r \rangle$) denotes a configuration whose lines are in regular neighborhoods of r linear generators of the quadric that form $(+1)$-pairs (respectively, (-1)-pairs); the subconfigurations lying in these neighborhoods are denoted by $A_1, \dots, A_r$. In the cases where the signs do not matter, we shall omit them.

A *mirror* configuration is a configuration which is rigidly isotopic to its mirror image.

2.2. Monodromy group. Let C be a nonsingular configuration of n lines in $\mathbb{R}P^3$ (C is viewed as a finite set of n elements). Given an element $\sigma \in S_C$ (the permutation group of C, isomorphic to S_n), we ask whether there exists a rigid isotopy $(h_t)_{t \in [0;1]}$ of C (in $\mathbb{R}P^3$) such that $h_0 = \mathrm{Id}$ and $h_1(L) = \sigma(L)$ for any line $L \in C$. The subset of permutations $\sigma \in S_C$ satisfying this property is a subgroup of S_C. This subgroup is called the *monodromy group* of C and will be denoted by $G(C)$. It is obvious that the monodromy group is a rigid isotopy invariant of configurations. Since this group depends on the rigid isotopy class k of C only, we will also denote it by $G(k)$. We note that two configurations that are mirror images of each other have the same monodromy group (even if they are not rigidly isotopic).

2.3. Multivariable Alexander polynomial. Here we recall a few notions and results about the multivariable Alexander polynomial. We will give only a brief (and incomplete) presentation, referring the reader to [Tu, Ha] for details.

We define the Alexander polynomial of a nonsingular configuration $C^* = \{L_1^*, \dots L_n^*\}$ endowed with a certain orientation and order. Let S^3 be the canonically oriented 3-sphere, and let $p : S^3 \to \mathbb{R}P^3$ be the double covering. The preimage $V = p^{-1}(L_1^*) \cup \dots \cup p^{-1}(L_n^*)$ is a link in S^3 consisting of n disjoint great circles of S^3. The order and orientation of C^* induce those of V. We define the Alexander polynomial $\Delta_{C^*}(t_1, \dots, t_n) \in \mathbb{Z}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ to be the Alexander polynomial $\Delta_V(t_1, \dots, t_n)$ of the (oriented and ordered) link V. From the known properties of the Alexander polynomial of ordered and oriented links (see [Tu, Ha]), we deduce that Δ_{C^*} is defined up to multiplication by $t_1^{\alpha_1} \dots t_n^{\alpha_n}$, where $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, and that

(1) $\Delta : C^* \mapsto \Delta_{C^*}$ is a topological invariant of ordered and oriented nonsingular configurations;

(2) if C'^* denotes the ordered and oriented configuration C^* with the orientation of the kth line reversed, then

$$\Delta_{C'^*}(t_1, \dots, t_n) = -\Delta_{C^*}(t_1, \dots, t_{k-1}, t_k^{-1}, t_{k+1}, \dots, t_n).$$

Among the results about the Alexander polynomial, of special interest to us is the iteration theorem, proved in full generality by Sumners and Woods [SW], and also by Turaev [Tu]. We shall apply a special case of that theorem to configurations of lines in $\mathbb{R}P^3$. For convenience, we state the version of the theorem that will be used in Subsection 4.2.2. For a proof of the iteration theorem, see [Tu].

Proposition. *Let* $C = \{L_1, \dots, L_n\}$ $(n \geq 2)$ *be an ordered configuration of oriented lines in* $\mathbb{R}P^3$. *Let* L *be a line in* $\mathbb{R}P^3 \backslash L_1 \cup \cdots \cup L_n$. *Suppose* L *is adjacent to* L_n *in the extended configuration* $C'' = \{L_1, \dots, L_{n-1}, L, L_n\}$. *Let* $\Delta_C(t_1, \cdots, t_n)$, $\Delta_{C'}(t_1, \cdots, t_n)$, *and* $\Delta_{C''}(t_1, \cdots, t_n, t_{n+1})$ *be the Alexander polynomials of the ordered configurations* C, $C' = \{L_1, \cdots, L_{n-1}, L\}$, *and* C'', *respectively. We set* $T = t_1^{\mu_1} t_2^{\mu_2} \cdots t_{n-1}^{\mu_{n-1}} t_n^q$, *where* $\mu_i = \operatorname{lk}(L_i, L_n)$ *and* $q = \operatorname{lk}(L, L_n)$. *Then*

(1) $\Delta_{C'}(t_1, \dots, t_n) = \Delta_C(t_1, \dots, t_{n-1}, t_n^\epsilon) \frac{T^\epsilon - 1}{T - 1}$;

(2) $\Delta_{C''}(t_1, \dots, t_{n+1}) = \Delta_C(t_1, \dots, t_{n-1}, t_n^\epsilon t_{n+1})(T^\epsilon t_{n+1}^q - 1)$,

where the sign $\epsilon = \pm 1$ *is taken according to whether the homology classes represented by the oriented lines* L *and* L_n *coincide or are opposite.*

§3. Results

The general question about the monodromy group of configurations consists in determining how powerful it is, as a rigid isotopy invariant. Since we are mostly interested in finding out what the monodromy group distinguishes, we introduce the following notion. Let n be a fixed positive integer. We say that a subgroup G of S_n is *complete* if it has the following property: given any two nonsingular configurations C_1 and C_2, the relation $G \cong G(C_1) \cong G(C_2)$ implies that C_1 or its mirror image is rigidly isotopic to C_2. For example, all monodromy groups of configurations of 5 or less lines are complete. This result was first obtained by the author with the help of a systematic study of linking numbers of triples (see [D]). In the table below we shall make use of the notation introduced in §2.

number of lines	rigid isotopy type	monodromy group
	$\langle\pm3\rangle$	S_3
4	$\langle\pm4\rangle$	S_4
	$\langle\langle+2\rangle,\langle-2\rangle\rangle$	$S_2\times S_2$
5	$\langle\pm5\rangle$ $\langle\langle+3\rangle,\langle-2\rangle\rangle$	S_5 $S_3\times S_2$
	$\langle\langle-3\rangle,\langle+2\rangle\rangle$	$S_3\times S_2$
	$\langle+\langle1\rangle,\langle-2\rangle,\langle-2\rangle\rangle$	$S_2\times S_2$
	simple	D_5(dihedral)

There is only one simple configuration of 5 lines [Vi, VD] (see Figure 1). We describe only partial results for configurations of 6 lines:

number of lines	rigid isotopy type	monodromy group
6	$\langle\pm6\rangle$	S_6
	$\langle\langle+4\rangle,\langle-2\rangle\rangle$	$S_4\times S_2$
	$\langle\langle-4\rangle,\langle+2\rangle\rangle$	$S_4\times S_2$
	$\langle\langle+3\rangle,\langle-3\rangle\rangle$	$S_3\times S_3$
	$\langle+\langle+2\rangle,\langle-2\rangle,\langle-2\rangle\rangle$	$S_2\times D_4$
	$\langle-\langle-2\rangle,\langle+2\rangle,\langle+2\rangle\rangle$	$S_2\times D_4$
	$\langle+\langle-3\rangle,\langle-2\rangle,\langle1\rangle\rangle$	$S_3\times S_2$
	$\langle-\langle+3\rangle,\langle+2\rangle,\langle1\rangle\rangle$	$S_3\times S_2$

Here D_4 denotes the dihedral group of order 8.

The computation of the monodromy group in the examples above (all of them but one are configurations of decomposable type) is relatively easy: all the results except for that concerning the simple configuration of five lines follow directly from Lemma 3 in Subsection 4.2.2. The monodromy group of the simple configuration of five lines is the dihedral group; this can be shown by studying the linking numbers of triples and with the help of some elementary geometry.

