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SPECTRAL AND SCATTERING THEORY BY THE CONJUGATE OPERATOR METHOD

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1. Introduction

Our purpose in this work is to study the spectral properties and the scattering theory of selfadjoint operators of the form $H = h(P) + V$ (where $P = -i\nabla$ is the momentum operator) using the variant of Mourre's method that we developed in our recent Note [BG 1] (detailed proofs of the results of this Note will be presented elsewhere). Here we would like to show the power of our abstract theory when it is applied to hamiltonians of the preceding form.

We have decided not to consider the most general hamiltonians of the form $h(P) + V$ to which our methods apply and this essentially for pedagogical reasons: this allows us to reduce to a reasonable amount the supplementary technical tools (see [BG 2] for a more general treatment). However, our assumptions on the function h largely cover all hypoelliptic polynomials. On the other hand, the perturbation V has a very general character. It is locally quite singular, because the sum $h(P) + V$ is assumed to exist only in form-sense. Since we consider only two-body situations here, V will be assumed form-compact with respect to $H_0 = h(P)$. Such a case has also been studied in [BMP] when H_0 is the laplacian, but their assumptions on V are much stronger than ours and the limiting absorption principle which they prove holds in a space much larger than here.

Let $\mathcal{G}$ be the form domain of H_0 (i.e. $\mathcal{G} = D(|H_0|^{1/2})$ equipped with the graph-norm), $\mathcal{G}^\star$ its adjoint space and identify $\mathcal{G} \subset \mathrm{H} \subset \mathcal{G}^\star$ as usual; here $\mathrm{H} = L^2(\mathbb{R}^n)$ and $H_0 = h(P)$ with h, let us say, a hypoelliptic polynomial (in fact a much larger class is considered, see Section 8). Let V be a symmetric, compact operator $\mathcal{G} \to \mathcal{G}^\star$ and H the selfadjoint operator defined by the sum $H_0 + V$ in the form-sense (see [RS]). We assume that $V = V_S + V_L$ where V_S, V_L are symmetric operators $\mathcal{G} \to \mathcal{G}^\star$. The operator V_S is the short-range part and the only condition we impose on it is the existence of a function $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta(x) > 0$ in some region $0 < a < |x| < b < \infty$ and $\eta(x) = 0$ otherwise such that

$$\int_1^\infty \|\eta(\frac{Q}{r})V_S\|_{B(\mathcal{G},\mathcal{G}^\star)}\, dr < \infty.$$

This allows both strong local singularities and a very slow short-range type decay at infinity. Let $\xi \in C^\infty(\mathbb{R}^n)$ be such that $\xi(x) = 0$ near zero and $\xi(x) = 1$ near infinity.

It is trivial to prove that the preceding assumption on V_S is equivalent to an Enss-type condition

$$\int_1^\infty \|\xi(\frac{Q}{r})V_S\|_{B(\mathcal{G},\mathcal{G}^\star)}\,dr < \infty.$$

The main point of our abstract results in [BG 1] is that we can treat long-range perturbations V_L of a very general nature. We think that it would be quite difficult to extend Enss'geometric analysis such as to cover our results (see [P] for an application of Enss'method to long-range perturbations of simply characteristic operators and [A] for an application of Mourre's method in a similar context). Here, V_L is any symmetric operator $\mathcal{G} \to \mathcal{G}^\star$ such that the commutators $[Q_j, V_L]$ and $[P_j, V_L]$ belong to $B(\mathcal{G},\mathcal{G}^\star)$ also ($j = 1, \ldots, n$; Q_j is the position observable) and

$$\sum_{j=1}^n \int_1^\infty \{\|\eta(\frac{Q}{r})[Q_j, V_L]\|_{B(\mathcal{G},\mathcal{G}^\star)} + \|\,|Q|\eta(\frac{Q}{r})[P_j, V_L]\,\|_{B(\mathcal{G},\mathcal{G}^\star)}\}\,\frac{dr}{r} < \infty.$$

Observe that V_L is not required to be a local operator and the preceding assumption allows quite strong local singularities and very slow (but regular) decay at infinity.

Under these assumptions we can make a complete spectral analysis of H: it has no singularly continuous spectrum; its eigenvalues which are not critical values of the function h are of finite multiplicity and can accumulate only at critical values. Moreover, and this is much more important, we can prove a strong form of the limiting absorption principle. Namely, let us define two Banach spaces $\mathcal{G}^\star_{1/2,1}$ and $\mathcal{G}_{-1/2,\infty}$ as follows. Choose a function $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta(x) > 0$ if $\frac{1}{2} < |x| < 2$ and $\theta(x) = 0$ otherwise and another function $\theta_0 \in C_0^\infty(\mathbb{R}^n)$ with $\theta_0(x) = 1$ if $|x| \leqslant 1$. Then $u \in \mathcal{G}^\star_{1/2,1}$ if and only if

$$\|\theta_0(Q)u\|_{\mathcal{G}^\star} + \int_1^\infty \|r^{1/2}\theta(\frac{Q}{r})u\|_{\mathcal{G}^\star}\,\frac{dr}{r} < \infty,$$

while $u \in \mathcal{G}_{-1/2,\infty}$ if and only if

$$\|\theta_0(Q)u\|_{\mathcal{G}} + \sup_{r\geqslant 1}\|r^{-1/2}\theta(\frac{Q}{r})u\|_{\mathcal{G}} < \infty.$$

These are Fourier transforms of a general type of Besov spaces. $\mathcal{G}^\star_{1/2,1}$ is slightly smaller than the domain of $< Q >^{1/2} = (1+|Q|^2)^{1/4}$ in $\mathcal{G}^\star$ and $\mathcal{G}_{-1/2,\infty}$ is the adjoint of $\mathcal{G}^\star_{1/2,1}$. Now we can state our form of the limiting absorption principle. Let $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z \geqslant 0\}$ and $Z(h) \subset \mathbb{R}$ be the (finite) set of critical values of h. For $\operatorname{Im} z \neq 0$, we have $(H-z)^{-1} \in B(\mathcal{G}^\star, \mathcal{G})$ and obviously we have $\mathcal{G}^\star_{1/2,1} \subset \mathcal{G}^\star$ continuously and densely and $\mathcal{G} \subset \mathcal{G}_{-1/2,\infty}$. Hence we may consider the application

$$z \mapsto (H-z)^{-1} \in B(\mathcal{G}^\star_{1/2,1}, \mathcal{G}_{-1/2,\infty})$$

which is holomorphic for $\operatorname{Im} z \neq 0$; our result is that it extends to a weak* -continuous function on $\mathbb{C}_\pm \setminus (Z(h) \cup \sigma_p(H))$, where $\sigma_p(H)$ is the set of eigenvalues of H.

This fact has deep consequences in scattering theory, for example it allows us to prove existence and completeness of relative wave operators. Denote by $\overset{\circ}{\mathcal{G}}_{-1/2,\infty}$ the closure of $C_0^\infty(\mathbb{R}^n)$ in $\mathcal{G}_{-1/2,\infty}$ and let $U : \mathcal{G} \to \mathcal{G}^*$ be a symmetric operator which is short-range in the sense defined above, i.e.

$$\int_1^\infty \|\xi(\frac{Q}{r})U\|_{B(\mathcal{G},\mathcal{G}^*)}\, dr < \infty,$$

and, moreover, extends to a continuous operator $U : \overset{\circ}{\mathcal{G}}_{-1/2,\infty} \to \mathcal{G}^*_{1/2,\infty}$. In fact, if U is a local operator, then these conditions are equivalent. Let $\widetilde{H} = H + U \equiv H_0 + V_L + V_S + U$ and E_c (resp. $\widetilde{E}_c$) the projection onto the orthogonal of the subspace generated by the eigenvalues of H (resp. $\widetilde{H}$). Then the wave operators

$$W_\pm = s - \lim_{t \to \pm\infty} e^{i\widetilde{H}t} e^{-iHt} E_c$$

exist and are complete: $W_\pm^* W_\pm = \widetilde{E}_c$.

Even if H_0 is the Laplace operator, this rezult is much stronger than those of Lavine [L]. In Chapter 30 of [H], Hörmander proves a limiting absorption principle for long-range perturbations of elliptic operators (theorem 30.2.10), but it is clear that even for elliptic h our results are stronger than his (both in the local singularities allowed for V_L and V_S and in their decay at infinity). Observe that V_L and V_S do not have to be pseudo-differential operators in our theorems. Our result on existence and completeness of wave operators covers those from Chapter 14 of [H] in the hypoelliptic case. In fact, abstract theorems of our Note [BG 1] give more general results in the simply characteristic case also; this will be shown in a later publication (see the division theorem from Section 4 of [BG 1] and [A] for a different point of view).

The reader should not, however, remain with the impression that our purpose here was just to generalise known results. In fact, we want to demonstrate the power of the conjugate operator method, its unifying character, and so its potential ability to treat hamiltonians outside the scope of the other methods of spectral and scattering analysis.

Another goal of this article is to develop some abstract tools which will enable us to verify the regularity assumptions imposed on the hamiltonian in [BG 1]. We refer, more precisely, to Sections 3 and 6. The tauberian theorem which we prove in Section 3 can be used in order to give simple and explicit characterizations of the real interpolation spaces which appear in our construction, but it also allows us to state simple and general conditions on the perturbation V implying the A–regularity of H (in the sense of Sections 6 and 7). We have to acknowledge that our approach to these problems has been very much influenced by the work of H. S. Shapiro (see [S 1], [S 2]) so as probably much of the modern work on Besov spaces (see [Pe]).

Finally, let us mention that several proofs which are only sketched in this paper, will be presented in detail and in a more general context in [BG2]. We have added a purely expository section on real interpolation: this is now a very technical subject and it is rather difficult to a nonspecialist to find his way through a book devoted to it. On the other hand, it is an essential tool in all our work. We hope that the "résumé" we made in Section 2 will be of some help to the reader.

2. Summary of Real Interpolation

Real interpolation is a very powerful instrument which does not seem to be fully appreciated by the mathematical-physics community. In this section we shall recall main definitions and results of this theory. In the next one we shall outline the connection with Littlewood-Paley theory in a context (more general than usual) which is important in our applications. This will involve a tauberian theorem with many useful consequences.

If X, Y are Banach spaces, we denote by $B(X,Y)$ the Banach space of linear continuous operators $X \to Y$ and we put $B(X) = B(X,X)$. The *adjoint* $X^\star$ of X is the Banach space of continuous, *anti-linear* forms $\varphi : X \to \mathbb{C}$ equipped with the norm

$$||\varphi||_{X^\star} = \sup\{|\varphi(x)| \quad | \quad ||x||^X \leqslant 1\}.$$

The anti-duality of $(X, X^\star)$ is the sesquilinear form $< \mid >: X \times X^\star \to \mathbb{C}$ given by $< x|\varphi >= \varphi(x)$; observe that it is anti-linear in the first variable and

$$||x||_X = \sup < x|\varphi > \quad | \quad \varphi \in X^\star, \quad ||\varphi||_{X^\star} \leqslant 1, \tag{1}$$

$$||\varphi||_{X^\star} = \sup < x|\varphi > \quad | \quad x \in X, \quad ||x||_X \leqslant 1. \tag{2}$$

We always identify (linearly and isometrically, because of (1) $X \subset X^{\star\star} \equiv (X^\star)^\star$ by defining $x(\varphi) = \varphi(x)^\star \equiv < x|\varphi >^\star$. This amounts to saying that the antiduality of $(X^\star, X^{\star\star})$ satisfies $< \varphi|x >=< x|\varphi >^\star$ for $x \in X \subset X^{\star\star}$. X is reflexive if $X = X^{\star\star}$. If X is a Hilbert space with scalar product $(\cdot|\cdot)_X$ (antilinear in the first variable), then Riesz lemma says that $x \to (\cdot|x)_X$ is a linear, bijective isometry of X onto $X^\star$ called the *Riesz isomorphism.* Sometimes we identify X with $X^\star$ through this isomorphism, i.e. we take $X^\star = X$ the antiduality being $< x|y >= (x|y)_X$.

