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ПРЕОБРАЗОВАНИЕ ЛАПЛАСА ДЛЯ L -ФУНКЦИЙ ДИРИХЛЕ

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Аннотация

Пусть χ характер Дирихле по модулю q . L - функция Дирихле $L(s, \chi)$ в полуплоскости $\sigma > 1$ определяемая рядом

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

и мероморфно продолжается на всю комплексную плоскость. Если χ -неглавный характер, то функция $L(s, \chi)$ является целой. В случае главного характера функция $L(s, \chi)$ имеет единственный простой полюс в точке $s = 1$. L - функции Дирихле играют важную роль при исследовании распределения простых чисел в арифметических прогрессиях, поэтому их аналитические свойства заслуживают пристального внимания. В приложениях часто нужны моменты L - функций Дирихле, асимптотическое поведение которых очень сложное. При исследовании моментов применяются различные методы, один из которых основан на применении преобразований Меллина. В свою очередь, преобразования Меллина используют преобразования Лапласа. В статье получены явные формулы для преобразования Лапласа функции $|L(s, \chi)|^2$ в критической полосе. Эти формулы расширяют формулы, доказанные в [3] на критической прямой $\sigma = \frac{1}{2}$.

Ключевые слова: L -функция Дирихле, преобразование Лапласа, преобразование Меллина, дзета-функция Римана.

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THE LAPLACE TRANSFORM OF DIRICHLET L -FUNCTIONS

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Abstract

Let χ be a Dirichlet character modulo q . The Dirichlet L -function $L(s, \chi)$ is defined in the half-plane $\sigma > 1$ by the series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and has a meromorphic continuation to the whole complex plane. If χ is a non-principal character, then the function $L(s, \chi)$ is entire one. In the case of the principal character, the function $L(s, \chi)$ has unique simple pole at the point $s = 1$. Dirichlet L - functions play an important role in the investigations of the distribution of prime numbers in arithmetical progressions, therefore, their analytic properties deserve a constant attention. In applications, often the moments of Dirichlet L -functions are used, whose asymptotic behaviour is very complicated. For investigation of moments, various methods are applied, one of them is based on the application of Mellin transforms. On the other hand, Mellin transforms use Laplace transforms. In the paper, the formulae for the Laplace transform of the function $|L(s, \chi)|^2$ in the critical strip are obtained. They extend the formulae obtained in [3] on the critical line $\sigma = \frac{1}{2}$.

Keywords: Dirichlet L -function, Laplace transform, Mellin transform, Riemann zeta-function.

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1. Introduction

Let $s = \sigma + it$ be a complex variable. The Laplace transform $\mathfrak{L}(s, f)$ of a function f is defined by

$$\mathfrak{L}(s, f) = \int_0^\infty f(x)e^{-sx} dx$$

provided that the integral exists for $\sigma > \sigma_0$ with some $\sigma_0 \in \mathbb{R}$. It is well known that Laplace transforms are very useful integral transforms having applications in various fields of mathematics and in practice. Analytic number theory is not an exception, here Laplace transforms are applied for the investigation of mean values (moments) of zeta and L -functions. The classical example given in the monograph [15] says that if $f(x) \geq 0$ for $x \in (0, \infty)$, and, for some $m \geq 0$,

$$\int_0^\infty f(x)e^{-\delta x} dx \sim \frac{1}{\delta} \log^m \frac{1}{\delta}$$

as $\delta \rightarrow 0$, then

$$\int_0^T f(x) dx \sim T \log^m T$$

as $T \rightarrow \infty$. This has been applied for the moments of the Riemann zeta-function $\zeta(s)$