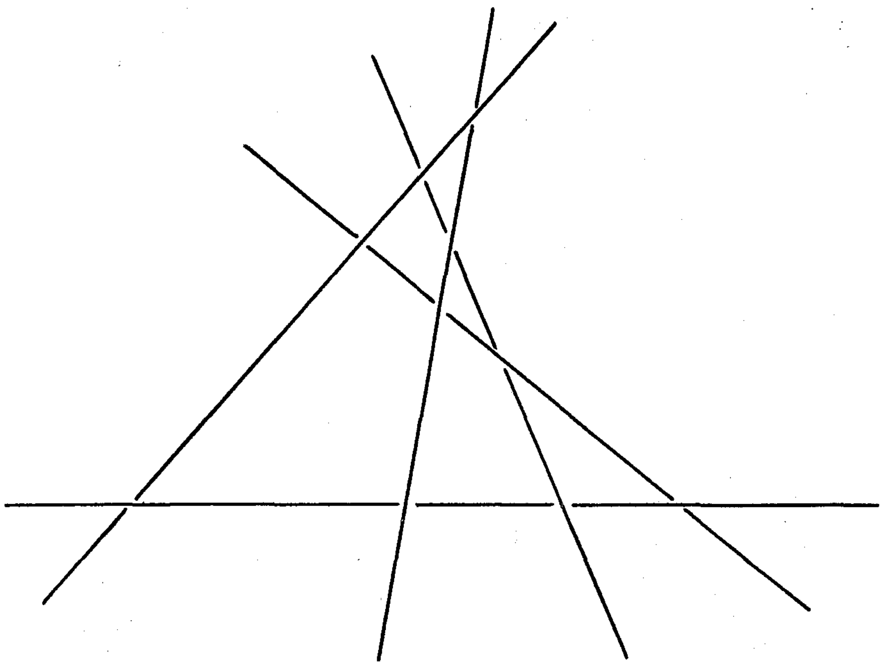

Figure 1. Simple configuration of 5 lines.

Recently, V. M. Mazurovskiĭ has announced a classification of the mondromy groups of the configurations of 6 and 7 lines ([BM1, BM2]; cf. [MP]). According to Mazurovskiĭ, there is a subgroup of S_6 that is not complete. On the other hand, we wish to give general conditions of completeness that are independent of the number of lines.

The following result establishes that the full symmetric group is complete. More precisely, it shows that the Hopf configurations can be defined in terms of the monodromy group only. It is clear that the Hopf configurations realize the maximal monodromy group, i.e., the full symmetric group. Indeed, any two lines in a Hopf configuration are adjacent; in particular, there exists a rigid isotopy exchanging them, whence we see that any transposition can be realized in this way. The result below shows that the Hopf configurations are the only ones having that property.

Theorem 1. *A nonsingular configuration of n lines is Hopf if and only if its monodromy group is S_n, the full symmetric group on n elements.*

The next result, which generalizes Theorem 1, shows that direct products of symmetric groups $S_n \times S_p$, as subgroups of S_{n+p}, are also complete.

Theorem 2. *Let n and p be integers such that $n, p \geq 2$. Then a nonsingular configuration of $N = n + p$ lines is $\langle\langle +n\rangle, \langle -p\rangle\rangle$ or $\langle\langle -n\rangle, \langle +p\rangle\rangle$ if and only if its monodromy group is isomorphic to $S_n \times S_p$.*

We couclude this section by stating an "obvious" conjecture.

Conjecture. *Any finite group is the monodromy group of some configuration.*

The problem with this conjecture is that it does not say anything about the relation between the finite group in question (or even its order) and the number of lines needed to geometrically realize it. Therefore, assuming the conjecture to be true, we complete it by the following algorithmic problem.

Question. *Given a finite group G, find the minimal number $b(G)$ of lines such that there exists a configuration of $b(G)$ lines whose monodromy group is G.*

§4. Proofs of the theorems

4.1. Proof of Theorem 1. We must prove only that if the monodromy group $G(\mathcal{C})$ of some configuration $\mathcal{C}$ of n lines is isomorphic to the symmetric group S_n, then $\mathcal{C}$ is Hopf, that is, $\mathcal{C} = \langle \pm n\rangle$.

We proceed by induction on the number n of lines. For $n = 1, 2, 3$, all configurations are Hopf. Let $\mathcal{C}$ be a configuration of $n \geq 4$ lines such that $G(\mathcal{C}) \cong S_n$, and let A and B be two lines in $\mathcal{C}$. There exists a rigid isotopy h_t of $\mathcal{C}$ such that $h_0 = \mathrm{Id}$, $h_1(A) = B$, and $h_1(L) = L$ for any line $L \in \mathcal{C} \setminus \{A, B\}$. Thus, $\mathrm{lk}(A, C, D) = \mathrm{lk}(B, C, D)$ for any $C,\ D \in \mathcal{C} \setminus \{A, B\}$. Since $\mathrm{lk}(A, C, D)\,\mathrm{lk}(A, B, D) = \mathrm{lk}(B, C, D)\,\mathrm{lk}(A, B, C)$, it follows that A and B are homologous. Hence $\mathcal{C}$, is homologically trivial. The monodromy group of any subconfiguration of $n-1$ lines of $\mathcal{C}$ is isomorphic to S_{n-1}, and thus, by induction hypothesis, must be $\langle \varepsilon(n-1)\rangle$. Lemma 1 below applies. Hence, $\mathcal{C}$ must be of type $\langle \varepsilon n\rangle$, because the other two possible types are clearly not homologically trivial for $n \geq 4$. This is the required result. •

Lemma 1. *Let $\mathcal{C}$ be a nonsingular configuration of N lines that contains a subconfiguration $\mathcal{C}'$ of $N-1$ lines of type $\langle +(N-1)\rangle$. Then $\mathcal{C}$ must be of one of the following types:*

(i) $\langle +N\rangle$;
(ii) $\langle\langle +(N-2)\rangle, \langle -2\rangle\rangle$;
(iii) *one of the following types $\mathcal{C}_k$: $\mathcal{C}_k = \langle -\langle 1\rangle, \langle +k\rangle, \langle +(N-k-1)\rangle\rangle$,* $2 \leq k \leq N-3$.

Proof. By assumption, there is a nonsingular quadric Q (a one-sheet hyperboloid if we think of the ambient space as affine) such that the $N-1$ lines of C' are linear generators of the same family for Q. Let A be the line such that $\{A\} \cup C' = C$ (completing C' to C). We consider the position of A with respect to Q. There are four possible cases:

(1) the line A does not intersect Q;
(2) the line A intersects Q at a single point which is a tangent point;
(3) the line A intersects Q at exactly two points;
(4) the line A lies on Q.

(We note that if and Q A has more than three points in common, then actually $A \subset Q$, i.e., this is the situation described in case (4).) In case (4), since the configuration is nonsingular, the line A belongs to the same family of linear generators that contains the other $N-1$ lines; then the configuration C is Hopf, of type $\langle +N \rangle$. By means of simple rigid isotopies (moving only A), cases (1) and (2) are reduced to cases (3) and (4). So, we are left with examining the configurations of type (3). The $N-1$ lines of the subconfiguration C' determine a partition of $Q \setminus C'$ in $N-1$ parts $Q_1, \dots, Q_{N-1}$. Let $P_1, P_2 \in A \cap Q$ be the two intersection points of A with Q. If P_1 and P_2 are in one and the same part Q_i for some i, then it is easy to see that, by a rigid isotopy of C keeping all the lines except A fixed, we can move A to a linear generator in the complement of the other lines; this is case (4). Now, assume that P_1 and P_2 are in different parts Q_i and Q_j with $i \neq j$. The boundary of Q_i (respectively, Q_j) consists of two lines of C', say, A_i and B_i (respectively, A_j and B_j). If Q_i and Q_j are adjacent regions (i.e., $j \equiv 1 \mod N$), the boundaries of Q_i and Q_j have one line L in common (see Figure 2 below), e.g. $L = A_j = B_i$. By slightly pushing A and L off Q, we see that there are exactly two adjacency classes: the lines A and L form one class and the rest of the lines the other one. Hence, the isotopy type of C is $\langle\langle +(N-2)\rangle, \langle -2\rangle\rangle$. If Q_i and Q_j are not adjacent regions, then $N \geqslant 4$. The boundary of $Q_i \cup Q_{i+1} \cup \cdots \cup Q_j$ consists of two lines. There are exactly three adjacency classes (Figure 2): the k lines included in the interior of $Q_i \cup Q_{i+1} \cup \dots Q_j$ form one class, the other lines but A form the second class, and A forms a class consisting of exactly one element. Then the isotopy type of C is $\langle -\langle 1\rangle, \langle +k\rangle, \langle +(N-k+1)\rangle\rangle$ for some $k \geqslant 2$. This completes the proof. •

The above proof also implies the following statement.