It will be convenient for us to present the interpolation theory not in the category of Banach spaces, but in that of *banachisable* spaces, or B-spaces. More precisely, we shall say that X is a B-*space* if X is a complex topological vector space such that there is a norm on X whose associated topology coincides with the initial topology on X and for which X is a Banach space. A norm with this property will be called a *compatible norm on* X. If there is a compatible norm on X which derives from a scalar product, then we say that X is a *H-space* (or *hilbertisable space*). If X, Y are B-spaces, then $X = Y$ means equality as topological vector space. We write $Y \subset X$ if Y is a vector subspace of X and the topology of Y is finner than that induced by X (we then also say that Y is continuously embedded in X). If $Y \subset X$ and the subset Y of X is dense in X, we write "$Y \subset X$ densely". Observe that $Y \subset X$ and $X \subset Y$ is equivalent to $Y = X$. From closed graph theorem it follows that if Y, X are B-spaces, $Y \subset X$ and Y is equal to X as a set, then Y coincides with X as B-space, i.e. $Y = X$. More generally, if Y_1, Y_2, X are B-spaces such that $Y_1 \subset X$, $Y_2 \subset X$ and Y_1 is included in Y_2 as a set, then $Y_1 \subset Y_2$; in particular, if Y_1, Y_2 coincide as sets, then they coincide as B-space (here, X may be any Hausdorff topological vector space). Hence, if Y is a vector subspace of X, then there is at most one B-space structure on Y such that $Y \subset X$.

If X, Y are B-spaces, then the spaces of linear continuous operators $X \to Y$ (which we continue to denote by $B(X,Y)$ is a B-space also. Similarly $X^\star$, the adjoint space.

Observe that for an H-space X there is no canonical isomorphism of X onto X^*. If X, Y are B-spaces and $Y \subset X$ *densely*, then we naturally have $X^* \subset Y^*$ (densely if Y is reflexive).

More generally, if $Y \subset X_1$ and $X_2 \subset Z$ are B-spaces and $Y \subset X_1$ densely, we have $B(X_1, X_2) \subset B(Y, Z)$. Finally, closed graph theorem shows that if $Y_1 \subset X_1$ and $Y_2 \subset X_2$ are B-spaces, $T \in B(X_1, X_2)$ and $T(Y_1) \subset Y_2$, then $T \in B(Y_1, Y_2)$.

One can develop the interpolation theory for any pair X, Y of B-spaces which are embedded in some Hausdorff topological vector space. One may find detailed presentations of this theory in [BL], [KPS] and [T]; see also [Pe] for recent references. We shall be concerned here with the case $Y \subset X$. Then for each real $\theta \in (0,1)$ and $p \in [1, \infty]$ the real interpolation provides us with a new B-space $(X, Y)_{\theta,p}$ such that the following properties hold:

(a) $Y \subset (X, Y)_{\theta,p} \subset X$;

(b) if $0 < \theta_1 < \theta_2 < 1$ then for all p_1, p_2 $\quad (X, Y)_{\theta_2,p_2} \subset (X, Y)_{\theta_1,p_1}$;

(c) if $1 \leqslant p_1 < p_2 \leqslant \infty$ then $\quad (X, Y)_{\theta,p_1} \subset (X, Y)_{\theta,p_2}$;

(d) Y is dense in $(X, Y)_{\theta,p}$ if $1 \leqslant p < \infty$.

Let $0 < \theta_1 < \theta_2 < 1$ and $p_1, p_2 \in [1, \infty]$. Then $Z_j = (X, Y)_{\theta_j,p_j}$ are B-spaces such that $Y \subset Z_2 \subset Z_1 \subset X$. In particular, one may interpolate between Z_2 and Z_1. The *reiteration theorem* says that no new spaces appear, more precisely, if $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$ then

$$((X, Y)_{\theta_1,p_1}, (X, Y)_{\theta_2,p_2})_{\theta,p} = (X, Y)_{\sigma,p} \tag{3}$$

where $\sigma = (1 - \theta)\theta_1 + \theta\theta_2$.

Let us assume now that $Y \subset X$ *densely*. Then $X^* \subset Y^*$, so we may interpolate and construct $(Y^*, X^*)_{\theta,p} \subset Y^*$. On the other hand, if $p < \infty$, we know that $Y \subset (X, Y)_{\theta,p}$ densely, hence we also have $((X, Y)_{\theta,p})^* \subset Y^*$. Again no new space is obtained by taking the adjoint: one can show that

$$((X, Y)_{\theta,p})^* = (Y^*, X^*)_{1-\theta,p'} \tag{4}$$

if $1 \leqslant p < \infty$, with $p' = \frac{p}{p-1}$. On the other hand, for $p = \infty$ let us denote $(X, Y)^\circ_{\theta,\infty}$ the closure of Y in $(X, Y)_{\theta,\infty}$. Then

$$((X, Y)^\circ_{\theta,\infty})^* = (Y^*, X^*)_{1-\theta,1} \tag{5}$$

Let us mention that, if Y is not dense in X, then denoting by $\overset{\circ}{X}$ the closure of Y in X we have $Y \subset \overset{\circ}{X}$ and

$$(X, Y)_{\theta,p} = (\overset{\circ}{X}, Y)_{\theta,p}. \tag{6}$$

One may also introduce the relative (Gagliardo) completion $\widehat{Y}$ of Y in X: it is the set of elements $x \in X$ which are limits in X of sequences bounded in Y. Then $\widehat{Y}$ has a natural B-space structure (see [KPS]), $Y \subset \widehat{Y} \subset X$ and

$$(X, Y)_{\theta,p} = (X, \widehat{Y})_{\theta,p} = (\overset{\circ}{X}, \widehat{Y})_{\theta,p}. \tag{7}$$

Let X, Y, Z be B-spaces such that $Y \subset Z \subset X$ and let $0 < \theta < 1$. Then one has $(X,Y)_{\theta,1} \subset Z$ if and only if there are compatible norms $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$ on X, Y, Z such that

$$\|y\|_Z \leqslant \|y\|_Y^{\theta} \|y\|_X^{1-\theta} \qquad (y \in Y). \tag{8}$$

In particular this is true for $Z = (X,Y)_{\theta,p}$. Furthermore, $Z \subset (X,Y)_{\theta,\infty}$ if and only if there are norms as before such that for each $z \in Z$ there is a family $\{y_\varepsilon\}_{0<\varepsilon<1}$, $y_\varepsilon \in Y$ with

$$\|y_\varepsilon\|_Y \leqslant \varepsilon^{-(1-\theta)} \|z\|_Z, \qquad \|y_\varepsilon - z\|_X \leqslant \varepsilon^{\theta} \|z\|_Z. \tag{9}$$

This last property makes the connection with approximation theory. A B-space Z with $Y \subset Z \subset X$ is called of class θ if $(X,Y)_{\theta,1} \subset Z \subset (X,Y)_{\theta,\infty}$. A more general form of the reiteration theorem is: if $Y \subset Z_j \subset X$, and Z_j is of class θ_j with $0 < \theta_1 < \theta_2 < 1$ then $Z_2 \subset Z_1$ and $(Z_1, Z_2)_{\theta,p} = (X,Y)_{\sigma,p}$ with $\sigma = (1-\theta)\theta_1 + \theta\theta_2$.

Let us consider now two pairs X_1, Y_1 and X_2, Y_2 of B-spaces such that $Y_1 \subset X_1, Y_2 \subset X_2$. Let $T \in B(X_1, X_2)$ be such that $T(Y_1) \subset Y_2$. As we know, this implies $T \in B(Y_1, Y_2)$. The interpolation theorem says that for all $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$ one also has $T(X^1_{\theta,p}) \subset X^2_{\theta,p}$, where we have denoted $X^j_{\theta,p} = (X_j, Y_j)_{\theta,p}$. This may also be written down as

$$B(X_1, X_2) \cap B(Y_1, Y_2) \subset B(X^1_{\theta,p}, X^2_{\theta,p}). \tag{10}$$

Moreover, there are compatible norms on all these spaces such that

$$\|T\|_{B(X^1_{\theta,p}, X^2_{\theta,p})} \leqslant \|T\|^{1-\theta}_{B(X_1,X_2)} \|T\|^{\theta}_{B(Y_1,Y_2)}. \tag{11}$$

As a consequence of this theorem, let us mention the following fact: Let X, Y, E, F be some B-spaces such that $F \subset Y \subset X$ and $F \subset E \subset X$. Take $T = 1_E : E \to X$ the inclusion map. Then $T(F) \subset Y$, so that

$$F \subset (E,F)_{\theta,p} \subset (X,Y)_{\theta,p} \subset X. \tag{12}$$

There are many ways of constructing the spaces $(X,Y)_{\theta,p}$ and each one is useful in different occasions; see the cited references for an exposition of these methods. In our subsequent work two of these methods will be especially useful and we think that they provide some real insight into the structure of the interpolation spaces, so we shall make some comments about them now. A third method (namely the trace method) is more useful for the proof of the abstract results stated in Section 7, but since our main purpose here are the applications, we do not speak about it. In Section 3, we shall give an abstract variant of the Littlewood-Paley type characterization of Besov spaces because of its usefulness in applications.

We begin with the so-called "positive operator" approach (Komatsu [Ko] being a very good reference). Let Λ be a closed, densely defined operator in the Banach space X such that each strictly positive real number r is in the resolvent set of Λ and $\|(\Lambda+r)^{-1}\|_{B(X)} \leqslant cr^{-1}$ for all $r > 0$ and some constant c. One defines, for each real $s > 0$ and each $p \in [1,\infty]$ a space $D^s_p(\Lambda) \equiv D^s_p(\Lambda; X)$ as follows: choose some integer $\ell > s$ and say that $x \in D^s_p(\Lambda)$ if

$$\|x\|_{D^s_p(\Lambda)} \equiv \|x\|_X + \left[\int_0^\infty \|r^s [\Lambda(\Lambda+r)^{-1}]^\ell x\|_X^p \frac{dr}{r} \right]^{1/p} < \infty. \tag{13}$$

In the case $p = \infty$, the second term has to be interpreted as $\sup_{r>0} \|r^s[\Lambda(\Lambda + r)^{-1}]^\ell x\|$. One can show that the condition $\|x\|_{D_p^s} < \infty$ does not depend on ℓ, so one can always take ℓ to be the smallest integer strictly larger than s (i.e. $\ell = [s] + 1$). Notice that only the behaviour of the integrand at infinity is relevant, so one could replace $\int_0^\infty$ by $\int_{r_0}^\infty$ for any $r_0 > 0$: the advantage then is that the definition is non-trivial even if $s = 0$ (then $\ell = 1$ by definition), a limit situation which occurs in our arguments but is not studied by Komatsu. Then D_p^s is a B-space ((13 is a compatible norm on it) such that for each integer $n > s$, $D(\Lambda^n) \subset D_p^s(\Lambda) \subset X$ (and densely if $p < \infty$). The integer powers Λ^m of Λ are closed densely defined operator and $D(\Lambda^m)$ is "almost" equal to $D_p^m(\Lambda)$. More precisely, the power Λ^s can be defined as a closed, densely defined operator in X for any real $s > 0$ and then

$$D_1^s(\Lambda) \subset D(\Lambda^s) \subset D_\infty^s(\Lambda). \tag{14}$$

In general, $D(\Lambda^s) \neq D_p^s(\Lambda)$ ($\forall p \in [1, \infty]$). But if X is a Hilbert space, Λ is a positive self-adjoint operator and Λ^s is defined according to the usual functional calculus, then $D(\Lambda^s) = D_2^s(\Lambda)$ (this is an easy exercise).

Now, we may state one of the main results of [Ko]: for any integer $m \geqslant 1$ and real $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$:

$$(X, D(\Lambda^m))_{\theta,p} = D_p^{\theta m}(\Lambda). \tag{15}$$

Here $D(\Lambda^m)$ is the B-space embedded (densely) in X equal to the domain of Λ^m. Observe that for $m = 1$ one may take $\ell = 1$ in (13). But for $m = 2$ and $\theta = 1/2$ we are forced to take $\ell = 2$ in the definition of D_p^1. We hope that this clarifies the definition of the classes in Section 5 below and of the notion of short-range perturbation given in Section 8.