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \tag{1}$$

with $k = 1$ and $k = 2$ in [15] and [1]. We remind that the function $\zeta(s)$ is defined, for $\sigma > 1$, by the series

$$\zeta(s) = \sum_{m=1}^\infty \frac{1}{m^s},$$

and by analytic continuation elsewhere. We observe that more precise formulae for moments (1) require those for $\mathfrak{L}(s, |\zeta|^{2k})$. In [9], applications of Laplace transforms for mean values of more general Dirichlet series are given. Let $d(m)$ be the number of divisors of m , γ denote the Euler constant, and let

$$\Delta(x) = \sum'_{m \leq x} d(m) - x(\log x + 2\gamma - 1) - \frac{1}{4},$$

where "'' means that the last term in the sum is to be halved if x is an integer. In [6], an asymptotic formula for the Laplace transform

$$\int_0^\infty \Delta^2(x)e^{-\frac{x}{T}} dx$$

was obtained. In [5], the Laplace transform was applied to give a simple proof for the classical Voronoi identity

$$\Delta(x) = -\frac{2}{\pi} \sum_{m=1}^\infty d(m) \left(\frac{x}{m}\right)^{1/2} \left(K_1(4\pi\sqrt{xm}) + \frac{\pi}{2} Y_1(4\pi\sqrt{xm})\right),$$

where K_1 and Y_1 are the Bessel functions. A very good survey on applications of the Laplace transforms in the theory of the Riemann zeta-function is given in [7].

We remind one more formula used in [1] and [12] for the investigation of the mean square of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$, namely,

$$\mathfrak{L}(s, |\zeta|^2) = ie^{is/2} \left(\gamma - \log 2\pi - \left(\frac{\pi}{2} - s\right) i\right) + 2\pi e^{-is/2} \sum_{m=1}^\infty d(m)e^{-2\pi ime^{-is}} + \lambda(s),$$

where the function $\lambda(s)$ is analytic in the strip $|\sigma| < \pi$. Moreover, in any fixed strip $|\sigma| \leq \theta$ with $0 < \theta < \pi$, the estimate

$$\lambda(s) = O((1 + |s|)^{-1})$$

is true. In [10], the above formula was extended to the critical strip, i.e., the formula for $\int_0^\infty |\zeta(\varrho + ix)|^2 e^{-sx} dx$ with a fixed $\frac{1}{2} < \varrho < 1$ has been obtained.

Now let χ be a Dirichlet character modulo q , and let $L(s, \chi)$ denote the corresponding Dirichlet L -function defined, for $\sigma > 1$, by the series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

If χ is a non-principal character, then $L(s, \chi)$ is analytically continuable to an entire function, while if χ_0 the principal character modulo q , then

$$L(s, \chi_0) = \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \zeta(s),$$

where p denotes a prime number, i.e. $L(s, \chi_0)$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right).$$

Analytic theory of Dirichlet L -functions can be found in [4], [8] and [11]. In [3], the formulae for the Laplace transform

$$\mathfrak{L}(s, |L(\chi)|^2) = \int_0^\infty \left|L\left(\frac{1}{2} + ix, \chi\right)\right|^2 e^{-sx} dx.$$

were obtained. This note is a continuation of [3], and is devoted to the Laplace transform

$$\mathfrak{L}_\varrho(s, |L(\chi)|^2) = \int_0^\infty |L(\varrho + ix, \chi)|^2 e^{-sx} dx,$$

where $\varrho, \frac{1}{2} < \varrho < 1$, is a fixed number.

For the statement of the results, we need some notation. Denote by $G(\chi)$ the Gauss sum, i.e.,

$$G(\chi) = \sum_{l=1}^q \chi(l) e^{2\pi il/q}.$$

Let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

$$E(\chi) = \begin{cases} \epsilon(\chi) & \text{if } a = 0, \\ \epsilon_1(\chi) & \text{if } a = 1, \end{cases}$$

where

$$\epsilon(\chi) = \frac{G(\chi)}{\sqrt{q}}, \quad \epsilon_1(\chi) = -\frac{G(\chi)}{\sqrt{q}}.$$

As usual, denote by $\Gamma(s)$ the Euler gamma-function, and by $\mu(m)$ the Möbius function. Moreover,

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha, \alpha \in \mathbb{C},$$

is the generalized divisor function.