Proposition 1. *A homologically trivial configuration is Hopf (this was proved also in* [KM]*). In particular, the configurations whose monodromy groups are the full symmetric group can be identified with the homologically trivial configurations.*

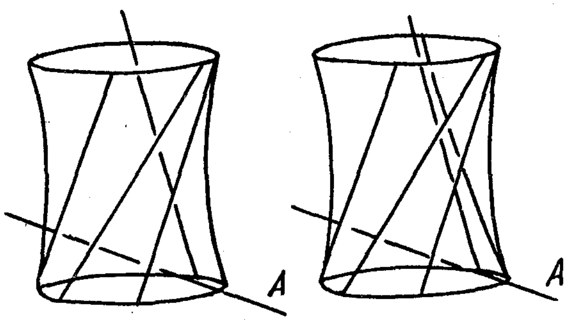

Figure 2.

Remark. Also, the proof classifies the configurations of 4 lines (then, it coincides with the proof in [VD]); this is no surprise because 4 is the smallest number n for which a non Hopf configuration of n lines exists.

4.2 Proof of Theorem 2.

4.2.1. **Preliminary result.** Let N be a positive integer and $\mathcal{A}_N$ the $\mathbb{Z}$-ring of Laurent polynomials with integral coefficients in N variables. Given $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}^N$, we denote by t^α the monomial $t_1^{\alpha_1} \cdots t_N^{\alpha_N}$. We call $\lfloor \alpha \rfloor = \alpha_1 + \cdots + \alpha_N$ the *reduced degree* of t^α. We denote by $S_N \times \mathcal{A}_N \to \mathcal{A}_N, (\sigma, P) \mapsto \sigma \cdot P$, the action of the symmetric group S_N on $\mathcal{A}_N$ induced by the natural action of S_N on $\mathbb{Z}^N$.

Lemma 2. *Let $P = \sum_{\alpha \in \Lambda} a_\alpha t^\alpha \in \mathcal{A}_N$ and $\sigma \in S_N$. Let $k, c \in \mathbb{Z}$. Assume that*

(1) *there exists $\epsilon \in \{\pm 1\}$ and $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{Z}^N$ such that*

$$\sigma \cdot P = \epsilon t^\beta P;$$

(2) *there is a unique monomial $M_\alpha = a_\alpha t^\alpha$ with $\alpha \in \Lambda$ such that $a_\alpha = c$ and $\lfloor \alpha \rfloor = k$.*

Then $\epsilon = 1, \lfloor \beta \rfloor = 0$, and $\beta = \sigma(\alpha) - \alpha$. In particular, if the constant term of P is the only monomial of P of reduced degree 0, then P is a fixed point of σ.

Proof. The action of S_N does not change the reduced degree of the monomials. Thus, from our first assumption it follows that $\lfloor \beta \rfloor = 0$.

By (1), there is a monomial $M_\gamma = a_\gamma t^\gamma$, $\gamma \in \Lambda$, such that $\sigma \cdot M_\alpha = a_\alpha t^{\sigma(\alpha)} = \epsilon t^\beta \cdot M_\gamma = \epsilon a_\gamma t^{\beta+\gamma}$. We deduce that $a_\alpha = \epsilon a_\gamma$ and $\sigma(\alpha) = \beta + \gamma$. Since $\lfloor \beta \rfloor = 0$, we have $\lfloor \sigma(\alpha) \rfloor = \lfloor \alpha \rfloor = \lfloor \gamma \rfloor$. Assumption (2) implies that $M_\gamma = M_\alpha$. Thus, $a_\alpha t^{\sigma(\alpha)} = \epsilon a_\alpha t^{\beta+\alpha}$, whence $\epsilon = 1$ and $\sigma(\alpha) = \beta + \alpha$. If the only monomial of P of reduced degree 0 is the constant term a_0, then $\beta = \sigma(0) - 0 = 0$, which completes the proof. •

4.2.2. *Computation of the monodromy group.*

The first part of the following lemma proves a half of Theorem 2.

Lemma 3.

(1) *Let $n, p \geq 2$ be positive integers. The monodromy group of any configuration of type $\langle\langle +n \rangle, \langle -p \rangle\rangle$ (respectively, $\langle\langle -n \rangle, \langle +p \rangle\rangle$) is isomorphic to $S_n \times S_p$.*

(2) *Let n, p, q be integers such that $n \geq 2, p \geq 1, q \geq 2$. If $n \neq q$, then the monodromy group of configurations of type $\langle -\langle +n \rangle, \langle -p \rangle, \langle +q \rangle\rangle$ (respectively, $\langle +\langle -n \rangle, \langle +p \rangle, \langle -q \rangle\rangle$) is isomorphic to $S_n \times S_p \times S_q$. If $n = q$, then the monodromy group of $\langle -\langle +n \rangle, \langle -p \rangle, \langle +q \rangle\rangle$ is isomorphic to $S_p \times G$, where G is a proper subgroup of S_{2n} containing $S_n \times S_n$ as a normal subgroup of index 2.*

Remark. Part (1) of the lemma was first proved in [D] merely by using the linking numbers of triples. Here we use the multivariable Alexander polynomial (cf. §2) to give a short proof.

Proof.

(1) Since $\langle\langle -n \rangle, \langle +p \rangle\rangle$ is the mirror image of $\langle\langle +n \rangle, \langle -p \rangle\rangle$, it suffices to prove the result for $C = \langle\langle +n \rangle, \langle -p \rangle\rangle$. There is an orientation and order on the lines of C (cf. Figure 3 below) such that the resulting oriented and ordered configuration C^* has the following Alexander polynomial:

$$\Delta_{C^*}(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+p}) = (t_1 \ldots t_{n+p} - 1)^{n-1}(t_1 \ldots t_n t_{n+1}^{-1} \ldots t_{n+p}^{-1} - 1)^{p-1}.$$

The above formula is obtained by repeated application of the iteration theorem (see the Proposition in Subsection 2.3), where the first n variables correspond to the n lines of the Hopf configuration $\langle +n \rangle$ contained in C.

Let $\sigma \in G(C)$. There exists $\epsilon \in \{-1, 1\}$ and $(\beta_1, \ldots, \beta_{n+p}) \in \mathbb{Z}^{n+p}$ such that

$$\Delta_{C^*}(t_{\sigma(1)}, \ldots, t_{\sigma(n+p)}) = \varepsilon\, t_1^{\beta_1} \ldots t_{n+p}^{\beta_{n+p}} \Delta_{C^*}(t_1, \ldots, t_n, t_{n+1}, \ldots, t_{n+p}).$$

This condition is equivalent to

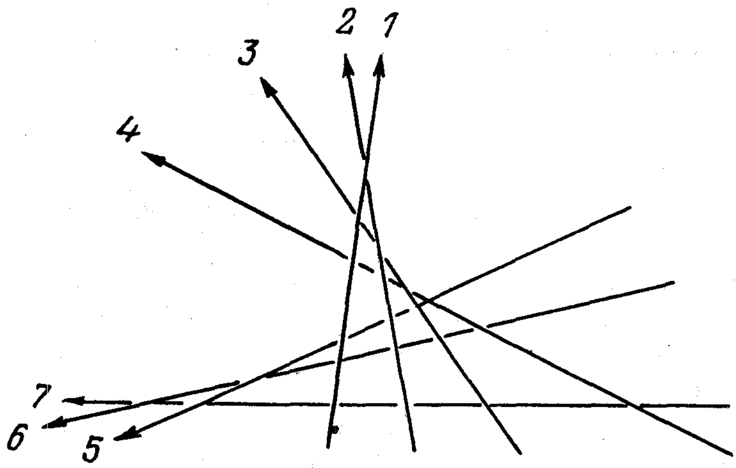

Figure 3. Ordered and oriented configuration C^* $(n = 4, p = 3)$.

$$\sigma \cdot P = \varepsilon t_1^{\beta_1} \cdots t_{n+p}^{\beta_{n+p}} P,$$

where $P(t_1, \dots, t_{n+p}) = (t_1 \dots t_n t_{n+1}^{-1} \dots t_{n+p}^{-1} - 1)^{p-1}$.

If $n \neq p$, the constant term of P (it is $(-1)^{p-1}$) is the only monomial of P whose reduced degree is 0. Lemma 2 shows that P is fixed by σ. Therefore, $\sigma \in S_{\{1,\dots,n\}} \times S_{\{n+1,\dots,n+p\}}$.