Another description of $(X, D(\Lambda^m))_{\theta,p}$ is possible when Λ is the infinitesimal generator of a bounded C_0-semigroup $\{e^{i\Lambda t}\}_{t\geqslant 0}$ in X, and this point of view will be quite fruitfull for us. In this case the assumptions we put on Λ are automatically satisfied (r must be uplaced by $+ir$, but this is irrelevant here). Then the norm (13) is iquivalent to (see again Komatsu):

$$\|x\|_X + \Big[\int_0^\infty \Big\|\frac{(e^{i\Lambda t} - 1)^\ell}{t^s} x\Big\|_X^p \frac{dt}{t}\Big]^{1/p} \tag{16}$$

where clearly, only the behaviour at zero of the integrand is relevant, so we could replace $\int_0^\infty$ by $\int_0^1$. With this change, we may allow s to be zero (then $\ell = 1$ by definition) and $e^{i\Lambda t}$ can be an arbitrary C_0-group. The space $D_p^s(\Lambda)$, when looked at from the point of view of the definition (16), should be denoted $C^{s,p}(\Lambda) \equiv C^{s,p}(\Lambda; X)$. This explains our notations in Section 5 (where $C^1 \equiv C^{1,1}$ and $C^1 \equiv C^{1,\infty}$): an element $x \in C^{s,p}(\Lambda)$ is characterized by a regularity property of the function $t \to e^{i\Lambda t} x \in X$ (it is of class $C^{s,p}$ in the usual, differential sense). On the other hand, (13) describes the elements $x \in D_p^s$ in terms of their "behaviour at infinity in the spectral representation of Λ". This is quite obvious if Λ is a self-adjoint operator in a Hilbert space X and is clarified in general by the Littlewood-Paley type theory which we present in the next Section.

3. A Tauberian Theorem and Littlewood-Paley Description of Interpolation Spaces

We shall prove in this section a tauberian theorem which plays a role in several of our arguments. Let X be a Banach space and $\{T_k \mid k \in \mathbb{R}^n\}$ a strongly continuous representation of $\mathbb{R}^n$ in X *which is of polynomial growth.* Let $N \in [0, \infty)$ and $M < \infty$ such that

$$||T_k||_{B(X)} \leqslant M \cdot 2^{N/2} < k >^N \qquad (k \in \mathbb{R}^n). \tag{17}$$

We have included the factor $2^{N/2}$ because $\sqrt{2}\cdot < k + p > \leqslant \sqrt{2} < k > \cdot\sqrt{2} < p >$. We shall not give a precise definition of the generator Λ of T_k but rather think of it as a symbol which helps in understanding the functional calculus which will be developed below. Formally $T_k = e^{i\Lambda k}$ and $\Lambda k = \Lambda_1 k_1 + \ldots + \Lambda_n k_n$. Assuption (17) allows us to construct a rich enough functional calculus for Λ. More precisely, let $\mathcal{M}_N(\mathbb{R}^n)$ denote the set of bounded continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ whose Fourier transforms $\hat{f}$ are measures such that:

$$|f|_N := \frac{2^{N/2}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} < k >^N |\hat{f}|(dk) < \infty.$$

Clearly, $\mathcal{M}_N$ is a Banach algebra with unit (with usual operations of addition and multiplication) which contains $\mathcal{S}(\mathbb{R}^n)$. Observe that, if we define $f^\sigma(x) = f(x\sigma)$ for $\sigma \geqslant 0$, then $f^\sigma \in \mathcal{M}_N$ and

$$|f^\sigma|_N \leqslant \max(1, \sigma^N)|f|_N. \tag{18}$$

Now we define for each $f \in \mathcal{M}_N(\mathbb{R}^n)$

$$f(\Lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} T_k \hat{f}(dk). \tag{19}$$

We obviously get a unital homomorphism $f \mapsto f(\Lambda)$ of $\mathcal{M}_N$ into $B(X)$ such that $||f(\Lambda)||_{B(X)} \leqslant M|f|_N$. We denote $f^\sigma(\Lambda) \equiv f(\sigma\Lambda)$ and observe that $f(\sigma\Lambda) \to f(0)Id_X$ strongly in $B(X)$ as $\sigma \to 0$, and

$$||f(\sigma\Lambda)||_{B(X)} \leqslant M|f|_N \max(1, \sigma^N). \tag{20}$$

Let us fix a real number $a > 1$ and a function $\theta \in C_0^\infty(\mathbb{R}^n)$ with the following properties:

(a) $\theta(x) > 0$ if $a^{-1} < |x| < a$ and $\theta(x) = 0$ otherwise;

(b) $\sum_{j\in\mathbb{Z}} \theta(a^j x) = 1$ if $x \neq 0$.

We shall denote $\theta_j(x) = \theta(a^j x)$ if $j \geqslant 1$ and

$$\theta_0(x) = \sum_{j=-\infty}^{0} \theta(a^j x).$$

Observe that $\theta_0 \in C^\infty(\mathbb{R}^n)$, $\theta_0(x) = 0$ if $|x| \leqslant a^{-1}$ and $\theta_0(x) = 1$ if $|x| \geqslant 1$. Since $\theta_0 - 1 \in C_0^\infty$, we have $\theta_0 \in \mathcal{M}_N$. Moreover, for $j \geqslant 1$ we have $\theta_j \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{M}_N$ and its support equals $\{a^{-j-1} \leqslant |x| \leqslant a^{-j+1}\}$. Clearly

$$\sum_{j=0}^{\infty} \theta_j(x) = 1 \quad \text{if} \quad x \neq 0. \tag{21}$$

In the sum, only successive functions have supports with non-disjoint interiors.

Theorem 3.1. *Suppose $\rho \in \mathcal{M}_N$ has the following property: there is a real number $\ell > N$ and a C^∞ function $w : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ with $w(\sigma x) = \sigma^\ell w(x)$ for $x \neq 0$ and $\sigma > 0$, such that $\rho(x) = w(x)\rho_0(x)$ for some $\rho_0 \in \mathcal{M}_N$ and all $x \neq 0$.*

Then there is a constant $c < \infty$ such that for all $\varepsilon \in (0,1)$ and all $u \in X$:

$$||\rho(\varepsilon\Lambda)u||_X \leqslant c\varepsilon^N ||u||_X + c||\theta_0(\varepsilon\Lambda)u||_X + c\varepsilon^\ell \int_{\varepsilon a^{-1}}^1 \tau^{-\ell} ||\theta(\tau\Lambda)u|| \frac{d\tau}{\tau}. \tag{22}$$

Proof. *Since* $\mathcal{M}_N$ is an algebra, the functions $\theta_j\rho$ belong to $\mathcal{M}_N$ for all $j \geqslant 0$ and because of (21) we have $\rho(x) = \sum_{j=0}^{\infty} \rho(x)\theta_j(x)$ for all $x \neq 0$. Since the elements of $\mathcal{M}_N$ are continuous functions, if two such functions are equal outside zero, then they are equal in $\mathcal{M}_N$. So we shall have:

$$\rho = \sum_{j=0}^{\infty} \rho\theta_j \quad \text{in} \quad \mathcal{M}_N, \tag{23}$$

if we prove that the series is convergent in $\mathcal{M}_N$. This, in turn, will follow from the estimate

$$|\rho\theta_j|_N \leqslant c a^{(N-\ell)j}, \tag{24}$$

which we prove now.

Let us define $\tilde{w}(x) = w(x)\theta(x)$. Then $\tilde{w} \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{M}_N$ and for all $\sigma > 0$:

$$\rho(x)\theta^\sigma(x) = \rho_0(x)w(x)\theta(\sigma x) = \sigma^{-\ell}\rho_0(x)w(\sigma x)\theta(\sigma x) = \sigma^{-\ell}\rho_0(x)\tilde{w}(\sigma x)$$

or:

$$\rho\theta^\sigma = \sigma^{-\ell}\rho_0\tilde{w}^\sigma. \tag{25}$$

Using (18) and assuming $\sigma \geqslant 1$ we get

$$|\rho\theta^\sigma|_N \leqslant c\sigma^{N-\ell}, \tag{26}$$

of which (24) is a consequence.

From now on we fix some $\varepsilon \in (0,1)$ and denote by c various constants independent of ε, j and $u \in X$. From (23) and (24) we get:

$$\rho(\varepsilon\Lambda) = \sum_{j=0}^{\infty} \rho(\varepsilon\Lambda)\theta_j(\varepsilon\Lambda) \tag{27}$$

the series being absolutely convergent in $B(X)$. Hence

$$||\rho(\varepsilon\Lambda)u||_X \leqslant c||\theta_0(\varepsilon\Lambda)u||_X + \sum_{j=1}^{\infty} ||\rho(\varepsilon\Lambda)\theta(\varepsilon a^j\Lambda)u||_X \tag{28}$$

because $||\rho(\varepsilon\Lambda)||_{B(X)} \leqslant M|\rho^\varepsilon|_N \leqslant M|\rho|_N \leqslant c$.

We shall estimate differently the terms in (28) with $\varepsilon a^j \geqslant 1$ and those with $\varepsilon a^j < 1$. If $\varepsilon\sigma \geqslant 1$ we have by (25) and (18):

$$\begin{aligned}||\rho(\varepsilon\Lambda)\theta(\varepsilon\sigma\Lambda)||_{B(X)} &= \sigma^{-\ell}||\rho_0(\varepsilon\Lambda)\tilde{w}(\varepsilon\sigma\Lambda)||_{B(X)} \leqslant \\ &\leqslant \sigma^{-\ell}M|\rho_0^\varepsilon\tilde{w}^{\varepsilon\sigma}|_N \leqslant \\ &\leqslant \sigma^{-\ell}M|\rho_0^\varepsilon|_N|\tilde{w}^{\varepsilon\sigma}|_N \leqslant \\ &\leqslant \sigma^{-\ell}M|\rho_0|_N(\varepsilon\sigma)^N|\tilde{w}|_N = c\varepsilon^N\sigma^{N-\ell}. \end{aligned} \tag{29}$$

Hence

$$\sum_{j\geqslant 1,\varepsilon a^j\geqslant 1} ||\rho(\varepsilon\Lambda)\theta(\varepsilon a^j\Lambda)||_{B(X)} \leqslant c\varepsilon^N\sum a^{j(N-\ell)} \leqslant c\varepsilon^N(1-a^{N-\ell})^{-1},$$

which, when substituted into (28), gives

$$||\rho(\varepsilon\Lambda)u||_X \leqslant c||\theta_0(\varepsilon\Lambda)u||_X + c\varepsilon^N||u||_X + \sum_{j\geqslant 1,\varepsilon a^j<1} ||\rho(\varepsilon\Lambda)\theta(\varepsilon a^j\Lambda)u||_{B(X)}. \tag{30}$$

In order to estimate the terms in the remaining sum, let $w_0 \in C_0^\infty(\mathbb{R}^n)$ be such that $\tilde{w} \equiv w\theta = w_0\theta$. Then (25)implies

$$\rho\theta^\sigma = \sigma^{-\ell}\rho_0\tilde{w}^\sigma = \sigma^{-\ell}\rho_0 w_0^\sigma\theta^\sigma,$$

so, for $\varepsilon\sigma < 1$

$$\begin{aligned}||\rho(\varepsilon\Lambda)\theta(\varepsilon\sigma\Lambda)u||_X &= \sigma^{-\ell}||\rho_0(\varepsilon\Lambda)w_0(\varepsilon\sigma\Lambda)\theta(\varepsilon\sigma\Lambda)u||_X \leqslant \\ &\leqslant \sigma^{-\ell}M|\rho_0^\varepsilon w_0^{\varepsilon\sigma}|_N||\theta(\varepsilon\sigma\Lambda)u||_X \leqslant c\sigma^{-\ell}||\theta(\varepsilon\sigma\Lambda)u||_X.\end{aligned}$$

Using this, we obtain in (30)

$$||\rho(\varepsilon\Lambda)u||_X \leqslant c||\theta_0(\varepsilon\Lambda)u||_X + c\varepsilon^N||u||_X + c\varepsilon^\ell \sum_{j\geqslant 1,\varepsilon a^j<1} (\varepsilon a^j)^{-\ell}||\theta(\varepsilon a^j\Lambda)u||_X. \tag{31}$$

The proof is finished by showing that the last sum is dominated by an integral. Let

$$\varphi_b(\tau) = \tau^{-b}||\theta(\tau\Lambda)u||_X$$

for $\tau > 0$. The sum in (33) is ($j \geqslant 1$ is an integer):

$$\varepsilon^\ell \sum_{\varepsilon a\leqslant\varepsilon a^j<1} \varphi_\ell(\varepsilon a^j) = \frac{a\varepsilon^\ell}{a-1} \sum_{\varepsilon a\leqslant\varepsilon a^j<1} \varphi_{\ell+1}(\varepsilon a^j)[\varepsilon a^j - \varepsilon a^{j-1}]. \tag{32}$$