ТЕОРЕМА 1. *Let $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$, $\varrho, \frac{1}{2} < \varrho < 1$, be a fixed number, and χ be a primitive character modulo $q > 1$. Then*

$$\mathfrak{L}_\varrho(s, |L(\chi)|^2) = \frac{2\pi i^a e^{-is(1-\varrho)}}{E(\chi)\sqrt{q}} \sum_{m=1}^{\infty} \frac{\chi(\bar{m})\sigma_{2\varrho-1}(m)}{m^{2\varrho-1}} \exp\left\{-\frac{2\pi im}{q} e^{-is}\right\} + \lambda_\varrho(s, \chi),$$

where the function λ_ϱ is analytic in the strip $\{s \in \mathbb{C} : |\sigma| < \pi\}$, and, for $|\sigma| \leq \theta, 0 < \theta < \pi$, the estimate

$$\lambda_\varrho(s, \chi) = O((1 + |s|)^{-1})$$

is valid.

ТЕОРЕМА 2. *Let $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$, $\varrho, \frac{1}{2} < \varrho < 1$, be a fixed number, and χ_0 be a principal character modulo $q > 1$. Then*

$$\begin{aligned} \mathfrak{L}_\varrho(s, |L(\chi_0)|^2) &= (2\pi i)^{2\varrho-1} i \Gamma(2-2\varrho) \zeta(2-2\varrho) e^{is(1-\varrho)} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{nm^{2\varrho-1}} \\ &+ i e^{is\varrho} \zeta(2\varrho) \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{n^{2\varrho}} + 2\pi e^{-is(1-\varrho)} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{mn^{2\varrho-1}} \sum_{k=1}^{\infty} \frac{\sigma_{2\varrho-1}(k)}{k^{2\varrho-1}} \\ &\exp\left\{-2\pi i e^{-is} \frac{nk}{m}\right\} + \lambda_\varrho(s, \chi_0). \end{aligned}$$

where the function $\lambda_\varrho(s, \chi_0)$ has the same properties as $\lambda_\varrho(s, \chi)$ in Theorem 1.

Note that if $q = 1$, then the formula of Theorem 2 implies that of [10].

2. Lemmas

We remind the following results on the functions $L(s, \chi)$ and $\zeta(s)$.

LEMMA 1. *If χ is a primitive character modulo q , then*

$$L(1-s, \bar{\chi}) = E^{-1}(\chi) 2^{1-s} \pi^{-s} q^{s-\frac{1}{2}} \Gamma(s) \cos\left(\frac{\pi a}{2} - \frac{\pi s}{2}\right) L(s, \chi).$$

For the proof, see [13].

LEMMA 2. *The function $\zeta(s)$ satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s).$$

Proof of lemma is given, for example, in [15].

LEMMA 3. *For any character χ modulo q , the estimate*

$$L(s, \chi) = O((q|t|)^c), \sigma \geq 1 - c,$$

where $0 < c \leq \frac{1}{2}$, and $|t| \geq 2$, is valid.

The lemma is Theorem 5.4 from [13].

LEMMA 4. *If f is a multiplicative function, then*

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p)).$$

The lemma is Theorem 2.18 in [2].

Now we recall the Mellin formula.

LEMMA 5. *Suppose that $c > 0$. Then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)b^{-s} ds = e^{-b}.$$

Proof of the formula can be found in [14].

LEMMA 6. *For $\sigma > \max\{1, \Re(\alpha + 1)\}$,*

$$L(s, \chi)L(s - \alpha, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{\alpha}(m)}{m^s},$$

and

$$\zeta(s)\zeta(s - \alpha) = \sum_{m=1}^{\infty} \frac{\sigma_{\alpha}(m)}{m^s}.$$

PROOF. For $\sigma > \max\{1, \Re(\alpha + 1)\}$, we have that

$$\begin{aligned} L(s, \chi)L(s - \alpha, \chi) &= \sum_{k=1}^{\infty} \frac{\chi(k)k^{\alpha}}{k^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{d|m} \chi(d)d^{\alpha} \chi\left(\frac{m}{d}\right) \\ &= \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{d|m} d^{\alpha} = \sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{\alpha}(m)}{m^s}. \end{aligned}$$