If $n = p$, the only monomials of P with coefficients ± 1 are $(-1)^{p-1}$ and $t_1^{p-1} \cdots t_p^{p-1} t_{p+1}^{-(p-1)} \cdots t_{2p}^{-(p-1)}$. Hence, if P is not invariant under σ, then

$$\epsilon = (-1)^{p-1} \quad \text{and} \quad \beta = (\underbrace{-(p-1), \dots, -(p-1)}_{p \text{ times}}, \underbrace{p-1, \dots, p-1}_{p \text{ times}}).$$

Then

$$\begin{aligned}
&(t_{\sigma(1)} \cdots t_{\sigma(p)} t_{\sigma(p+1)}^{-1} \cdots t_{\sigma(2p)}^{-1} - 1)^{p-1} \\
&\quad = (-1)^{p-1} t_1^{-(p-1)} \cdots t_p^{-(p-1)} t_{n+1}^{p-1} \cdots t_{2p}^{p-1} \cdot (t_1 \cdots t_p t_{p+1}^{-1} \cdots t_{2p}^{-1} - 1)^{p-1} \\
&\quad = (-1)^{p-1} (1 - t_1^{-1} \cdots t_p^{-1} t_{p+1} \cdots t_{2p})^{p-1} \\
&\quad = (t_1^{-1} \cdots t_p^{-1} t_{p+1} \cdots t_{2p} - 1)^{p-1}.
\end{aligned}$$

Therefore, $\sigma(\{1,\dots,p\}) = \{p+1,\dots,2p\}$ and $\sigma(\{p+1,\dots,2p\}) = \{1,\dots,p\}$. This is a contradiction, because if $i,j,k \in \{1,\dots,p\}$ and $i',j',k' \in \{p+1,\dots,2p\}$, then

$$\mathrm{lk}(L_i, L_j, L_k) = 1 \neq -1 = \mathrm{lk}(L_{i'}, L_{j'}, L_{k'})$$

Thus, a permutation σ with the property mentioned above certainly does not belong to the monodromy group.

As a result, in both cases we obtain the necessary condition $\sigma \in S_{\{1,\dots,n\}} \times S_{\{n+1,\dots,n+p\}}$. This condition is also sufficient, because any two lines L_i, L_j with i, j both in $\{1,\dots,n\}$ or both in $\{n+1,\dots,n+p\}$ are adjacent.

(2) We can endow the lines of the configuration $\mathcal{C} = \langle +\langle -n\rangle, \langle +p\rangle, \langle -q\rangle\rangle$ with an orientation and order (cf. Figure 4) such that the resulting oriented and ordered configuration $\mathcal{C}^*$ has the following Alexander polynomial:

$$\begin{aligned}
&\Delta_{\mathcal{C}^*}(t_1,\dots,t_{n+p+q}) \\
&\quad = (t_1^{-1}\dots t_n^{-1} t_{n+1}\dots t_{n+p+q} - 1)^{n-1}(t_1\dots t_{n+p+q} - 1)^p \\
&\quad \times (t_1\dots t_{n+p} t_{n+p+1}^{-1}\dots t_{n+p+q}^{-1} - 1)^{q-1}.
\end{aligned}$$

Again, this formula is obtained using the iteration theorem (see Proposition in Subsection 2.3). Here the n variables $t_1,\dots,t_n$ correspond to the Hopf configuration $\langle -n\rangle$ contained in $\mathcal{C}$, the p variables $t_{n+1},\dots,t_{n+p}$ to the Hopf configuration $\langle +p\rangle$ contained in $\mathcal{C}$, and the q variables $t_{n+p+1},\dots,t_{n+p+q}$ to the Hopf configuration $\langle -q\rangle$ contained in $\mathcal{C}$.

Let $\sigma \in G(\mathcal{C})$. Then there exists $\varepsilon \in \{-1,1\}$ and $(\beta_1,\dots,\beta_{n+p+q}) \in \mathbb{Z}^{n+p+q}$ such that

$$\Delta_{\mathcal{C}^*}(t_{\sigma(1)},\dots,t_{\sigma(n+p+q)}) = \varepsilon\, t_1^{\beta_1}\dots t_{n+p+q}^{\beta_{n+p+q}} \Delta_{\mathcal{C}^*}(t_1,\dots,t_{n+p+q}).$$

For simplicity of notation, we put $u = t_1\cdots t_n, v = t_{n+1}\cdots t_{n+p}$, and $w = t_{n+p+1}\cdots t_{n+p+q}$. The equality above is equivalent to

$$\sigma \cdot P = \epsilon t^{\beta} \cdot P,$$

where $P(t_1,\dots,t_{n+p+q}) = (u^{-1}vw - 1)^{n-1}(uvw^{-1} - 1)^{q-1}$. The only monomials of P whose coefficient is ± 1 are given in the following table.

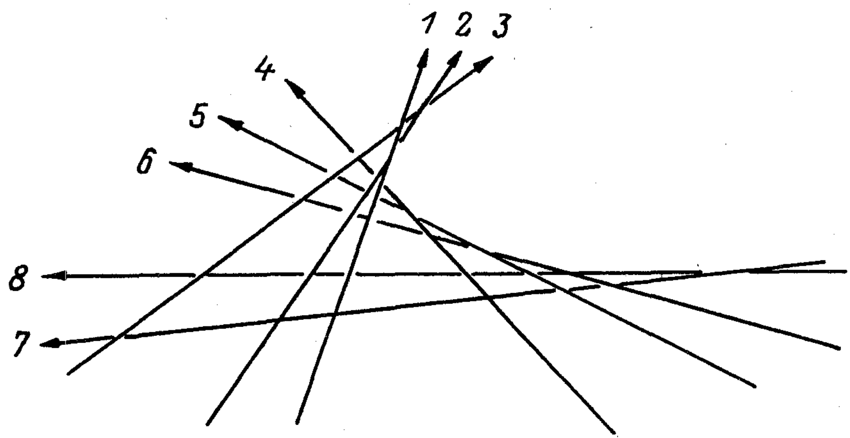

Figure 4. Ordered and oriented configuration C^* $(n = 3, p = 3, q = 2)$.

monomials	reduced degrees
$(-1)^{n+q-2}$	0
$(-1)^{n-1}u^{q-1}v^{q-1}w^{-(q-1)}$	$(n+p-q)(q-1) = a$
$(-1)^{q-1}u^{-(n-1)}v^{n-1}w^{n-1}$	$(-n+p+q)(n-1) = b$
$u^{q-n}v^{n+q-2}w^{n-q}$	$(n+p-q)(q-1) + (-n+p+q)(n-1) = a+b$

If $a > 0$ and $b > 0$, then $a+b$ is distinct from a, b, and 0. It follows that $(-1)^{n+q-2}$ is the only monomial with coefficient ± 1 having reduced degree 0. Hence, Lemma 2 applies, and P is fixed by σ.

If $a < 0$ or $b < 0$, we observe that a is distinct from $b, a+b$, and 0. It follows that $(-1)^{n-1}u^{q-1}v^{q-1}w^{-(q-1)}$ is the only monomial with coefficient ± 1 having reduced degree a. Similarly, $(-1)^{q-1}u^{-(n-1)}v^{n-1}w^{n-1}$ is the only monomial with coefficient ± 1 having reduced degree b. Applying Lemma 2 twice, we deduce that $\beta = 0$ and P is fixed by σ.

There remain the cases where $a = 0$ and $b = 0$. Since they are symmetric (exchange the role of n and q), it suffices to consider the case $a = 0$. Since $q \geq 2$, we have $q = n+p$. There are exactly two monomials with coefficient ± 1 having reduced degree 0 (respectively, b): $(-1)^{n+q-2}$ and $(-1)^{n-1}u^{q-1}v^{q-1}w^{-(q-1)}$ (respectively, $(-1)^{q-1}u^{-(n-1)}v^{n-1}w^{n-1}$ and $u^{q-n}v^{n+q-2}w^{n-q}$). Assume that $\sigma \cdot P \neq P$. Then, necessarily,

$$\epsilon = (-1)^{q-1} \quad \text{and} \quad \beta = (\underbrace{-(q-1), \dots, -(q-1)}_{n+p \text{ times}}, \underbrace{q-1, \dots, q-1}_{q \text{ times}}).$$

Thus, we obtain the relation

$$\sigma \cdot P = (u^{-1}vw - 1)^{n-1}(u^{-1}v^{-1}w - 1)^{q-1}.$$

Since P possesses a monomial of degree $(\underbrace{0, \dots, 0}_{n \text{ times}}, \underbrace{2, \dots, 2}_{p \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}})$, and the polynomial on the right-hand side does not, even up to a permutation, we see that there is no permutation σ satisfying the above relation. This is a contradiction; therefore, P must be fixed by σ.