It is straightforward to prove that for $0 < t < s < at$ and for all $\lambda \in \mathbb{R}$

$$\theta(s\lambda) = \sum_{k=-1}^{2} \theta(s\lambda)\theta(a^k t\lambda).$$

Then this remains true at the operator level, i.e. one may replace λ by Λ. Clearly we shall then get

$$\varphi_0(s) \leqslant \sum_{k=-1}^{2} ||\theta(s\Lambda)\theta(a^k t\Lambda)u||_X \leqslant \sum_{k=-1}^{2} ||\theta(s\Lambda)||_{B(X)}\varphi_0(a^k t) \leqslant$$
$$\leqslant M \cdot \max(1, s^N)|\theta|_N \sum_{k=-1}^{2} \varphi_0(a^k t).$$

It is obvious now that for each $T \in (0, \infty)$ and each ℓ there is $c < \infty$ such that for $0 < t < s < at < T$

$$\varphi_{\ell+1}(s) \leqslant c \sum_{k=-1}^{2} \varphi_{\ell+1}(a^k t). \tag{33}$$

If $\tau \in (\varepsilon a^{j-1}, \varepsilon a^j)$ then $\tau < \varepsilon a^j < a\tau$, so that,

$$\varphi_{\ell+1}(\varepsilon a^j) \leqslant c \sum_{k=-1}^{2} \varphi_{\ell+1}(a^k \tau).$$

Hence we can dominate (32) by

$$\frac{a\varepsilon^\ell}{a-1} \sum_{\varepsilon a \leqslant \varepsilon a^j < 1} \int_{\varepsilon a^{j-1}}^{\varepsilon a^j} c \sum\nolimits_{k=-1}^{2} \varphi_{\ell+1}(a^k \tau)\, dr \leqslant$$
$$\leqslant \frac{ca\varepsilon^\ell}{a-1} \int_\varepsilon^1 \sum\nolimits_{k=-1}^{2} \varphi_{\ell+1}(a^k \tau)\, d\tau \leqslant$$
$$\leqslant c\varepsilon^\ell \int_{\varepsilon a^{-1}}^{a^2} \varphi_{\ell+1}(\tau)\, d\tau.$$

The contribution of the integral on $(1, a^2)$ can be neglected because $\varepsilon^\ell \leqslant \varepsilon^N$. •

The above theorem has many useful consequences, both theoretically (e.g. in our proof of the result of [BG1]) and in applications. For example, one may deduce from it the usual description of Besov spaces. Also, an explicit and useful description of interpolation spaces (including limit cases such as $\theta = 0$) can be obtained. The following corollary is relevant in this context:

Corollary 3.2. *Let $\sigma \in [0, N]$ be a real number. Under the assumptions of Theorem 3.1, there is a finite constant c such that for all $u \in X$:*

$$\int_0^1 ||\rho(\varepsilon\Lambda)u||_X \frac{d\varepsilon}{\varepsilon^{1+\sigma}} \leqslant c||u||_X + c\int_0^1 ||\theta_0(\varepsilon\Lambda)u||_X \frac{d\varepsilon}{\varepsilon^{1+\sigma}}. \tag{34}$$

if $\sigma > 0$, then we also have.

$$\int_0^1 ||\rho(\varepsilon\Lambda)u||_X \frac{d\varepsilon}{\varepsilon^{1+\sigma}} \leqslant c||u||_X + \int_0^1 ||\theta(\varepsilon\Lambda)u||X \frac{d\varepsilon}{\varepsilon^{1+\sigma}}. \tag{35}$$

Proof. The first estimate follows by integration of (22) if one takes into account that $\theta_0(ax) = 1$ on $\operatorname{supp}\theta$, so that

$$||\theta(\tau\Lambda)u|| = ||\theta(\tau\Lambda)\theta_0(a\tau\Lambda)u|| \leqslant c||\theta_0(a\tau\Lambda)u||$$

for $0 < \tau \leqslant$ const. To get (35), we prove that the last term in (34) is dominated by the last term in (35) by using $||\theta_0(\varepsilon\Lambda)u||_X \leqslant \sum_{j=-\infty}^{0} ||\theta(\varepsilon a^j\Lambda)u||_X$. •

As an example, let $n = 1$ and $\rho(t) = \left(\frac{t}{t+i}\right)^\ell$; the expression (13) clearly shows the interest of results such as (34), (35).

We shall need in a moment another easy application of the tauberian theorem. The next proposition is proved by another method in an appendix of [BGM].

Proposition 3.3. *Let (Y, X) be a Friedrichs couple, i.e. X, Y are Hilbert spaces such that $Y \subset X$ densely. Let Λ be any selfadjoint operator in X with the domain equal to Y (such an operator always exists). Denote by F the spectral measure of Λ and put $F_0 = F([-1,+1])$, $F_j = F([-2^j, -2^{j-1}) \cup (2^{j-1}, 2^j])$ for $j = 1, 2, \ldots$. Then for everg $0 < \sigma < 1$, $1 \leqslant p \leqslant \infty$ we have the following compatible norm on the interpolation space $(X, Y)_{\sigma,p} \equiv X_{\sigma,p}$*

$$||u||_{X_{\sigma,p}} = \left[\sum_{j=0}^{\infty} ||2^{j\sigma} F_j u||_X^p\right]^{1/p} \tag{36}$$

with the usual modification if $p = \infty$.

Proof. Λ generates a unitary group $T_t = e^{i\Lambda t}$ in X, hence we may take any $N \geqslant 0$ in (17). For the description of $(X, Y)_{\sigma,p} = (X, D(\Lambda))_{\sigma,p}$ we use (15) (with $m = 1$) and (13) with $\ell = 1$, $s = \sigma$ and Λ replaced by $i\Lambda$. As explained there, we may take the integral over $(1, \infty)$. After the change of the variable $\varepsilon = \frac{1}{\tau}$, we get the following compatible norm on $X_{\sigma,p}$:

$$||u||_X + \left[\int_0^1 ||\varepsilon^{-\sigma} \frac{\varepsilon\Lambda}{\varepsilon\Lambda + i} u||_X^p \frac{d\varepsilon}{\varepsilon}\right]^{1/p}. \tag{37}$$

We shall consider only the case $1 \leqslant p < \infty$ (the case $p = \infty$ is rather trivial and only $1 \leqslant p \leqslant 2$ will be of interest to us). Take $\rho(t) = \frac{t}{t+i}$ and some N with $\sigma < N < 1$ in (22). Then

$$||\frac{\varepsilon\Lambda}{\varepsilon\Lambda + i} u||_X \leqslant c\varepsilon^N ||u||_X + ||\theta_0(\varepsilon\Lambda)u||_X + c\varepsilon \int_{\varepsilon a^{-1}}^1 \tau^{-1} ||\theta(\tau\Lambda)u||_X \frac{d\tau}{\tau} =$$

$$= c\varepsilon^N ||u||_X + ||\theta_0(\varepsilon\Lambda)u||_X + c\int_{a^{-1}}^{\varepsilon^{-1}} ||\theta(\varepsilon t\Lambda)u||_X \frac{dt}{t^2}$$

Let χ_ε be the characteristic function of the interval $[a^{-1}, \varepsilon^{-1}]$. We get

$$\Big[\int_0^1 ||\varepsilon^{-\sigma}\frac{\varepsilon\Lambda}{\varepsilon\Lambda+i}u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p} \leqslant$$

$$\leqslant c||u||_X + \Big[\int_0^1 ||\varepsilon^{-\sigma}\theta_0(\varepsilon\Lambda)u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p} + + c\int_{a^{-1}}^\infty \frac{dt}{t^2}\Big[\int_0^1 ||\chi_\varepsilon(t)\varepsilon^{-\sigma}\theta(\varepsilon t\Lambda)u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p}. \tag{38}$$

The last integral becomes, after the change of variable $\varepsilon \mapsto \frac{\tau}{t}$:

$$\int_{a^{-1}}^\infty \frac{dt}{t^2}\Big[\int_0^1 t^{\sigma p}||\tau^{-\sigma}\theta(\tau\Lambda)u||_X^p \frac{d\tau}{\tau}\Big]^{1/p} = \int_{a^{-1}}^\infty \frac{dt}{t^{2-\sigma}}\Big[\int_0^1 ||\varepsilon^{-\sigma}\theta(\varepsilon\Lambda)u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p}.$$

Then, as in the proof Corollary 3.2, one estimates the first integral in the right-hand side of (38) in terms of a similar integral with θ_0 replaced by θ. So we get

$$\Big[\int_0^1 ||\varepsilon^{-\sigma}\frac{\varepsilon\Lambda}{\varepsilon\Lambda+i}u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p} \leqslant c||u||_X + c\Big[\int_0^1 ||\varepsilon^{-\sigma}\theta(\varepsilon\Lambda)u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p}. \tag{39}$$

It is trivial to prove that the right-hand side of (39) is dominated by (37). So the expression

$$||u||_X + \Big[\int_0^1 ||\varepsilon^{-\sigma}\theta(\varepsilon\Lambda)u||_X^p \frac{d\varepsilon}{\varepsilon}\Big]^{1/p} \tag{40}$$

is a compatible norm on $X_{\sigma,p}$. Taking $a = 2$, we easily see that one may replace here $\theta(\varepsilon\Lambda)$ by the spectral projection of Λ associated to the set $\frac{1}{2\varepsilon} < |x| < \frac{2}{\varepsilon}$. If we take $r = \frac{1}{\varepsilon}$ as a new variable, one gets an integral form of (36) which can be replaced by a discrete version without difficulty. •

The preceding proof shows the usefulness of the tauberian theorem, but we shall need (36) in order to prove a result which is fundamental for us. Before this, we shall recall a concept related to the geometry of Banach spaces introduced by B. Maurey and G. Pisier [MP]. Let $\{w_n\}_{n\geqslant 1}$ be any sequence of independent, centered, Bernoulli variables (i.e. w_n are independent random variables on some probability space, which take only values ± 1 and have mean value zero; for example w_n can be the n-th Rademacher function on the interval $[0,1]$). We shall denote by dt the probability measure on the space where all w_n are defined. The definition below is independent of the choice of $\{w_n\}$.

Definition 3.4. Let X be a Banach space and $q > 0$ a real number. One says that X *is of cotype* q if there is $\alpha \in (0,\infty)$ and a constant $c < \infty$ such that for any finite family $x_1, \dots, x_N \in X$

$$\Big[\sum_{n=1}^N ||x_n||_X^q\Big]^{1/q} \leqslant c\Big[\int ||\sum_{n=1}^N x_n w_n(t)||_X^\alpha dt\Big]^{1/\alpha}. \tag{41}$$

Remarks.

1) A general form of Khintchine's inequality due to J.-P. Kahane and G. Pisier shows that if the above property holds for some α, then it holds for all α. This will be important for us below. Note that q cannot be smaller than 2.