The case of the Riemann zeta-function can be found in [15]. □

3. Proof of Theorems

It is sufficient to prove the theorems for a slightly different integrals. ДОКАЗАТЕЛЬСТВО. [Proof of Theorem 1] Consider the function

$$\begin{aligned} \lambda_{\varrho}(s, \chi) &= \int_0^{\infty} |L(\varrho + ix, \chi)|^2 e^{-sx} dx \\ &\quad - \frac{e^{-is(1-\varrho)}}{2i^{1-a}} \int_{\varrho-i\infty}^{\varrho+i\infty} \frac{L(z, \chi)L(2\varrho - z, \bar{\chi})e^{-i(1-2\varrho+z)(\frac{\pi}{2}-s)}}{\cos\left(\frac{\pi a}{2} - \frac{\pi(1-2\varrho+z)}{2}\right)} dz. \end{aligned} \tag{2}$$

Suppose $a = 0$, then we find that

$$\begin{aligned}
 \lambda_\varrho(s, \chi) &= \int_0^\infty |L(\varrho + ix, \chi)|^2 e^{-sx} dx \\
 &- e^{-is(1-\varrho)} \int_{-\infty}^\infty \frac{L(\varrho + ix, \chi)L(\varrho - ix, \bar{\chi}) \exp\{-\frac{\pi i}{2}(1-\varrho) + \frac{\pi x}{2} + is(1-\varrho) - sx\}}{\exp\{\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{2}(1-\varrho) + \frac{\pi x}{2}\}} dx \\
 &= \int_0^\infty \frac{|L(\varrho + ix, \chi)|^2 e^{-sx} \exp\{\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{2}(1-\varrho) + \frac{\pi x}{2}\}} dx \\
 &- \int_0^\infty \frac{|L(\varrho + ix, \bar{\chi})|^2 e^{sx} \exp\{-\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{2}(1-\varrho) + \frac{\pi x}{2}\} + \exp\{-\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\}} dx. \tag{3}
 \end{aligned}$$

If $a = 1$, similarly as above, we find that

$$\begin{aligned}
 \lambda_\varrho(s, \chi) &= \int_0^\infty \frac{|L(\varrho + ix, \chi)|^2 e^{-sx} \exp\{\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{2}(1-\varrho) + \frac{\pi x}{2}\}} dx \\
 &+ \int_0^\infty \frac{|L(\varrho + ix, \bar{\chi})|^2 e^{sx} \exp\{-\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{2}(1-\varrho) + \frac{\pi x}{2}\} - \exp\{-\frac{\pi i}{2}(1-\varrho) - \frac{\pi x}{2}\}} dx. \tag{4}
 \end{aligned}$$

By estimates for $L(s, \chi)$ from Lemma 3, both these integrals are uniformly convergent on compact subsets of the strip $\{s \in \mathbb{C} : |\sigma| < \pi\}$, thus, the function $\lambda_\varrho(s, \chi)$ is analytic in this region. Suppose that $|\sigma| \leq \theta$, where $0 < \theta < \pi$ is fixed. If $|s|$ is small, then the integrals in (3) and (4) are bounded. If $|s|$ is large, then integrating by parts and using the estimate from Lemma 3, we obtain that

$$\lambda_\varrho(s, \chi) = O(|s|^{-1}).$$

So we have that, for $|\sigma| \leq \theta$, $0 < \theta < \pi$,

$$\lambda_\varrho(s, \chi) = O((1 + |s|)^{-1}).$$

From (2), we deduce that

$$\mathfrak{L}_\varrho(s, |L(\chi)|^2) = \frac{e^{-is(1-\varrho)}}{2i^{1-a}} \int_{\varrho-i\infty}^{\varrho+i\infty} \frac{L(z, \chi)L(2\varrho - z, \bar{\chi})e^{-i(1-2\varrho+z)(\frac{\pi}{2}-s)}}{\cos\left(\frac{\pi a}{2} - \frac{\pi(1-2\varrho+z)}{2}\right)} dz + \lambda_\varrho(s, \chi).$$