As a result, in all cases we obtain the necessary condition $\sigma \cdot P = P$. It is not hard to see that the monomial $u^{-n+q}v^{n+q-2}w^{n-q}$ must then be fixed by σ. It follows that if $n \neq q$, then $\sigma \in S_{\{1,\dots,n\}} \times S_{\{n+1,\dots,n+p\}} \times S_{\{n+p+1,\dots,n+p+q\}}$. This is also a sufficient condition for σ to belong to the monodromy group, because any pair of lines L_i, L_j with i, j both in $\{1, \dots, n\}$, both in $\{n+1, \dots, n+p\}$, or both in $\{n+p+1, \dots, n+p+q\}$ are adjacent. If $n = q$, there is an extra possibility for σ. Besides the permutations in $S_{\{1,\dots,n\}} \times S_{\{n+1,\dots,n+p\}} \times S_{\{n+p+1,\dots,n+p+q\}}$, there are permutations σ that exchange $\{1, \dots, n\}$ and $\{n+p, \dots, 2n+p\}$, namely,

$$\sigma(\{1, \dots, n\}) = \{n+p, \dots, 2n+p\}, \qquad \sigma(\{n+p, \dots, 2n+p\}) = \{1, \dots, n\}. \quad (1)$$

Consequently, $\sigma \in S_{\{n+1,\dots,n+p\}} \times H$, where H is generated by the elements of $S_{\{1,\dots,n\}} \times S_{\{n+p+1,\dots,2n+p\}}$ and the elements of $S_{\{1,\dots,2n+p\}}$ with property (1). Since the derived configuration of C is $\langle +3 \rangle$, there is a rigid isotopy exchanging the two Hopf subconfigurations $\langle -n \rangle$ and $\langle -p \rangle$. Within the same adjacency class, all permutations can be realized by rigid isotopies. By composing such rigid isotopies, we can construct a rigid isotopy of C realizing any permutation σ with property (1). It is immediately verified that the group $G(C)$ satisfies the claimed properties. •

4.2.3. The remaining half of Theorem 2. The converse is proved again by induction on the number of lines, but the proof requires the following elementary though technical algebraic result: up to conjugacy, there is only one embedding $S_n \times S_p \to S_{n+p}$ if n or p is greater than 3 (we refer to the appendix of this paper for the precise statement and the proof, as well as the list of nonstandard embeddings). From this result, we deduce the following lemma.

Lemma 4. *Let C be a nonsingular configuration of $n+p$ lines with $n, p \geq 2$ such that the monodromy group $G(C)$ is isomorphic to $S_n \times S_p$. There is an ordering $L_1, \dots, L_{n+p}$ of the lines of C such that $G(C) = S_{\{L_1,\dots,L_n\}} \times S_{\{L_{n+1},\dots,L_{n+p}\}}$.*

Remark. If G is the monodromy group of a configuration of N lines, then there is a natural embedding $G \hookrightarrow S_N$. Lemma 4 says that if G is isomorphic to the product $S_n \times S_p$ with $n + p = N$, then the embedding is standard. Hence the geometry of configurations of lines implies a certain algebraic rigidity of the monodromy group.

Proof. If n or p is greater than 3, the result immediately follows from the product embedding theorem presented in the Appendix. If not, there can be nonstandard embeddings $S_n \times S_p \to S_{n+p}$ (see the list in the Appendix). We show directly that no monodromy group can be the image of a nonstandard embedding. For $n = p = 2$, the only configuration whose monodromy group is isomorphic to $S_2 \times S_2$ is $\langle\langle +2\rangle, \langle -2\rangle\rangle$, and the corresponding monodromy group is embedded in S_4 in a standard way. In what follows, we index the lines in the same way as in the description of the nonstandard embeddings in the Appendix. For $n = p = 3$, we inspect the linking numbers, using the fact that the elements of the monodromy group preserve the linking numbers of the triples. We observe that $\mathrm{lk}(L_1, L_5, L_6) = \mathrm{lk}(L_2, L_3, L_4)$, and the linking numbers of all other triples of lines are equal. Using the fact that homology is a transitive relation, we deduce that the configuration is homologically trivial. By Propositon 1, the monodmony group is the full summetric group, a contradiction. For the other nonstandard embeddings, we observe that the subconfiguration $\{L_3, \dots, L_{n+2}\}$ is homologically trivial, hence Hopf by Proposition 1, and we twice apply Lemmas 1 and 3. First, we show that the subconfiguration $\{L_2, \dots, L_{n+2}\}$ turns out to be different from the nonstandard embeddings of $S_2 \times S_n$. •

Theorem 2 is true for $N \leq 5$ (see the known classification in §3). Assume that the theorem is true for the configurations of $N - 1$ lines. Let C be a configuration of $N = n + p$ lines ($N \geq 6$) whose monodromy group satisfies $G(C) \cong S_n \times S_p$. By Lemma 4, there is a certain ordering $L_1, \dots, L_N$ of the lines of C such that $G(C) = S_{\{L_1,\dots,L_n\}} \times S_{\{L_{n+1},\dots,L_{n+p}\}}$. There is no loss of generality in assuming that

$n \leq p$. Let $\mathcal{C}' = \mathcal{C} \setminus \{L_{n+p}\}$; this is a subconfiguration of $\mathcal{C}$ which consists of $N-1$ lines. Clearly, $S_{\{L_1,\dots,L_n\}} \times S_{\{L_{n+1},\dots,L_{n+p-1}\}} \subseteq G(\mathcal{C}')$. Now the proof splits into two cases: (1) there is at least one permutation $\tau \in G(\mathcal{C}')$ such that τ sends some L_i, $1 \leq i \leq n$, to L_j, $n+1 \leq j \leq n+p-1$; (2) there is no such permutation.
Case 1: $S_{\{L_1,\dots,L_n\}} \times S_{\{L_{n+1},\dots,L_{n+p-1}\}} \subsetneq G(\mathcal{C}') \subseteq S_{\{L_1,\dots,L_{n+p-1}\}}$. We claim that $\mathcal{C}' = \langle \pm(N-1) \rangle$. To see this, we first apply the following lemma (the proof is straightforward and we omit it).

Lemma 5. *Let A and B two disjoint finite sets, and let $G = S_A \times S_B \subset S_{A \cup B}$ (natural embedding). Let $\tau \in S_{A\cup B}$ be such that $\tau(A) \cap B \neq \varnothing$. We denote by H the subgroup of $S_{A\cup B}$ generated by the elements of G and τ. Then either $H = S_{A\cup B}$, or τ exchanges A and B (i.e., $\tau(A) = B$ and $\tau(B) = A$, which implies that $|A| = |B|$).*

If $G(\mathcal{C}')$ is the full symmetric group $S_{\{L_1,\dots,L_{n+p-1}\}} \cong S_{N-1}$, then we apply Theorem 1 to obtain the claimed result. If $G(\mathcal{C}')$ is not the full symmetric group, then from Lemma 5 it follows that $G(\mathcal{C}')$ contains at least one element τ exchanging the sets $A = \{L_1, \dots, L_n\}$ and $B = \{L_{n+1}, \cdots, L_{n+p-1}\}$. This implies that $p-1 = n \geq 3$. Using τ and multiplication by elements of S_A and S_B, respectively, we immediately see that $G(\mathcal{C}')$ sends any triple of elements of A to any triple of elements of B. In the same way, $G(\mathcal{C}')$ sends any triple $L_\alpha, L_\beta, L_\gamma$ with $\alpha, \beta \in \{1, \dots, n\}$ and $\gamma \in \{n+1, \dots, 2n\}$ to any triple $L_{\alpha'}, L_{\beta'}, L_{\gamma'}$ with $\alpha', \beta' \in \{n+1, \dots, 2n\}$ and $\gamma' \in \{1, \dots, n\}$. Since the elements of the monodromy group preserve the linking number of triples, we see that any two lines with one line in A and the other in B are homologous. Since homology is a transitive relation, it follows that all lines in $\mathcal{C}'$ are homologous, i.e., $\mathcal{C}'$ is homologically trivial. Proposition 1 implies that $\mathcal{C}'$ is Hopf, which is the required result.