2) With a special choice of $\{w_n\}_{n\geqslant 1}$, one gets the following equivalent form of the above definition: there exist $\alpha \in (0,\infty)$ and $c<\infty$ such that for any $x_1,\dots,x_N \in X$:

$$\Big[\sum_{n=1}^{N} \|x_n\|_X^q\Big]^{1/q} \leqslant c\Big[2^{-N} \sum_{\varepsilon_1,\cdots,\varepsilon_N=\pm 1} \Big\|\sum_{n=1}^{N} \varepsilon_n x_n\Big\|^{\alpha}\Big]^{1/\alpha}. \tag{42}$$

We are interested in this notion because of the following result of G. Pisier [Pi]:

Theorem 3.5. *Let X,Y be Banach spaces such that X^* and Y are of cotype 2 and X or Y has the approximation property. Then any bounded operator $U : X \to Y$ is factorisable through a Hilbert space, i.e. there is a Hilbert space H and there are bounded operators $V : X \to H$ and $W : H \to Y$ such that $U = WV$.*

In order to use this theorem, we shall need to know that some interpolation spaces are of cotype 2. Each Hilbert space has this property, because one has the equality in (42) with $q=\alpha=2$ and $c=1$. But one has much more:

Proposition 3.6. *Let (X,Y) be a Friedrichs couple rm (see Proposition 3.3). Then $(X,Y)_{\sigma,p}$ is of cotype 2 and has the bounded approximation property if $0<\sigma<1$ and $1\leqslant p\leqslant 2$.*

Proof. The assertion concerning the bounded approximation property (see [Pi]) follows easily from the interpolation property. We shall now use the following easily proven assertion: if $\mathcal{H}$ is a Hilbert space, (Ω,μ) is a σ-finite measure space and $1\leqslant p\leqslant 2$, then $L^p(\Omega,\mathcal{H};\mu)$ is of cotype 2 (see [MP]). Now let $\{\mathcal{H}_i\}_{i\in I}$ be a countable family of Hilbert spaces and define $\ell^p(\{\mathcal{H}_i\}_{i\in I})$ as the Banach space of all sequences $x=\{x_i\}_{i\in I}$ with $x_i\in\mathcal{H}_i$ and

$$\|x\| = \Big(\sum_{i\in I} \|x_i\|_{\mathcal{H}_i}^p\Big)^{1/p} < \infty.$$

If $\mathcal{H}=\oplus_{i\in I}\mathcal{H}_i$ (Hilbert direct sum) and $\Omega = I$ equipped with the discrete measure, then $\ell^p(\{\mathcal{H}_i\}_{i\in I})$ is obviously a closed subspace of $L^p(\Omega,\mathcal{H};\mu)$. Since a closed subspace of a space of cotype 2 has also cotype 2, $\ell^p(\{\mathcal{H}_i\}_{i\in I})$ is of cotype 2 if $1\leqslant p\leqslant 2$. The proof is finished by an application of Proposition 3.3. •

Corollary 3.7. *If $(Y_1,X_1),(Y_2,X_2)$ are Friedrichs couples, then any bounded operator*

$$T : (X_1,Y_1)^0_{\sigma,\infty} \to (X_2,Y_2)_{\sigma,1}$$

is factorisable through a Hilbert space.

4. Unitary Groups in Friedrichs Couples

In our approach to the conjugate operator method the main object is a triplet $(\mathcal{G}, \mathcal{H}; W)$ consisting of two (complex, separable) Hilbert spaces $\mathcal{G}, \mathcal{H}$ such that $\mathcal{G}$ is continuously and densely embedded in $\mathcal{H}$ and $W = \{W_\alpha\}_{\alpha\in\mathbb{R}}$ is a strongly continuous unitary group in $\mathcal{H}$ which leaves $\mathcal{G}$ invariant: $W_\alpha \mathcal{G} = \mathcal{G}$ for all $\alpha \in \mathbb{R}$. The ordered pair $(\mathcal{G}, \mathcal{H})$ will be called a *Friedrichs couple.* We shall say that a family W verifying the preceding conditions is a *unitary group in the Friedrichs couple* $(\mathcal{G}, \mathcal{H})$. The purpose of this section is to describe some constructions and objects which are naturally associated to the structure $(\mathcal{G}, \mathcal{H}; W)$.

From now on we shall identify $\mathcal{H} \cong \mathcal{H}^\star$ with the help of the Riesz isomorphism. This induces a canonical continuous and dense embedding $\mathcal{H} \subset \mathcal{G}^\star$, so that we get a triplet of spaces $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^\star$. It is clear that we shall have continuous embeddings $B(\mathcal{H}) \subset B(\mathcal{G}, \mathcal{G}^\star)$, $B(\mathcal{G}) \subset B(\mathcal{G}, \mathcal{G}^\star)$, etc. The Banach space $B(\mathcal{G}, \mathcal{G}^\star)$ will play a special role later on and we shall denote it by $\mathcal{X}$. Observe that $\mathcal{X}$ is provided with an isometric involution $T \mapsto T^\star$ which is preserved by the embedding $B(\mathcal{H}) \subset \mathcal{X}$.

Let us denote $W_\alpha^+ = W_\alpha|_\mathcal{G}$ considered as an operator in $\mathcal{G}$. Closed graph theorem implies that $W_\alpha^+ \in B(\mathcal{G})$. It is easily seen that the operator $(W_\alpha^+)^\star$ (the operator in $B(\mathcal{G}^\star)$ which is the adjoint of the operator W_α^+ acting in $\mathcal{G}$) is an extension of $W_{-\alpha}$, i.e. $(W_\alpha^+)^\star|_\mathcal{H} = W_{-\alpha}$. Since $\mathcal{H}$ is dense in $\mathcal{G}^\star$, this extension is uniquely defined by the requirement that it be continuous in $\mathcal{G}^\star$. Let us denote by W_α^- the unique continuous extension of W_α to $\mathcal{G}^\star$ (so $W_\alpha^- = (W_{-\alpha}^+)^\star$). It is clear that $\{W_\alpha^\pm\}_{\alpha\in\mathbb{R}}$ are groups of operators in $B(\mathcal{G})$, resp. $B(\mathcal{G}^\star)$. One can prove that they are even C_0-groups (i.e. $\alpha \mapsto W_\alpha^+ \in B(\mathcal{G})$ is strongly continuous for example).

Let A be the self-adjoint operator in $\mathcal{H}$ which generates W_α, i.e. $W_\alpha = e^{iA\alpha}$; we shall denote by $D(A; \mathcal{H})$ its domain. Let A_+, A_- be the generators of W_α^+, W_α^- in $\mathcal{G}, \mathcal{G}^\star$; we shall denote by $D(A; \mathcal{G})$, $D(A; \mathcal{G}^\star)$ their domains. One has

$$D(A; \mathcal{G}^\star) = \{u \in \mathcal{G}^\star \,|\, \alpha \mapsto W_\alpha^- \in \mathcal{G}^\star \quad \text{has a derivative for } \alpha = 0\}, \tag{43'}$$

$$u \in D(A; \mathcal{G}^\star) \Rightarrow A_- u = \lim_{\alpha\to 0} \frac{W_\alpha^- u - u}{ia}. \tag{43''}$$

Furthermore, it is easy to show that

$$D(A; \mathcal{H}) = \{u \in \mathcal{H} \,|\, u \in D(A; \mathcal{G}^\star) \quad \text{and} \quad A_- u \in \mathcal{H}\} \tag{44}$$

and $Au = A_- u$ if $u \in D(A; \mathcal{H})$. Also

$$D(A; \mathcal{G}) = \{u \in \mathcal{G} \,|\, u \in D(A; \mathcal{G}^\star) \quad \text{and} \quad A_- u \in \mathcal{G}\} \tag{45}$$

and $A_+ u = A_- u$ if $u \in D(A; \mathcal{G})$. Hence A and A_+ are just restrictions of A_- to subspaces of its domain. On the other hand, these subspaces are dense in $D(A; \mathcal{G}^\star)$ (with respect to the graph topology associated to A_-): this is a consequence of Nelson's lemma (see Theorem 1.9 in [D]). In other words A_+ and A, considered as operators in $\mathcal{G}^\star$ are closable and their closure is just A_-.

It is clear from the preceding discussion that there is no meaning in making a distinction between the operators A_+, A and A_- and also between the C_0-groups which they generate $W_\alpha^+, W_\alpha, W_\alpha^-$. From now on we shall denote them by the same symbol: A, resp. W_α. Hence W_α is a C_0-group in $\mathcal{G}^\star$, which leaves $\mathcal{H}$ and $\mathcal{G}$ invariant and is unitary in $\mathcal{H}$. A is the generator of W defined on $D(A;\mathcal{G}^\star)$ (see (43)) and its restriction to $D(A;\mathcal{H})$ (defined by (44)) is a self-adjoint operator in $\mathcal{H}$, etc.

Let us denote by $\mathcal{F}$ the space $D(A;\mathcal{G}^\star)$ equipped with the graph-norm associated to A:

$$||u||_{\mathcal{F}} = [||u||^2_{\mathcal{G}^\star} + ||Au||^2_{\mathcal{G}^\star}]^{1/2}. \tag{46}$$

Then $\mathcal{F}$ is a Hilbert space such that $\mathcal{F} \subset \mathcal{G}^\star$ continuously and densely. $D(A;\mathcal{G})$ and $D(A;\mathcal{H})$ are dense sobspace of $\mathcal{F}$, but $\mathcal{G}$ and $\mathcal{H}$ are not comparable with $\mathcal{F}$. We may then construct new Banach spaces by real interpolation. The space which plays the main role in our approach is

$$\mathcal{E} = (\mathcal{G}^\star, \mathcal{F})_{1/2,1}. \tag{47}$$

We can use the results of Section 2 in order to give rather explicit descriptions of the space $\mathcal{E}$. The first one is in terms of W : $\mathcal{E}$ is the set of all $u \in \mathcal{G}^\star$ such that

$$\int_0^1 ||W_\alpha u - u||_{\mathcal{G}^\star} \frac{d\alpha}{\alpha^{3/2}} < \infty. \tag{48}$$

For the second description, observe that there is $r_0 \geqslant 0$ such that $A+ir$ is an isomorphism of $\mathcal{F}$ onto $\mathcal{G}^\star$ for all $|r| \geqslant r_0$; then $\mathcal{E}$ is the space of all $u \in \mathcal{G}^\star$ such that

$$\int_{r_0}^\infty ||A(A+ir)^{-1}u||_{\mathcal{G}^\star} \frac{dr}{\sqrt{r}} < \infty. \tag{49}$$

$\mathcal{E}$ is a Banach space, non-reflexive in general, such that $\mathcal{F} \subset \mathcal{E} \subset \mathcal{G}^\star$, both embeddings being continuous and dense. This implies that $\mathcal{G} \subset \mathcal{E}^\star \subset \mathcal{F}^\star$, with continuous embeddings. $\mathcal{G}$ is not dense in $\mathcal{E}^\star$ (in general) and we denote by $\mathcal{E}^{\star 0}$ the closure of $\mathcal{G}$ in $\mathcal{E}^\star$. Then $(\mathcal{E}^{\star 0})^\star = \mathcal{E}$, an important fact in our applications. One can also describe $\mathcal{E}^\star$ by an interpolation procedure

$$\mathcal{E}^\star = (\mathcal{G}, \mathcal{F}^\star)_{1/2,\infty}. \tag{50}$$

5. Examples

Let $\mathcal{H} = L^2(\mathbb{R}^n), \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ be the space of temperate test functions and $\mathcal{S}^\star = \mathcal{S}^\star(\mathbb{R}^n)$ its anti-dual (the space of temperate distribution). Usual identifications $\mathcal{S} \subset \mathcal{H} \cong \mathcal{H}^\star \subset \mathcal{S}^\star$ are made. We denote by the Fourier transform (acting in $\mathcal{S}^\star$) defined according to the rule:

$$(\mathcal{F}f)(k) = \hat{f}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ikx} f(x)\,dx.$$

We also put $(\mathcal{F}^\star f)(k) := \hat{f}(-k)$. Q_j denotes the operators of multiplication by the variable x_j and $P_j = -i\partial/\partial x_j$, both acting in $\mathcal{S}^\star$. Then $Q = (Q_1, \dots, Q_n)$ and $P = (P_1, \dots, P_n)$ are vector-operators and we may form the operators $Qk =$

$\sum_{j=1}^{n} Q_j k_j$, $Px = \sum_{j=1}^{n} P_j x_j$ for each $k, x \in \mathbb{R}^n$. Observe that $\{e^{iQk}\}_{k\in\mathbb{R}^n}$, $\{e^{iPx}\}_{x\in\mathbb{R}^n}$ are n-parameter groups of operators on $\mathcal{S}^*$ which become strongly continuous unitary groups acting an $\mathcal{H}$. More generally we denote $f(Q)$ the operator of multiplication by a function f and $f(P) = \mathcal{F}^* f(Q)\mathcal{F}$. If $f \in C^\infty_{\text{pol}}(\mathbb{R}^n)$ (i.e. f is C^∞ and all its derivatives have at most polynomial growth at infinity), then $f(Q)$ and $f(P)$ are continuous operators in $\mathcal{S}^*$. It is easily shown that

$$e^{-iQk} f(P) e^{iQk} = f(P+k) \quad \text{for each } k \in \mathbb{R}^n. \tag{51}$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^m ($m \geqslant 1$ integer) such that its derivatives of order m are bounded functions. We shall denote:

$$\widetilde{h}_m(k) = \left[\sum\nolimits_{|\alpha|\leqslant m} |h^{(\alpha)}(k)|^2\right]^{1/2}. \tag{52}$$

If h is a polynomial of degree m, then $\widetilde{h}_m$ is usually denoted by $\hat{h}$. Let

$$\mathcal{H}^h = \{u \in \mathcal{H} \mid h(P)u \in \mathcal{H}\}, \tag{53}$$

with norm $||u||_{\mathcal{H}^h} = (||u||^2 + ||h(P)u||^2)^{1/2}$.