Therefore, using Lemma 1, we find that

$$\mathfrak{L}_\varrho(s, |L(\chi)|^2) = \frac{e^{-is(1-\varrho)}}{2i^{1-a}} \int_{\varrho-i\infty}^{\varrho+i\infty} \frac{L(z, \chi)L(2\varrho - z, \bar{\chi})e^{-i(1-2\varrho+z)(\frac{\pi}{2}-s)}}{\cos\left(\frac{\pi a}{2} - \frac{\pi(1-2\varrho+z)}{2}\right)} dz + \lambda_\varrho(s, \chi)$$

$$\begin{aligned}
&= \frac{e^{-is(1-\varrho)}}{E(\chi)\sqrt{qi^{1-a}}} \int_{\varrho-i\infty}^{\varrho+i\infty} L(z, \chi)L(z-2\varrho+1, \chi)\Gamma(1-2\varrho+z) \\
&\times \exp\left\{-i(1-2\varrho+z)\left(\frac{\pi}{2}-s\right)\right\} \frac{q^{1-2\varrho+z}}{2\pi^{1-2\varrho+z}} dz + \lambda_{\varrho}(s, \chi) \\
&= \frac{e^{-is(1-\varrho)}}{E(\chi)\sqrt{qi^{1-a}}} \int_{\varrho-i\infty}^{\varrho+i\infty} L(z, \chi)L(z-2\varrho+1, \chi)\Gamma(1-2\varrho+z) \left(\frac{2\pi i}{q}e^{-is}\right)^{-1+2\varrho-z} dz \\
&+ \lambda_{\varrho}(s, \chi). \tag{5}
\end{aligned}$$

Now we move the line of integration in (5) to the right and use Lemma 5. If χ is a non-principal character, the integrand in (5) is a regular function. Therefore, by Lemma 6,

$$\begin{aligned}
&\frac{e^{-is(1-\varrho)}}{E(\chi)\sqrt{qi^{1-a}}} \int_{\varrho-i\infty}^{\varrho+i\infty} L(z, \chi)L(z-2\varrho+1, \chi)\Gamma(1-2\varrho+z) \left(\frac{2\pi i}{q}e^{-is}\right)^{-1+2\varrho-z} dz + \lambda_{\varrho}(s, \chi) \\
&= \frac{e^{-is(1-\varrho)}}{E(\chi)\sqrt{qi^{1-a}}} \int_{2-i\infty}^{2+i\infty} L(z, \chi)L(z-2\varrho+1, \chi)\Gamma(1-2\varrho+z) \left(\frac{2\pi i}{q}e^{-is}\right)^{-1+2\varrho-z} dz + \lambda_{\varrho}(s, \chi) \\
&= \frac{e^{-is(1-\varrho)}}{E(\chi)\sqrt{qi^{1-a}}} \int_{2-i\infty}^{2+i\infty} \Gamma(1-2\varrho+z) \sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{2\varrho-1}(m)}{m^z} \left(\frac{2\pi i}{q}e^{-is}\right)^{-1+2\varrho-z} dz + \lambda_{\varrho}(s, \chi) \\
&= \frac{e^{-is(1-\varrho)}}{E(\chi)\sqrt{qi^{1-a}}} \sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{2\varrho-1}(m)}{m^{2\varrho-1}} \int_{2-i\infty}^{2+i\infty} \Gamma(1-2\varrho+z) \left(\frac{2\pi im}{q}e^{-is}\right)^{-1+2\varrho-z} dz + \lambda_{\varrho}(s, \chi) \\
&= \frac{2\pi e^{-is(1-\varrho)}i^a}{E(\chi)\sqrt{q}} \sum_{m=1}^{\infty} \frac{\chi(m)\sigma_{2\varrho-1}(m)}{m^{2\varrho-1}} \exp\left\{-\frac{2\pi im}{q}e^{-is}\right\} + \lambda_{\varrho}(s, \chi).
\end{aligned}$$

□ This and (5) prove the theorem.