Thus, now all that remains to be seen is the effect of adding one line (L_{n+p}) to $\mathcal{C}'$. Then Lemma 1 applies: $\mathcal{C}$ cannot be Hopf because its monodromy group would be S_N, and $\mathcal{C}$ (or its mirror image) cannot be of type $\langle -\langle 1 \rangle, \langle +k \rangle, \langle +(N-k-1) \rangle \rangle$ because Lemma 4 would imply that its monodromy group is either $S_k \times S_{N-k-1}$ (this contradicts the fact that $n+p = N$) or a group which is clearly not isomorphic to the direct product of two symmetric groups. Therefore, $\mathcal{C}$ (or its mirror image) must coincide with $\langle \langle +(N-2) \rangle, \langle -2 \rangle \rangle$. This concludes the analysis of Case (1).
Case 2: $G(\mathcal{C}') = S_{\{L_1,\dots,L_n\}} \times S_{\{L_{n+1},\dots,L_{n+p-1}\}}$. By the induction hypothesis, $\mathcal{C}'$ (or its mirror image) is $\langle \langle +n \rangle, \langle -(p-1) \rangle \rangle$. Then there exist two disjoint quadrics Q and Q' (which are the boundaries of disjoint neighborhoods) such that we can put $n-1$ lines to Q and the remaining p lines to Q'. Again we need to know the effect of adding one line to that configuration. The following lemma is proved by using the same kind of argument as in the proof of Lemma 1.

Lemma 6. *Let C be a nonsingular configuration of N lines that contains a subconfiguration of $N-1$ lines of type $\langle\langle +r\rangle, \langle -s\rangle\rangle$ with $N-1=r+s$ $(r,s\geq 2)$. Then C has one of the following properties:*

(i) $C=\langle\langle +(r+1)\rangle, \langle -s\rangle\rangle$;
(ii) $C=\langle\langle +r\rangle, \langle -(s+1)\rangle\rangle$;
(iii) $C=\langle +\langle +(r-1)\rangle, \langle -2\rangle, \langle -s\rangle\rangle$;
(iv) $C=\langle -\langle +r\rangle, \langle +2\rangle, \langle -(s-1)\rangle\rangle$;
(v_k) $C=\langle\langle +\langle +(r-k)\rangle, \langle -s\rangle\rangle, \langle -\langle 1\rangle, \langle +k\rangle\rangle\rangle$ *for some* $2\leq k\leq r-1$;
(vi_k) $C=\langle\langle -\langle -(s-k)\rangle, \langle +r\rangle\rangle, \langle +\langle 1\rangle, \langle -k\rangle\rangle\rangle$ *for some* $2\leq k\leq s-1$;
(vii) *C contains the simple configuration of 5 lines as a subconfiguration.*

Only the configurations described in (i) and (ii) have the monodromy group isomorphic to a product $S_n\times S_p$ with $N=n+p$. Cases (iii) and (iv) are ruled out by Lemma 4. For cases (v_k) and (vi_k), we observe that there is one line which cannot be exchanged with any other line by an element of the monodromy group. This is readily checked with the help of linking numbers. Therefore, those cases are ruled out by Lemma 4. Case (vii) is ruled out by Lemma 4, because the monodromy group of the simple configuration of five lines is D_5. That concludes the analysis of Case (2) and the proof of the theorem. •

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Appendix. An embedding theorem for direct products of symmetric groups

A.1. Introduction

An embedding of a group G in a group H is an injective homomorphism ϕ: $G\hookrightarrow H$. Let n and p be natural numbers, $n\leq p$. We say that an embedding of the symmetric group S_n in the symmetric group S_p is *standard* if the image of the embedding acts exactly on n elements. It is well known that all embeddings of S_n into S_{n+1} are standard except for $n=5$ (see, e.g., [Bu]). Our purpose in this note is to extend this result to the case of products of symmetric groups. Despite the simplicity of the result, the author has not succeeded in finding it in the literature.

A.2. Results

First, we recall some definitions and notation. If E is a set, $|E|$ denotes the cardinality of E. If x is a rational number, $[x]$ denotes the greatest integer n not exceeding x. Let G be a subgroup of the symmetric group S_n. If $\sigma \in G$ is a permutation, then the *support* of σ is defined as $\mathrm{supp}_G(\sigma) = \{1 \leq x \leq n,\ \sigma(x) \neq x\}$. Also, we define the (relative) support of the subgroup G in S_n by $\mathrm{Supp}_{S_n}(G) = \bigcup_{\sigma \in G} \mathrm{Supp}_{S_n}(\sigma)$. Let $j : S_n \hookrightarrow S_{n+m}$ and $k : S_m \hookrightarrow S_{n+m}$ be the embeddings induced by the natural inclusions $\{1, \ldots, n\} \subset \{1, \ldots, n+m\}$ and $\{n+1, \ldots, n+m\} \subset \{1, \ldots, n+m\}$, respectively. There is the natural embedding $S_n \times S_p \to S_{n+m}$ given by $(\sigma, \tau) \mapsto j(\sigma)k(\tau)$. Then we say that an embedding $\phi\colon S_n \times S_m \to S_{n+m}$ is *standard* if ϕ is inner conjugated to the natural embedding by an element of S_{n+m}.

Theorem (Embedding Theorem). *Let n and m be natural numbers, $n, m > 1$. All embeddings $S_n \times S_m \hookrightarrow S_{n+m}$ are standard except in the following cases:*

- $n = 2$ *or* $m = 2$;
- $n = 3$ *and* $m = 3$.

Classification of nonstandard embeddings ϕ: $S_n \times S_m \hookrightarrow S_{n+m}$. In what follows, we denote by G (respectively, H) the subgroup $\{1\} \times S_m$ (respectively, $S_n \times \{1\}$) and we naturally identify G (respectively, H) with S_m (respectively, S_n). We list all nonstandard embeddings up to conjugacy by an element of S_{n+m}. By $(a_1 \ldots a_n)$ we denote the n-cycle sending a_1 to a_2, etc. We adopt the convention that permutations act on the right.

n = m = 2:

1)

$$\phi(G) = \{\mathrm{id}, (12)(34)\};$$
$$\phi(H) = \{\mathrm{id}, (13)(24)\};$$
$$\phi(S_2 \times S_2) = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \subset S_4;$$

2)

$$\phi(G) = \{\mathrm{id}, (12)(34)\};$$
$$\phi(H) = \{\mathrm{id}, (12)\}; \qquad \phi(S_2 \times S_2) = \{\mathrm{id}, (12), (34), (12)(34)\} \subset S_4.$$

m = 2, n ≥ 3:

1)

$$\phi(G) = \{\mathrm{id}, (12)\};$$

as a set,

$$\phi(H) = \{(12)\sigma, \sigma \in S_{\{3,\dots,n+2\}} \setminus A_{\{3,\dots,n+2\}}\} \cup A_{\{3,\dots,n+2\}},$$

where $S_{\{3,\dots,n+2\}}$ (respectively, $A_{\{3,\dots,n+2\}}$) denotes the symmetric group (respectively, the alternating group) on $\{3,\dots,n+2\}$. If n is odd, $\phi(H)$ is generated by (12)(34) and $(34\dots n+2)$; if n is even, $\phi(H)$ is generated by (12)(34) and $(12)(34\dots n+2)$.

n = m = 3:

1)

$$\phi(G) = \{\mathrm{id}, (12)(35)(46), (13)(26)(45), (14)(25)(36), (156)(243), (165)(234)\};$$
$$\phi(H) = \{\mathrm{id}, (12)(36)(45), (13)(25)(46), (14)(26)(35), (156)(234), (165)(243)\}.$$

Remark. The nonstandard embedding $\phi : S_3 \times S_3 \hookrightarrow S_6$ is given by the left and right actions of S_3 on itself (since these actions commute, so do the corresponding automorphisms (elements of $S_{S_3} \cong S_6$), so that ϕ is an embedding of $S_3 \times S_3$ in S_6). A similar observation applies to the first embedding $\phi : S_2 \times S_2 \hookrightarrow S_4$ on the list.