The operator $h(P)$ is self-adjoint on $\mathcal{H}$ with the domain equal to $\mathcal{H}^h$. It is straightforward to prove that $\mathcal{H}^h$ is invariant under the group e^{iQk} if and only if there is a constant c such that

$$\widetilde{h}_m(k) \leqslant c(1 + |h(k)|). \tag{54}$$

If this is the case, the n-parameter group induced by e^{iQk} in $\mathcal{H}^h$ is strongly continuous and of polynomial growth. In this section we shall fix a function h with the above properties and we shall put

$$\mathcal{G} = \{u \in \mathcal{H} \mid < h(P) >^{1/2} u \in \mathcal{H}\}, \tag{55}$$

where $< h(P) >^{1/2} = (1 + |h(P)|^2)^{1/4}$ (we systematically use the notation $< x > = \sqrt{1+|x|^2}$ when it makes sense). Then $\mathcal{G}$ is a Hilbert space with respect to the norm:

$$||u||_{\mathcal{G}} = || < h(P) >^{1/2} u||$$

and $(\mathcal{G}, \mathcal{H})$ is a Friedrichs couple. We have $\mathcal{S} \subset \mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^* \subset \mathcal{S}^*$ and $\mathcal{G}^*$ is the set of temperate distributions u such that $< h(P) >^{-1/2} u \in \mathcal{H}$.

Now we consider weighted versions of these spaces. For each $s \in \mathbb{R}$ let

$$\mathcal{G}_s = \{u \in \mathcal{S}^* \mid < Q >^s u \in \mathcal{G}\}, \tag{56}$$

$$||u||_{\mathcal{G}_s} = || < Q >^s u||_{\mathcal{G}}.$$

Then $\mathcal{G}_s$ is a Hilbert space, $\mathcal{G}_0 = \mathcal{G}$ and $\mathcal{G}_s \subset \mathcal{G}_t$ continuously and densely if $t \leqslant s$. Moreover, we have the interpolation property $(\mathcal{G}_t, \mathcal{G}_s)_{\theta,2} = \mathcal{G}_{(1-\theta)t+\Theta s}$.

In exactly the same way we define $\mathcal{G}^\star_s \equiv (\mathcal{G}^\star)_s$. We get a new family of spaces with properties similar to the above ones. The relation between these two families is $(\mathcal{G}_s)^\star = \mathcal{G}^\star_{-s}$ which is trivial to verify.

One must think of $\mathcal{G}_s$ as having a supplementary index 2: $\mathcal{G}_s \equiv \mathcal{G}_{s,2}$. Now we define $\mathcal{G}_{s,p}$ for all $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$ by real interpolation: if $s_1 < s < s_2$ and $s = (1-\theta)s_1 + \theta s_2$ with $0 < \theta < 1$, then we put

$$\mathcal{G}_{s,p} = (\mathcal{G}_{s_1}, \mathcal{G}_{s_2})_{\theta,p}. \tag{57}$$

The fact that the definition does not depend on s_1, s_2 is a consequence of the reiteration theorem. Similarly we define

$$\mathcal{G}^\star_{s,p} \equiv (\mathcal{G}^\star)_{s,p}. \tag{58}$$

From the description of the adjoint of an interpolation space we gave in section 2, we obtain

$$(\mathcal{G}_{s,p})^\star = \mathcal{G}^\star_{-s,p'}; \qquad (\mathcal{G}^\star_{s,p})^\star = \mathcal{G}_{-s,p'} \quad \text{if } s \in \mathbb{R} \text{ and } 1 \leqslant p < \infty. \tag{59}$$

$\mathcal{S}$ is dense in $\mathcal{G}_{s,p}$ and $\mathcal{G}^\star_{s,p}$ if $p < \infty$. For $p = \infty$, it is not, and we denote $\overset{\circ}{\mathcal{G}}_{s,\infty}, \overset{\circ}{\mathcal{G}}{}^\star_{s,\infty}$ the closure of $\mathcal{S}$ in these spaces. Then

$$(\overset{\circ}{\mathcal{G}}_{s,\infty})^\star = \mathcal{G}^\star_{-s,1}; \qquad (\overset{\circ}{\mathcal{G}}{}^\star_{s,\infty})^\star = \mathcal{G}_{-s,1}. \tag{60}$$

The preceding definitions and properties can be formalated assuming much less about h then we did, but the condition (54) allows us to give a much more explicit characterization of the spaces $\mathcal{G}_{s,p}, \mathcal{G}^\star_{s,p}$ in the spirit of the Littlewood-Paley theory. To do this, let us fix a radial function $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $\theta(x) > 0$ if $\frac{1}{2} < |x| < 2$ and $\theta(x) = 0$ otherwise. Also, let $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ if $|x| \leqslant 1$ and $\eta(x) = 0$ if $|x| \geqslant 2$. Then the following two norms are equivalent and are equivalent to the norm of $\mathcal{G}_{s,p}$:

$$1) \qquad ||\eta(Q)u||_{\mathcal{G}} + \Big[\sum\nolimits_{j=0}^{\infty} ||2^{js}\theta(2^{-j}Q)u||^p_{\mathcal{G}}\Big]^{1/p};$$

$$2) \qquad ||\eta(Q)u||_{\mathcal{G}} + \Big[\int_1^\infty ||r^s\theta(\frac{Q}{r})u||^p_{\mathcal{G}} \frac{dr}{r}\Big]^{1/p}.$$

Similar assertions hold for $\mathcal{G}^\star_{s,p}$.

We shall consider groups $W_\alpha = e^{iA\alpha}$ with generators of the form

$$A = \frac{1}{2}(F(P)Q + QF(P)) = F(P)Q + \frac{i}{2}(\operatorname{div} F)(P)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded Lipschitz function. Moreover, Fh' be a bounded function, uhicheasily that W_α leaves $\mathcal{G}$ invariant. So we have

$$\begin{aligned}\mathcal{F} = D(A;\mathcal{G}^\star) &= \{u \in \mathcal{G}^\star \mid Au \in \mathcal{G}^\star\} = \\ &= \{u \in \mathcal{G}^\star \mid \sum\nolimits_{j=1}^{n} F_j(P)Q_j u \in \mathcal{G}^\star\} \supset \\ &\supset \{u \in \mathcal{G}^\star \mid Q_j u \in \mathcal{G}^\star \text{ for all } j = 1, \dots, n\} \equiv \mathcal{G}^\star_1.\end{aligned} \tag{61}$$

The interest of the embedding $\mathcal{G}_1^\star \subset \mathcal{F}$ is that the space $\mathcal{G}_1^\star$ has been explicitly described above and so

$$\mathcal{E} = (\mathcal{G}^\star, \mathcal{F})_{1/2,1} \supset (\mathcal{G}^\star, \mathcal{G}_1^\star)_{1/2,1} = \mathcal{G}_{1/2,1}^\star. \tag{62}$$

This also implies by duality that $\mathcal{E}^\star \subset \mathcal{G}_{-1/2,\infty}$. In particular,

$$B(\mathcal{E}, \mathcal{E}^\star) \subset B(\mathcal{G}_{1/2,1}^\star, \mathcal{G}_{-1/2,\infty}).$$

So, once we have the limiting absorption principle in $B(\mathcal{E}, \mathcal{E}^\star)$, we shall have it also in $B(\mathcal{G}_{1/2,1}^\star, \mathcal{G}_{-1/2,\infty})$. Observe that $\mathcal{E}^{\star\circ} \subset \overset{\circ}{\mathcal{G}}_{-1/2,\infty}$.

Let h_1, h_2 be two different functions with the same properties as h. Denote by $\mathcal{G}(h_j)$ the corresponding $\mathcal{G}$ spaces. Then each continuous operator $T : \mathcal{G}(h_1)^\circ_{-1/2,\infty} \to \mathcal{G}(h_2)^\star_{1/2,1}$ will give by restriction a continuous operator $\mathcal{E}_1^{\star\circ} \to \mathcal{E}_2$. Observe that we are in the context of Corollary 3.7.

We shall give a simple estimate for the norm in $\mathcal{H}^h$ of the operator of multiplication by a function ψ. We begin by deriving a commutation formula between $\varphi(P)$ and $\psi(Q)$. If $\varphi : \mathbb{R}^n \to \mathbb{C}$ is a function of class $\mathbb{C}^m$ and $|\alpha| \leqslant m$, we define a continuous function $\varphi_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_\alpha(x, y) = |\alpha| \int_0^1 \varphi^{(\alpha)}(x + ty)(1-t)^{|\alpha|-1}\, dt. \tag{63}$$

Then $\varphi_\alpha(x, 0) = \varphi^{(\alpha)}(x)$, and Taylor's formula is

$$\varphi(x+y) = \sum\nolimits_{|\alpha|<m} \frac{y^\alpha}{\alpha!}\varphi^{(\alpha)}(x) + \sum\nolimits_{|\alpha|=m} \frac{y^\alpha}{\alpha!}\varphi_\alpha(x, y).$$

It is an obvious consequence of (51) that for $\varphi \in C^m_{\mathrm{pol}}(\mathbb{R}^n)$ and $k \in \mathbb{R}^n$ we shall have the following equality on the level of a operators $\mathcal{S} \to \mathcal{S}^\star$:

$$\varphi(P)e^{iQk} = \sum\nolimits_{|\alpha|<m} \frac{k^\alpha}{\alpha!} e^{iQk}\varphi^{(\alpha)}(P) + \sum\nolimits_{|\alpha|=m} \frac{k^\alpha}{\alpha!} e^{iQk}\varphi_\alpha(P, k). \tag{64}$$

Let $\psi \in C^m_{\mathrm{pol}}(\mathbb{R}^n)$ be such that its derivatives of order m are Fourier transforms of rapidly decreasing measures. Multiplying the preceding identity by $(2\pi)^{-n/2}\hat{\psi}(k)$ and integrating we get

$$\begin{aligned}\varphi(P)\psi(Q) = \sum\nolimits_{|\alpha|<m} \frac{(-i)^{|\alpha|}}{\alpha!}\psi^{(\alpha)}(Q)\varphi^{(\alpha)}(P) + \\ + \sum\nolimits_{|\alpha|=m} \frac{(-i)^{|\alpha)}}{\alpha!(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{iQk}\varphi_\alpha(P, k)(\mathcal{F}\psi^{(\alpha)})(k)\, dk\end{aligned} \tag{65}$$

as operators $\mathcal{S} \to \mathcal{S}^\star$.

Let $\psi \in S^{m-\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$ (i.e. $|\psi^{(\alpha)}(x)| \leqslant c_\alpha < x >^{m-\varepsilon-|\alpha|}$ for all α). Then $\mathcal{F}\psi^{(\alpha)}$ for $|\alpha| = m$ is bounded by $c|x|^{-n+\varepsilon}$ (so it is a locally integrable function) and coincides for $|x| \geqslant 1$ with a function in $\mathcal{S}$. Hence the preceding development may be used.

Take now $\varphi = h$ where h is as above and $u \in \mathcal{S}$:

$$||h(P)\psi(Q)u|| \leqslant \sum_{|\alpha|<m} \frac{1}{\alpha!}||\psi^{(\alpha)}||_{L^\infty(\mathbb{R}^n)}||h^{(\alpha)}(P)u||+ \\ + \sum_{|\alpha|=m} \frac{1}{\alpha!(2\pi)^{n/2}} \int_{\mathbb{R}^n} ||h_\alpha(P,k)u||\,|(\mathcal{F}\psi^{(\alpha)})(k)|\,dk. \tag{66}$$

This implies that

$$||\psi(Q)||_{B(\mathcal{H}^h)} \leqslant c\sum_{|\alpha|<m} ||\psi^{(\alpha)}||_{L^\infty(\mathbb{R}^n)} + c\sum_{|\alpha|=m} ||\mathcal{F}\psi^{(\alpha)}||_{L^1(\mathbb{R}^n)}. \tag{67}$$

But, if $\nu > n/2$ is any real number:

$$||\hat{f}||_{L^1} = ||<Q>^{-\nu}<Q>^{\nu}\hat{f}||_{L^1} \leqslant ||<Q>^{-\nu}||_{L^2}||\mathcal{F}<P>^{\nu}f||_{L^2} = c||f||_{\mathcal{H}^\nu}$$

where $\mathcal{H}^\nu$ is the usual Sobolev space.