The proof of Theorem 2 is more complicated. ДОКАЗАТЕЛЬСТВО. [Proof of Theorem 2] Define

$$\begin{aligned}
\lambda_{\varrho}(s, \chi_0) &= \int_0^{\infty} |L(\varrho + ix, \chi_0)|^2 e^{-sx} dx \\
&- \frac{e^{-is(1-\varrho)}}{2i} \int_{\varrho-i\infty}^{\varrho+i\infty} \frac{L(z, \chi_0)L(2\varrho-z, \chi_0)e^{-i(1-2\varrho+z)(\frac{\pi}{2}-s)}}{\cos \frac{\pi}{2}(1-2\varrho+z)} dz.
\end{aligned}$$

Then

$$\begin{aligned}
\mathfrak{L}_{\varrho}(s, |L(\chi_0)|^2) &= \int_0^{\infty} |L(\varrho + ix, \chi_0)|^2 e^{-sx} dx \\
&= \frac{e^{-is(1-\varrho)}}{2i} \int_{\varrho-i\infty}^{\varrho+i\infty} \frac{L(z, \chi_0)L(2\varrho-z, \chi_0)e^{-i(1-2\varrho+z)(\frac{\pi}{2}-s)}}{\cos \frac{\pi}{2}(1-2\varrho+z)} dz + \lambda_{\varrho}(s, \chi_0). \tag{6}
\end{aligned}$$

Since

$$L(z, \chi_0) = \zeta(z) \prod_{p|q} \left(1 - \frac{1}{p^z}\right),$$

Lemma 2 implies

$$\begin{aligned} L(2\varrho - z, \chi_0) &= L(1 - (1 - 2\varrho + z), \chi_0) \\ &= 2^{2\varrho - z} \pi^{-(1-2\varrho+z)} \cos \frac{\pi}{2} (1 - 2\varrho + z) \Gamma(1 - 2\varrho + z) \zeta(1 - 2\varrho + z) \prod_{p|q} \left(1 - \frac{1}{p^{2\varrho - z}}\right). \end{aligned} \quad (7)$$

Hence, in view of (6), we find

$$\begin{aligned} &\mathfrak{L}_\varrho(s, |L(\chi_0)|^2) \\ &= \frac{e^{-is(1-\varrho)}}{i} \int_{\varrho - i\infty}^{\varrho + i\infty} (2\pi)^{-1+2\varrho-z} \zeta(z) \zeta(1 - 2\varrho + z) \Gamma(1 - 2\varrho + z) e^{-i(1-2\varrho+z)(\frac{\pi}{2}-s)} \\ &\prod_{p|q} \left(1 - \frac{1}{p^z}\right) \prod_{p|q} \left(1 - \frac{1}{p^{2\varrho-z}}\right) dz + \lambda_\varrho(s, \chi_0). \end{aligned} \quad (8)$$

Using Lemma 4, we write the products in (8) in the form

$$\prod_{p|q} \left(1 - \frac{1}{p^z}\right) \prod_{p|q} \left(1 - \frac{1}{p^{2\varrho-z}}\right) = \sum_{m|q} \sum_{n|q} \mu(m) \mu(n) \frac{1}{m^{2\varrho}} \left(\frac{m}{n}\right)^z.$$

Therefore,

$$\begin{aligned} \mathfrak{L}_\varrho(s, |L(\chi_0)|^2) &= \frac{e^{-is(1-\varrho)}}{i} \sum_{m|q} \sum_{n|q} \frac{\mu(m) \mu(n)}{mn^{2\varrho-1}} \\ &\int_{\varrho - i\infty}^{\varrho + i\infty} \left(2\pi i e^{-is \frac{n}{m}}\right)^{-1+2\varrho-z} \zeta(z) \zeta(1 - 2\varrho + z) \Gamma(1 - 2\varrho + z) dz + \lambda_\varrho(s, \chi_0). \end{aligned} \quad (9)$$