Proof. We prove that the above list is complete (for the particular integers n and m indicated). The case $n = m = 2$ is straightforward. If $n = m = 3$, first we observe that if an element of order 3 in G or H is sent to a 3-cycle in S_6, then the embedding is standard (because the images of the elements of order 2 in G (respectively, H) must commute in S_6 with the images of elements of order 3 in H (respectively, G).) If all elements of order 3 in G and H are sent to double 3-cycles in S_6, then all elements of order 2 in G and H are sent to triple transpositions in S_6. Fix the image $\phi(c)$ of an element $c \in G$ of order 3 (it is a double 3-cycle). All images of the elements of order 2 in H (which we know to be triple transpositions) must commute with $\phi(c)$. Using this fact, we explicitly compute those 3 elements. Those elements generate $\phi(H)$. A symmetric remark yields the computation of $\phi(G)$. This construction shows that the embedding is well defined and nonstandard. In the remaining cases $m = 2$ and $n \geqslant 3$, from Lemma 1 below it follows that there exists an odd prime number p such that $\frac{3}{4}n \leqslant p \leqslant n$ and all p-cycles of H are sent to p-cycles in S_{n+2}. Since the image σ of the element of order 2 in G must commute in S_{n+2} with each image $\phi(c)$ of a p-cycle (which is a p-cycle), we deduce that the support of σ and the support of $\phi(c)$ are disjoint. By the injectivity of ϕ, $|\bigcup_c \mathrm{Supp}_{S_{n+2}}(\phi(c))| \geqslant n$

(the union is taken over all p-cycles $c \in H$). So, $|\mathrm{Supp}_{S_{n+2}}(\sigma)| \leqslant 2$, whence σ is a transposition. We may assume that $\sigma = (12)$ in S_{n+2}. Observe that the alternating group $A_n \subset H$ is generated by the p-cycles. It follows that $\phi(A_n)$ acts on exactly n elements, and its support in S_{n+2} is disjoint from $\mathrm{Supp}_{S_{n+2}}(\phi(G))$. Clearly, $\phi(A_n)$ and the image of any transposition $\tau \in H$ generate $\phi(H)$. If for any transposition $\tau \in H$, $\mathrm{Supp}_{S_{n+2}}(\phi(\tau)) \cap \{1,2\} = \varnothing$, then $\mathrm{Supp}_{S_{n+2}}(G) \cap \mathrm{Supp}_{S_{n+2}}(H) = \varnothing$ and the embedding ϕ is standard. Otherwise we deduce that the embedding coincides with the one given on the list above. •

A.3. Proof of the Theorem

Let ϕ be an embedding: $S_n \times S_m \hookrightarrow S_{n+m}$. We recall that G (respectively, H) denotes the subgroup $\{1\} \times S_m$ (respectively, $S_n \times \{1\}$), and we naturally identify G (respectively, H) with S_m (respectively, S_n). Thus, a permutation $\sigma \in S_m$ (respectively, S_n) is identified with the corresponding element $(1, \sigma) \in G$ (respectively, $(\sigma, 1) \in H$). The proof involves a result from number theory (see Lemma 1 below) and will be split into 4 steps. From now on we assume, without loss of generality, that $4 \leq m$ and $3 \lneq n \leq m$.

Lemma 1. *Set $\alpha = 4/3$. Let $N \geq 3$ be an integer. There exists a prime number p such that $N/\alpha \leq p \leq N$.*

Step 1. *There exists a positive integer p such that*

(i) $\frac{m}{\alpha} \leq p \leq m$, *where* $\alpha = \frac{4}{3}$;

(ii) *the image in S_{n+m} of any p-cycle in $G \cong S_m$ under ϕ is a p-cycle.*

Step 2. *Let $p > 2$ be a number satisfying condition* (ii) *of step* 1. *Then*

(iii) *for any p-cycle $c \in G$ we have*

$$\mathrm{Supp}_{S_{n+m}}(\phi(H)) \cap \mathrm{Supp}_{S_{n+m}}(\phi(c)) = \varnothing;$$

(iv)

$$\left|\bigcup_{p\text{-cycles } \sigma \in G} \mathrm{Supp}_{S_{n+m}}(\phi(\sigma))\right| = m;$$

(v)

$$|\mathrm{Supp}_{S_{n+m}}(\phi(H))| = n.$$

Step 3. *There exists a positive integer q such that*

(ii′) *the image under ϕ of any q-cycle belonging to $H \cong S_n$ is a q-cycle.*

Step 4. *Let q be a number satisfying condition* (ii′) *of step* 3. *Then*

(iii′) *for any q-cycle $c \in H$ we have*

$$\mathrm{Supp}_{S_{n+m}}(\phi(G)) \cap \mathrm{Supp}_{S_{n+m}}(\phi(c)) = \varnothing;$$

(iv′)

$$\left|\bigcup_{\text{q-cycles } \sigma \in H} \mathrm{Supp}_{S_{n+m}}(\phi(\sigma))\right| = n;$$

(v′)

$$|\,\mathrm{Supp}_{S_{n+m}}(\phi(G))| = m.$$

Proof of step 1. Let p be a positive integer, let c be any p-cycle in G, and let $\sigma = \phi(c)$ be the image under ϕ in of the p-cycle c. Let $Z_{S_{n+m}}(\sigma)$ denote the centralizer of σ, i.e., $Z_{S_{n+m}}(\sigma) = \{\tau \in S_{n+m},\ \tau\sigma = \sigma\tau\}$. Since ϕ is an embedding $S_n \times S_m \to S_{n+m}$, we have

$$|Z_{S_n \times S_m}(c)| \leq |Z_{S_{n+m}}(\sigma)|. \tag{2}$$

We treat the special case $m = 4$, $n = 3$ first. Choose $p = 4$, let c be any 4-cycle in G, and let $\sigma = \phi(c)$. Then, in S_7, σ is either a 4-cycle or a product of a 4-cycle θ and a transposition τ, with $\mathrm{Supp}_{S_7}(\theta) \cap \mathrm{Supp}_{S_7}(\tau) = \varnothing$. We want to exclude the latter case. Easy computations show that $|Z_{S_4 \times S_3}(c)| = 4 \cdot 3! = 24$, whereas $|Z_{S_7}(\theta\sigma)| = 4 \cdot 2 = 8$ which contradicts inequality (2).

In the general case, let p be any prime ≥ 3, let c be any p-cycle in G, and let $\sigma = \sigma_1 \ldots \sigma_l$ be the image under ϕ in S_{n+m} of the p-cycle c, written as a product of disjoint nontrivial cycles σ_i. Since p is prime, each σ_i is a p-cycle. It is easily seen that

$$|Z_{S_n \times S_m}(c)| = n!(m-p)!p$$

and

$$|Z_{S_{n+m}}(\sigma)| = p^l l!(n+m-lp)!.$$

Thus, dividing both sides by p, we see that inequality (2) is equivalent to

$$n!(m-p)! \leq p^{l-1} l!(n+m-lp)! \tag{3}$$

By Lemma 1, for any fixed $m \geq 3$, there exists a prime number $p \geq 3$ satisfying condition (i). We must show that $l = 1$. Suppose, to the contrary, that $l \geq 2$. Then

$lp \leq n+m$ implies $l \leq \frac{n+m}{p}$. Also, $n \leq m$ by our assumption, and $\frac{m}{p} \leq \alpha$ by property (i). Thus,

$$2 \leq l \leq \frac{2m}{p} \leq 2\alpha = \frac{8}{3} < 3.$$

Since l is an integer, the inequality obtained implies that $l = 2$. Now, the desired contradiction (to inequality (3)) follows from the lemma below.

Lemma 2. *Let* (n, m) *be a pair of integers such that* $3 \leq n \leq m$ *and* $4 \leq m$. *For any* $p \geq 3$ *satisfying condition* (i) *we have*

$$2p(n+m-2p)! < n!(m-p)!. \tag{4}$$

Proof. We distinguish the following two cases: $2p = n+m$ and $2p < n+m$.

Case (i): $2p = n+m$. The inequality is easily verified for $4 \leq m \leq 7$. For $m \geq 8$ we have $2m < [\frac{m}{2}]!$. Also,

$$n = 2p - m \geq 2\frac{m}{\alpha} - m = \left(\frac{2}{\alpha} - 1\right)m = \frac{m}{2} \geq \left[\frac{m}{2}\right].$$

Thus, for $m \geq 8$ we have

$$2p \leq 2m < \left[\frac{m}{2}\right]! \leq n! \leq n!(m-p)!,$$

which is the required inequality.