To conclude with, we get the estimate we were looking for:

$$||\psi(Q)||_{B(\mathcal{H}^h)} \leqslant c\sum_{|\alpha|<m} ||\psi^{(\alpha)}||_{L^\infty(\mathbb{R}^n)} + c\sum_{|\alpha|=m} ||\psi^{(\alpha)}||_{\mathcal{H}^\nu(\mathbb{R}^n)}. \tag{68}$$

for some constant c depending only on h and on $\nu > n/2$. By duality and interpolation, one may replace here $\mathcal{H}^\nu$ by $\mathcal{G}$ or $\mathcal{G}^\star$.

6. Regularity of an Operator with Respect to a Group

Let us consider a triplet $(\mathcal{G},\mathcal{H};W)$ as in Section 4, i.e. $(\mathcal{G},\mathcal{H})$ is a Friedrichs couple and $W_\alpha = e^{iA\alpha}$ is a strongly continuous unitary group in $\mathcal{H}$ which leaves $\mathcal{G}$ invariant. We recall the identification $\mathcal{G} \subset \mathcal{H} \cong \mathcal{H}^\star \subset \mathcal{G}^\star$ and the notation $\mathcal{X}$ for the Banach space $B(\mathcal{G},\mathcal{G}^\star)$. The action of the group W extends to $\mathcal{X}$

$$\mathcal{W}_\alpha(T) = W_\alpha T W_{-\alpha} \qquad \text{for } T \in \mathcal{X}. \tag{69}$$

Since W acts continuously in $\mathcal{G}$ and $\mathcal{G}^\star$, the function $\alpha \mapsto \mathcal{W}_\alpha(T) \in B(\mathcal{G},\mathcal{G}^\star)$ is strongly continuous, but not norm-continuous in general. Hence $\mathcal{W} = \{\mathcal{W}_\alpha\}_{\alpha\in\mathbb{R}}$ is *not* a C_0-group in the Banach space $\mathcal{X}$. It is, however, a dual group in the sense of Butzer-Berens [BB], because $\mathcal{X}$ may be identified with the dual of the Banach space $\mathcal{X}_\star$ of trace class operators $\mathcal{G}^\star \to \mathcal{G}$ and the topology $\sigma(\mathcal{X},\mathcal{X}_\star)$ coincides with the ultraweak topology of $\mathcal{X}$ which, on bounded subsets, is weaker than the strong topology. To conclude with, there is enough structure in order to develop for $\mathcal{W}$ a theory parallel to that of C_0-groups, the main difference being that the domain of the infinitesimal generator is only strongly dense in $\mathcal{X}$.

We find it suggestive to denote the domain of the infinitesimal generator $\mathcal{A}$ of $\mathcal{W}$ by $C^1(\mathcal{A};\mathcal{X})$. So, we shall say that an operator $T \in \mathcal{X}$ is *of class* $C^1(\mathcal{A};\mathcal{X})$, and we shall write $T \in C^1(\mathcal{A};\mathcal{X})$ if one of the following equivalent conditions is fulfilled:

$$||\mathcal{W}_\alpha(T) - T||_{\mathcal{X}} \leqslant c|\alpha| \quad \text{for some } c < \infty \quad \text{and all } \alpha \in (-1,+1); \tag{70}$$

$$\liminf_{\varepsilon\to+0} \varepsilon^{-1} \|\mathcal{W}_\varepsilon(T) - T\|_{\mathcal{X}} < \infty. \tag{71}$$

The limit $\lim_{\varepsilon\to 0} \dfrac{\mathcal{W}_\varepsilon(T) - T}{i\varepsilon} = [A,T]$ exists in the strong topology of $B(\mathcal{G},\mathcal{G}^*)$; (72)

the sesquilinear form $< Au \mid Tv > - < u \mid TAv >$ on $D(A;\mathcal{G})$ is continuous in the topology induced by $\mathcal{G}$. (73)

The equality in (72) must be interpreted as a definition. Clearly, the operator $\mathcal{A}(T) \equiv [A,T] \in \mathcal{X}$ defined in this way coincides with the continuous operator $\mathcal{G} \to \mathcal{G}^*$ associated to the sesquilinear from (73) and we have for $\alpha \in \mathbb{R}$:

$$\mathcal{W}_\alpha(T) = T + i\int_0^\alpha \mathcal{W}_\beta([A,T])\, d\beta, \tag{74}$$

the integral being considered in the strong topology of $\mathcal{X}$. The proof of the equivalence of the four assertions (70)-(73) is in fact an easy consequence of this formula, which obviously holds in the sense of sesquilinear forms on $D(A;\mathcal{G})$.

One can define a subspace $C^1_n(A;\mathcal{X})$ by requiring that the limit (72) exist in the norm-topology of $\mathcal{X}$ (which is equivalent to the requirement that $\alpha \mapsto \mathcal{W}_\alpha(T) \in \mathcal{X}$ be norm-C^1). In our applications a slightly stronger condition is needed.

Definition 6.1. An operator $T \in \mathcal{X}$ is said to be *of class* $\mathcal{C}^1(A;\mathcal{X})$ (or $\mathcal{C}^1(A;\mathcal{G},\mathcal{G}^*)$) if

$$\int_0^1 \|\mathcal{W}_\alpha(T) + \mathcal{W}_{-\alpha}(T) - 2T\|_{\mathcal{X}} \frac{d\alpha}{\alpha^2} < \infty. \tag{75}$$

The condition $T \in \mathcal{C}^1(A;\mathcal{X})$ is only "slightly" stronger than $C^1_n(A;\mathcal{X})$, a precise meaning of this assertion being the following consequence of Theorem 4.4. from [K]: for $T \in \mathcal{X}$ the limit

$$\lim_{\varepsilon\to 0} \int_\varepsilon^1 (\mathcal{W}_\alpha(T) + \mathcal{W}_{-\alpha}(T) - 2T) \frac{d\alpha}{\alpha^2}$$

exists in norm in $\mathcal{X}$ if and only if $T \in C^1_n(A;\mathcal{X})$.

One may also understand the preceding definition from the point of view of interpolation theory. Let $C^2(A;\mathcal{X})$ be the space of all $T \in C^1(A;\mathcal{X})$ such that $\mathcal{A}(T) \in C^1(A;\mathcal{X})$ also (so that $\mathcal{A}^2(T) = [A,[A,T]] \in \mathcal{X}$ is well defined) and let $C^2_n(A;\mathcal{X})$ be the subspace of those T such that $\mathcal{A}(T) \in C^1_n(A;\mathcal{X})$. Equivalently, $C^2(A;\mathcal{X})$, (resp. $C^2_n(A;\mathcal{X})$ is the space of operators $T \in \mathcal{X}$ such that $\alpha \to \mathcal{W}_\alpha(T) \in \mathcal{X}$ is strongly (resp. norm) of class C^2. $C^1(A;\mathcal{X})$ and $C^2(A;\mathcal{X})$ are Banach spaces with respect to the natural norms

$$\|T\|_{C^1(A;\mathcal{X})} = \|T\|_{\mathcal{X}} + \|\mathcal{A}(T)\|_{\mathcal{X}}$$
$$\|T\|_{C^2(A;\mathcal{X})} = \|T\|_{\mathcal{X}} + \|\mathcal{A}^2(T)\|_{\mathcal{X}}$$

and $C^1_n(A;\mathcal{X}), C^2_n(A;\mathcal{X})$ are their closed subspaces. Taking into account the theory presented in Section 2, one can prove that

$$\mathcal{C}^1(A;\mathcal{X}) = (\mathcal{X}, C^2(A;\mathcal{X}))_{1/2,1} \equiv C^{1,1}(A;\mathcal{X}), \tag{76}$$

with a natural notation. In fact, if we put $C^\circ(A;\mathcal{X}) \equiv \mathcal{X}$ and $C_n^\circ(A;\mathcal{X})$ the closure of $C^1(A;\mathcal{X})$ in $\mathcal{X}$, then $\mathcal{W}_\alpha$ is a $C_\circ$-group in the Banach space $C_n^\circ(A;\mathcal{X})$ and $C_n^2(A;\mathcal{X})$ is the domain of the square of its generator, so that we clearly have

$$\mathcal{C}^1(A;\mathcal{X}) = (C_n^\circ(A;\mathcal{X}), C_n^2(A;\mathcal{X}))_{1/2,1}. \tag{77}$$

Here one can replace $C_n^\circ(A;\mathcal{X})$ by $C^\circ(A;\mathcal{X}) \equiv \mathcal{X}$ because of Theorem 1.6.2 of [T]. Using the notion of relative completion, one may replace $C_n^2(A;\mathcal{X})$ by $C^2(A;\mathcal{X})$. Observe that $C^{1,1}(A;\mathcal{X}) \equiv D_1^1(\mathcal{A}) \subset D(\mathcal{A}) \subset D_\infty^1(\mathcal{A})$, which gives another precise meaning to the assertion "$\mathcal{C}^1$ *is only slightly more restritive than* C^1".

Now let us explain how we verify the condition $T \in \mathcal{C}^1(A;\mathcal{X})$ in applications. Of course, the easiest way is to require $T \in C^2(A;\mathcal{X})$. In our examples, we require this for the kinetic part of the hamiltonian and for the part of the interaction which is locally smooth: essentially no decay at infinity will be needed. But for the part of the interaction which is locally singular, such an assumption implies a stronger decay at infinity than expected (roughly speaking, a decay like $|x|^{-2-\varepsilon}$, or $|x|^{-1-\varepsilon}$ should be sufficient for the short-range part).

In order to make a balance between "regularity of the decay" and "rapidity of the decay" for various parts of the interaction, we shall describe now other methods of verifying the $\mathcal{C}^1$ assumption: the first one applies to the so-called long-range part of the interaction and the second one to the short-range part.

Assume first that $T \in C^1(A;\mathcal{X})$ is known and denote $S = i[A,T] \equiv i\mathcal{A}(T)$. What condition on S implies $T \in \mathcal{C}^1$? Writing

$$\mathcal{W}_\alpha(T) + \mathcal{W}_{-\alpha}(T) - 2T = \int_0^\alpha [\mathcal{W}_\varepsilon(S) - \mathcal{W}_{-\varepsilon\alpha}(S)]\, d\varepsilon,$$

it is trivial to show that a sufficient condition is

$$\int_0^1 \|\mathcal{W}_\varepsilon(S) - S\|_{\mathcal{X}} \frac{d\varepsilon}{\varepsilon} < \infty. \tag{78}$$

In other terms, we see that if the function $\alpha \mapsto \mathcal{W}_\alpha(T) \in \mathcal{X}$ is derivable with Dini continuous (in norm) derivative, then $T \in \mathcal{C}^1(A;\mathcal{X})$. One may also write

$$\mathcal{W}_\varepsilon(S) - S = [W_\varepsilon - 1, S]W_{-\varepsilon} = 2\Big[S, \big(\sin\frac{A\varepsilon}{2}\big)^2\Big]W_{-\varepsilon} + i[\sin A\varepsilon, S]W_{-\varepsilon},$$

hence for S a symmetric operator (78) is obviously a consequence of

$$\int_0^1 \|\sin A\varepsilon \cdot S\|_{\mathcal{X}} \frac{d\varepsilon}{\varepsilon} < \infty. \tag{79}$$

It is slightly more difficult to show that this assumption is a consequence of

$$\int_{r_0}^\infty \|A(A+ir)^{-1}S\|_{\mathcal{X}} \frac{dr}{r} < \infty, \tag{80}$$

where r_0 is chosen so that $A + ir$ is an isomorphism of $\mathcal{F} = D(A;\mathcal{G}^\star)$ onto $\mathcal{G}^\star$ for all $|r| \geqslant r_0 > 0$. To prove this, one shows that $||f(A\varepsilon)||_{B(\mathcal{G}^\star)} \leqslant$ const for $0 < \varepsilon \leqslant 1$ and $f(t) = \sin t + i\,\frac{\sin t}{t}$, and then observes that $\sin A\varepsilon = A\varepsilon(A\varepsilon + i)^{-1} f(A\varepsilon)$ for ε small.