Now we move the line of integration in the last integral to the right. The integrand in (9) has simple poles at the points $z = 1$ and $z = 2\varrho$. Clearly,

$$\begin{aligned} &\operatorname{Res}_{z=1} \Gamma(1 - 2\varrho + z) \zeta(z) \zeta(1 - 2\varrho + z) \left(2\pi i e^{-is \frac{n}{m}}\right)^{-1+2\varrho-z} \\ &= \Gamma(2 - 2\varrho) \zeta(2 - 2\varrho) \left(2\pi i e^{-is \frac{n}{m}}\right)^{-2+2\varrho} \end{aligned} \quad (10)$$

and

$$\operatorname{Res}_{z=2\varrho} \Gamma(1 - 2\varrho + z) \zeta(z) \zeta(1 - 2\varrho + z) \left(2\pi i e^{-is \frac{n}{m}}\right)^{-1+2\varrho-z} = \frac{\zeta(2\varrho) e^{is} m}{2\pi i n}. \quad (11)$$

Finally, having in mind formulae (10), (11) and Lemma 6, we deduce from (9) that

$$\begin{aligned}
\mathfrak{L}_\varrho(s, |L(\chi_0)|^2) &= (2\pi i)^{2\varrho-1} \Gamma(2-2\varrho) \zeta(2-2\varrho) e^{is(1-\varrho)} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{nm^{2\varrho-1}} \\
&+ ie^{is\varrho} \zeta(2\varrho) \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{n^{2\varrho}} + \frac{e^{-is(1-\varrho)}}{i} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{mn^{2\varrho-1}} \int_{2-i\infty}^{2+i\infty} \left(2\pi i e^{-is} \frac{n}{m}\right)^{-1+2\varrho-z} \\
&\times \zeta(z) \zeta(1-2\varrho+z) \Gamma(1-2\varrho+z) dz + \lambda_\varrho(s, \chi_0) \\
&= (2\pi i)^{2\varrho-1} \Gamma(2-2\varrho) \zeta(2-2\varrho) e^{is(1-\varrho)} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{nm^{2\varrho-1}} \\
&+ ie^{is\varrho} \zeta(2\varrho) \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{n^{2\varrho}} + \frac{e^{-is(1-\varrho)}}{i} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{mn^{2\varrho-1}} \sum_{k=1}^{\infty} \frac{\sigma_{2\varrho-1}(k)}{k^{2\varrho-1}} \\
&\times \int_{2-i\infty}^{2+i\infty} \Gamma(1-2\varrho+z) \left(2\pi i e^{-is} \frac{n}{m} k\right)^{-1+2\varrho-z} dz + \lambda_\varrho(s, \chi_0) \\
&= (2\pi i)^{2\varrho-1} \Gamma(2-2\varrho) \zeta(2-2\varrho) e^{is(1-\varrho)} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{nm^{2\varrho-1}} \\
&+ ie^{is\varrho} \zeta(2\varrho) \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{n^{2\varrho}} + \frac{e^{-is(1-\varrho)}}{i} \sum_{m|q} \sum_{n|q} \frac{\mu(m)\mu(n)}{mn^{2\varrho-1}} \sum_{k=1}^{\infty} \frac{\sigma_{2\varrho-1}(k)}{k^{2\varrho-1}} \\
&\times \exp\left\{-2\pi i e^{-is} \frac{nk}{m}\right\} + \lambda_\varrho(s, \chi_0).
\end{aligned}$$

□ The theorem is proved.

4. Conclusions

In the paper, it is obtained that the Laplace transform

$$\int_0^\infty |L(\varrho + ix, \chi)|^2 e^{-sx} dx$$

with a fixed ρ , $\frac{1}{2} < \rho < 1$, can be expressed by elementary functions including the generalized divisor function

$$\sum_{d|m} d^{2\rho-1}.$$

The formulae obtained have a different form for a primitive character and the principal character, and can be applied for the investigation of Mellin transforms of $|L(\varrho + ix, \chi)|^2$.

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

1. Atkinson A.A., The mean value of the zeta-function on the critical line // Quart. J. Math. Oxford. 1939. Vol. 10. P. 122-128.
2. Apostol T.M., Introduction to Analytic Number Theory, Springer, Berlin, 1976.
3. Balčiūnas A., Laurinčikas A. The Laplace transform of Dirichlet L-functions // Nonlinear Anal. Model. Control. 2012. Vol. 17. P.127-138.