Case (ii): $2p < n+m$. The inequality is fulfilled for $m = 4,\ 5$. For $m \geq 6$, we rewrite the inequality we want to prove as follows:

$$2p < \binom{n}{2p-m}(m-p)!(2p-m)!\,.$$

Since $m > 0$ and $m/\alpha = 3m/4 \leq p$, we have $m < 3m/2 \leq 2p$, whence $2p - m > 0$. Now, $2p - m > 0$ and $2p - m < n$ imply that $\binom{n}{2p-m} \geq n$.

Also $2p - m \geq 2(m/\alpha) - m = (\frac{2}{\alpha} - 1)m \geq [(\frac{2}{\alpha} - 1)m] = [m/2]$. Therefore, it suffices to show that $2p < n(m-p)![m/2]!$. By assumption, $n > 2p - m$. Since $(m-p)! \geq 1$, it suffices to show that $2p < \frac{1}{2}m[m/2]!$ or that $4p < m[m/2]!$. But it is easy to check that $4p \leq 4m < m[m/2]!$, because $4 < [m/2]!$ for any $m \geq 6$. This completes the proof of Lemma 2. •

Proof of step 2. We set $E = \bigcup_{\text{p-cycles } \sigma \in G} \text{supp}_{S_{n+m}}(\phi(\sigma))$. Using the injectivity of ϕ and counting the p-cycles, we easily verify that $|E| \geq m$. Next, since G and H commute elementwise, the images of the p-cycles $c \in G$ commute in S_{n+m} with the images of the transpositions $\tau \in H$. The image of any p-cycle in G is a p-cycle in S_{n+m}, and the image of a transposition in H is a product of transpositions (with disjoint supports) in S_{n+m}. Let $\sigma = \phi(c) = (a_1 \ldots a_p)$ be the image of a p-cycle $c \in G$ (where $a_i \in \{1, \ldots, n+m\}$), and let $\rho = \phi(\tau)$ be the image of a transposition $\tau \in H$. Then

$$\begin{aligned}(a_1 \ldots a_p) = \sigma = \phi(c) &= \phi(\tau)\phi(c)\phi(\tau)^{-1} = \rho\sigma\rho^{-1} \\ &= \rho(a_1 \ldots a_p)\rho^{-1} \\ &= (\rho(a_1) \ldots \rho(a_p)).\end{aligned}$$

This implies that $\rho|_{\{a_1, \ldots, a_p\}}$ is a cyclic permutation on the p-uple $(a_1, \ldots, a_p)$, which is of order 1 or p. Since ρ is of order 2 and $p > 2$, this permutation must be of order 1, i.e., $\rho|_{\{a_1, \ldots, a_p\}} = \text{id}_{\{a_1, \ldots, a_p\}}$. Thus,

$$\text{Supp}_{S_{n+m}}(\rho) \cap \text{Supp}_{S_{n+m}}(\sigma) = \varnothing.$$

Since the images $\rho = \phi(\tau)$ of the transpositions $\tau \in H$ generate $\phi(H)$, the above relation implies property (iii). Since ϕ is an embedding, $\phi(H)$ is isomorphic to S_n; in particular, $|\text{Supp}_{S_{n+m}}(\phi(H))| \geq n$. Consequently,

$$|E| \leq n + m - |\text{Supp}_{S_{n+m}}(\phi(H))| \leq n + m - n = m.$$

As a result, we have $|E| = m$, which proves property (iv). Property (iii) shows that

$$|\text{Supp}_{S_{n+m}}(\phi(H))| \leq \text{Supp}_{S_{n+m}}(S_{n+m}) - |E| = n + m - m = n.$$

So, finally, $|\text{Supp}_{S_{n+m}}(\phi(H))| = n$. This proves property (v). •

Proof of step 3. Applying Lemma 1 to the integer $n \geq 3$, we obtain a prime number q such that $n/\alpha \leq q \leq n$ with $\alpha = \frac{4}{3}$. Now we pick any q-cycle $c \in H$. Then $\phi(c)$ is a product of, say, l distinct q-cycles with pairwise disjoint supports. Thus,

$lq \leq |\mathrm{Supp}_{S_{n+m}}(\phi(H))| = n$ by property (v), which implies that $l \leq \frac{n}{q} \leq \alpha = \frac{4}{3} < 2$. So, $l = 1$ and $\phi(c)$ is a q-cycle in S_{n+m}. •

Proof of step 4. It suffices to exchange the roles of p and q and of G and H in the proof of step 2. •

End of proof. Properties (iv) and (v′) show that

$$\bigcup_{\text{p-cycles } \sigma \in G} \mathrm{Supp}_{S_{n+m}}(\phi(\sigma)) = \mathrm{Supp}_{S_{n+m}}(\phi(G)).$$

Using property (iii), we obtain

$$\mathrm{Supp}_{S_{n+m}}(\phi(G)) \cap \mathrm{Supp}_{S_{n+m}}(\phi(H)) = \varnothing,$$

which implies that the embedding ϕ is standard. •

Proof of Lemma 1. We start with the following Chebyshev-like double inequality [RS]:

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) < \pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right) \quad \text{for all } x \geq 52, \qquad (5)$$

where $\pi(x) = |\{p \text{ prime}, p \leq x\}|$. Using (5), it can be shown that the number of primes between x and $\frac{4}{3}x$ (this number is equal to $\pi(\frac{4}{3}x) - \pi(x)$) is greater than 1 for all $x \geq 95$. Therefore, Lemma 1 is true for any $N \geq 190$. For $3 \leq N \leq 190$, the result is checked by inspection. •

References

[BM1] Borobia A., Mazurovskiĭ V. F., *On diagrams of configurations of 7 skew lines of* $\mathbb{R}^3$, Adv. Soviet Math., Amer. Math. Soc., Providence, RI (в печати).

[BM2] Borobia A., Mazurovskiĭ V. F., *Nonsingular configurations of 7 lines of the real projective 3-space*, Knot Theory and its Ramifications (в печати).

[Bu] Burnside W., *Theory of groups of finite order*, Dover Publications, Inc., New York, 1955 (Chapter XI).

[CP] Crapo H., Penne R., *Chirality and the isotopy classification of skew lines in projective three-space*, Adv. Math. 103 (1994), no. 1, 1–106.

[D] Deloup F., *Entrelacs de droites*, Mémoire de 1ère année de magistère, Strasbourg, 1992.

[Ha] Hartley R., *The Conway potential function for links*, Comment. Math. Helv. 58 (1983), 365–378.

[KM] Khashin S. I., Mazurovskiĭ V. F., *Stable equivalence of real projective configurations*, The Memorial Proceedings «Hilbert's 16-th Problem and Related Topics» dedicated to Prof.

D. A. Gudkov (V. I. Kharlamov, G. M. Polotovskiĭ, O. Ya. Viro, eds.), Adv. Soviet Math., Amer. Math. Soc., Providence, RI (в печати).

[MP] Мазуровский В. Ф., Павлов Н. Б., *Классификация упорядоченных неособых конфигураций $\leqslant 7$ прямых пространства $\mathbb{R}P^3$ с точностью до жестких изотопий*, Зап. науч. семин. ПОМИ **231** (1995), 269–285.

[Ma1] Мазуровский В. Ф., *Конфигурации шести скрещивающихся прямых*, Зап. науч. семин. ЛОМИ **167** (1988), 121–134.

[Ma2] Мазуровский В. Ф., *Неособые конфигурации k-мерных подпространств $(2k+1)$-мерного вещественного проективного пространства*, Вестн. Ленингр. ун-та. Сер. мат. мех. астроном. **1990**, вып. 3, 21–26.

[RS] Rosser J., Schoenfeld L., *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

[SW] Sumners D. W., Woods J. M., *The monodromy of reducible plane curves*, Invent. Math. **40** (1977), no. 2, 107–141.

[Tu] Тураев В. Г., *Кручение Райдемайстера в теории узлов*, Успехи мат. наук. **41** (1986), № 1, 97–147.

[VD] Виро О. Я., Дроботухина Ю. В., *Конфигурации скрещивающихся прямых*, Алгебра и анализ **1** (1989), № 4, 222–246.

[Vi] Виро О. Я., *Топологические задачи о прямых и точках трехмерного пространства*, Докл. АН СССР **284** (1985), № 5, 1049–1052.

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