4. Воронин А.А., Карацуба А.А., Дзета- функция Римана. Москва: Физматлит, 1994.
5. Ivič A., The Voronoi identity via the Laplace transform // Ramanujan J. 1988 Vol.2. P.39-45.
6. Ivič A., The Laplace transform of the square in the circle an divisor problems // Studia Sci. Math-Hung.1996. Vol.32. P.181-205.
7. Ivič A., The Laplace and Mellin transforms of powers of the Riemann zeta-function // Int. J. Math. Anal.2006. Vol.1-2 P.113-140.
8. Iwaniec H., Kowalski E., Analytic number theory Amer. Math. Soc., Colloq. Publ. Vol.53, 2004.
9. Jutila M., Mean values of Dirichlet series via Laplace transform, in Analytic number theory // London Math. Soc. Lecture Note, Cambridge Univ. Press, Cambridge, 1997. Vol. 247 P. 169-207.
10. Kačinskaitė R., Laurinčikas A., The Laplace transform of the Riemann zeta-function in the critical strip // Integral Transf. Spec. Funct. 2009. Vol.20 P. 643-648.
11. Карацуба А.А., Основы аналитической теории чисел. Москва: Наука 1983.
12. Lukkarinen M., The Mellin transform of the square of Riemann's zeta-function and Atkinson's formula Ann. Acad. Sci. Fenn. Math. Diss., Suomalainen Tiedeakatemia, Helsinki, 2005. Vol.140.
13. Prachar K., Primzahlverteilung. Göttingen, Heidelberg, Berlin: Springer-Verlag, 1957.
14. Titchmarsh E.C., Theory of Functions. Oxford University Press, Oxford, 1939.
15. Titchmarsh E.C., The Theory of the Riemann Zeta- Function 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.

REFERENCES

1. Atkinson A.A., The mean value of the zeta-function on the critical line // Quart. J. Math. Oxford. 1939. Vol. 10.P. 122-128.
2. Apostol T.M., Introduction to Analytic Number Theory, Springer, Berlin, 1976.
3. Balčiūnas A., Laurinčikas A. The Laplace transform of Dirichlet L-functions // Nonlinear Anal. Model. Control. 2012. Vol. 17. P.127-138.
4. Воронин А.А., Карацуба А.А., Дзета- функция Римана. Москва: Физматлит, 1994.
5. Ivič A., The Voronoi identity via the Laplace transform // Ramanujan J. 1988 Vol.2. P.39-45.
6. Ivič A., The Laplace transform of the square in the circle an divisor problems // Studia Sci. Math-Hung.1996. Vol.32. P.181-205.
7. Ivič A., The Laplace and Mellin transforms of powers of the Riemann zeta-function // Int. J. Math. Anal.2006. Vol.1-2 P.113-140.
8. Iwaniec H., Kowalski E., Analytic number theory Amer. Math. Soc., Colloq. Publ. Vol.53, 2004.
9. Jutila M., Mean values of Dirichlet series via Laplace transform, in Analytic number theory // London Math. Soc. Lecture Note, Cambridge Univ. Press, Cambridge, 1997. Vol. 247 P. 169-207.

10. Kačinskaitė R., Laurinčikas A., The Laplace transform of the Riemann zeta-function in the critical strip // *Integral Transf. Spec. Funct.* 2009. Vol.20 P. 643-648.
11. Карацуба А.А., Основы аналитической теории чисел. Москва: Наука 1983.
12. Lukkarinen M., The Mellin transform of the square of Riemann's zeta-function and Atkinson's formula *Ann. Acad. Sci. Fenn. Math. Diss., Suomalainen Tiedeakatemia, Helsinki, 2005. Vol.140.*
13. Prachar K., *Primzahlverteilung.* Göttingen, Heidelberg, Berlin: Springer-Verlag, 1957.
14. Titchmarsh E.C., *Theory of Functions.* Oxford University Press, Oxford, 1939.
15. Titchmarsh E.C., *The Theory of the Riemann Zeta- Function* 2nd ed., revised by D. R. Heath-Brown, Clarendon Press, Oxford, 1986.

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