By virtue of the explanations given in Sections 2,3, it is clear that (79) (80) can be interpreted as saying that "S improves the decay at infinity in the spectral representation of A". In fact, both conditions imply $S(\mathcal{G}) \subset D_1^0(A;\mathcal{X})$ and are implied by $S(\mathcal{G}) \subset D_\infty^\sigma(A;\mathcal{G}^\star)$, $\sigma > 0$. We shall now state a much deeper result whose proof may be found in [BG 2].

Theorem 6.2. *Let Λ be a closed operator in $\mathcal{G}^\star$ with domain included in $D(A;\mathcal{G}^\star)$ and having the following property: there are $c, r_0 > 0$ such that ir is in the resolvent set of Λ for $r \geqslant r_0$ and*

$$||(\Lambda + ir)^{-1}||_{B(\mathcal{G}^\star)} \leqslant cr^{-1}.$$

If T is a symmetric operator $\mathcal{G} \to \mathcal{G}^\star$ which is of class $C^1(A;\mathcal{G},\mathcal{G}^\star)$ and if $S = i[A,T]$ satisfies

$$\int_{r_0}^\infty ||\Lambda(\Lambda + ir)^{-1} S||_{\mathcal{X}} \frac{dr}{r} < \infty, \tag{81}$$

then T is of class $C^1(A;\mathcal{G},\mathcal{G}^\star)$.

Assume, furthermore, that Λ generates a C_0-group $\{e^{i\Lambda\alpha}\}_{\alpha\in\mathbb{R}}$ of polynomial growth in $\mathcal{G}^\star$. Let $\xi \in C^\infty(\mathbb{R})$ with $\xi(t) = 0$ near zero and $\xi(t) = 1$ near infinity. Then (81) is a consequence of

$$\int_1^\infty ||\xi(\frac{\Lambda}{r}) S||_{\mathcal{X}} \frac{dr}{r} < \infty. \tag{82}$$

The C^1 assumption for the long range part of the interaction will be verified using this theorem. Let us explain now how we treat the short range part for which we do not want to make any explicit assumption concerning the commutator $[A,T]$.

We shall use the identity

$$\begin{aligned} &W_\varepsilon(T) + W_{-\varepsilon}(T) - 2T = \\ &\quad = (W_\varepsilon - 2 + W_{-\varepsilon})TW_\varepsilon + W_\varepsilon T(W_\varepsilon - 2 + W_{-\varepsilon}) - 2(W_\varepsilon - 1)T(W_\varepsilon - 1) - \\ &\quad - 4\big(\sin\frac{A\varepsilon}{2}\big)^2 TW_\varepsilon - 4W_\varepsilon T\big(\sin\frac{A\varepsilon}{2}\big)^2 + 8W_{\varepsilon/2}\sin\frac{A\varepsilon}{2} T \sin\frac{A\varepsilon}{2} W_{\varepsilon/2}. \end{aligned}$$

Hence a symmetric operator $T : \mathcal{G} \to \mathcal{G}^\star$ is of class C^1 if

$$\int_0^1 [\,||(\sin A\varepsilon)^2 T||_{\mathcal{X}} + ||(\sin A\varepsilon)T(\sin A\varepsilon)||_{\mathcal{X}}]\,\frac{d\varepsilon}{\varepsilon^2} < \infty. \tag{83}$$

As before, one can show that one can replace here $\sin A\varepsilon$ by $A\varepsilon(A\varepsilon + i)^{-1} \equiv A(A + ir)$ with $r = \varepsilon^{-1}$. Observe that (83) implies $T(\mathcal{G}) \subset D_1^1(A;\mathcal{G}^\star)$ and is implied by $T(\mathcal{G}) \subset D_\infty^\sigma(A;\mathcal{G}^\star)$ if $\sigma > 1$, i.e. it means that T improves decay at infinity in the spectral representation of A by one power at least. Then one proves (see [BG 2]).

Theorem 6.3. *Let Λ be a closed, densely defined operator in $\mathcal{G}^*$ such that ir is in the resolvent set of Λ for each $r \geqslant 0$ and $||(\Lambda + ir)^{-1}||_{B(\mathcal{G}^*)} \leqslant c(1+r)^{-1}$. Assume that $D(\Lambda) \subset D(A;\mathcal{G}^*)$ and $D(\Lambda^2) \subset D(A^2;\mathcal{G}^*)$. If T is a symmetric operator $\mathcal{G} \to \mathcal{G}^*$ such that*

$$\int_0^\infty \{||(\frac{\Lambda}{\Lambda + ir})^2 T||_{\mathcal{X}} + ||(\frac{\Lambda}{\Lambda + ir}) T (\frac{\Lambda^*}{\Lambda^* - ir})||_{\mathcal{X}}\} \, dr < \infty, \tag{84}$$

then $T \in \mathcal{C}^1(A;\mathcal{G},\mathcal{G}^*)$.

Assume,furthermore, that Λ generates a C_0-group $\{e^{i\Lambda\alpha}\}_{\alpha\in\mathbb{R}}$ of polynomial growth in $\mathcal{G}^*$ and that $\eta \in C_0^\infty(\mathbb{R})$ with $\eta(t) > 0$ on some region $0 < a < |t| < b < \infty$ and $\eta(t) = 0$ otherwise. Then the finiteness of the first integral in (84) is a consequence of

$$\int_1^\infty ||\eta(\frac{\Lambda}{r}) T||_{\mathcal{X}} \, dr < \infty. \tag{85}$$

As it was already mentioned, we use this result in order to treat the short range part of the interaction. The second integral in (84) is treated by an interpolation argument. In applications we shall take Λ equal to $< Q >$, the operator of multiplication by $< x >= \sqrt{1+|x|^2}$.

7. Conjugate Operator Method

In this section we shall state the main result of our note [BG 1] and then we shall indicate a method of verifying Mourre's estimate. Let H be a self-adjoint operator in the Hilbert space $\mathcal{H}$ and denote $\mathcal{G} \equiv D(|H|^{1/2})$ its form-domain. We assume that $\mathcal{G}$ is equipped with a Hilbert structure equivalent to that induced by the graph-norm associated to $|H|^{1/2}$. For example, we could take $(||u||^2 + < u|\,|H|u >)^{1/2}$ as norm on $\mathcal{G}$. Then $\mathcal{G} \subset \mathcal{H}$ densely and we identify $\mathcal{G} \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{G}^*$ as usual.

H induces a continuous symmetric operator $\mathcal{G} \to \mathcal{G}^*$ which we denote by the same letter. If $\sigma(H)$ is the spectrum of the self-adjoint operator H in $\mathcal{H}$, then for each $z \in \mathbb{C} \setminus \sigma(H)$ the operator $(H - z) : \mathcal{G} \to \mathcal{G}^*$ is an isomorphism and its inverse $R(z) = (H - z)^{-1} \in B(\mathcal{G}^*, \mathcal{G})$ is the unique continuous extension to $\mathcal{G}^*$ of the resolvent of H in $\mathcal{H}$.

We shall introduce now the main object of Mourre's theory. The next definition is adapted to the context in which we work (only the form-domain of H is left invariant by the group W, that is compensated by a stronger condition on the commutator $B = i[H, A]$).

Definition 7.1. Let A be a self-adjoint operator in $\mathcal{H}$ and $I \subset \mathbb{R}$ an open subset. We shall say that *A is conjugate to H on I (in the form-sense)* if the next three conditions are fulfilled:

(1) The unitary group $W_\alpha = e^{iA\alpha}$ generated by A leaves $\mathcal{G}$ invariant;

(2) For each $u \in \mathcal{G}$, the derivative

$$\frac{d}{d\alpha} < W_\alpha u | H W_\alpha u > |_{\alpha=0} \equiv < u \,|\, Bu > \tag{86}$$

exists;

(3) There is a strictly positive number a and a compact operator $K : \mathcal{G} \to \mathcal{G}^\star$ such that for all $u \in \mathcal{G}$ with $E(I)u = u$:

$$< u \,|\, Bu >\geqslant a||u||^2 + < u \,|\, Ku > . \tag{87}$$

Here E is the spectral measure of H and (87) is the Mourre estimate (in a form suitable for our situation). By the closed graph theorem (and polarization formula), B is a continuous symmetric operator $\mathcal{G} \to \mathcal{G}^\star$ and (87) is equivalent to

$$E(I)BE(I) \geqslant aE(I) + K \tag{88}$$

as operators in $B(\mathcal{G}, \mathcal{G}^\star)$. If we have such an estimate with $K = 0$, we shall say that A *is strictly conjugate to H on I* (in form-sense).

Let us observe that W is a unitary group in the Friedrichs couple $(\mathcal{G}, \mathcal{H})$, in the sense of Section 4. Hence we can introduce the Banach space $\mathcal{E} = (\mathcal{G}^\star, D(A;\mathcal{G}^\star))_{1/2,1}$ satisfying $\mathcal{E} \subset \mathcal{G}^\star$ densely, so that $\mathcal{G} \subset \mathcal{E}^\star$. At the operator level, this implies that $B(\mathcal{G}^\star, \mathcal{G}) \subset B(\mathcal{E}, \mathcal{E}^\star)$. Since $R(z) \in B(\mathcal{G}^\star, \mathcal{G})$, we may then consider the function

$$\mathbb{C} \setminus \sigma(H) \in z \mapsto R(z) \in B(\mathcal{E}, \mathcal{E}^\star) \tag{89}$$

which is obviously holomorphic. Our main result concerns the boundary values on some real intervals of this function.

Theorem 7.2. *Assume that H has a conjugate operator A (in the form sense) on the open real set I and that H is of class $C^1(A; \mathcal{G}, \mathcal{G}^\star)$. Then H has in I at most a finite number of eigenvalues (counting multiplicities) and no singularly continuous spectrum. Let I° be the complement in I of the set of eigenvalues of H and $I^\circ_\pm = \{z \in \mathbb{C} \,|\, Re\, z \in I^\circ,\ \pm Im\, z \geqslant 0\}$. Then the function (89) (i.e. the resolvent of H considered as a $B(\mathcal{E}, \mathcal{E}^\star)$-valued application) extends to a weak*-continuous function on $I^\circ_\pm$.*

Let us recall that $\mathcal{E}^\star = (\mathcal{F}^\star, \mathcal{G})_{1/2,\infty}$ where $\mathcal{F} = D(A;\mathcal{G}^\star)$ has the property $\mathcal{G} \subset \mathcal{F}^\star$ (in fact $\mathcal{F}^\star$ is the completion of $\mathcal{G}$ for the norm $||(A+ir)^{-1}u||_\mathcal{G}$ if $r \in \mathbb{R}$ is large enough). We have $\mathcal{G} \subset \mathcal{E}^\star$ non-densely in general and we denote by $\mathcal{E}^{\star\circ}$ the closure of $\mathcal{G}$ in $\mathcal{E}^\star$. As we explained in Section 2, $(\mathcal{E}^{\star\circ})^\star = \mathcal{E}$. Observe that $R(\lambda \pm i\mu) \in B(\mathcal{G}^\star, \mathcal{G}) \subset B(\mathcal{E}, \mathcal{E}^{\star\circ})$ if $\mu \neq 0$ (but the weak*-limits only belong to $B(\mathcal{E}, \mathcal{E}^\star)$ in general).

Let us say that an operator $T : D(T) \subset \mathcal{H} \to \mathcal{K}$ ($\mathcal{K}$ a Hilbert space) is *locally H-smooth on an open real set J* if $TE(K)$ is H-smooth for each compact subset K of J (see Section XIII.7 in [RS]). Assume that $T : \mathcal{E}^{\star\circ} \to \mathcal{K}$ is a bounded operator, so that $T^\star : \mathcal{K} \cong \mathcal{K}^\star \to \mathcal{E}$ is also bounded. Then, under the assumptions of the theorem, we have for $\lambda \in I^\circ$ and $\mu \neq 0$

$$||TR(\lambda + i\mu)T^\star||_{B(\mathcal{K})} \leqslant ||T||_{B(\mathcal{E}^{\star\circ},\mathcal{K})} ||R(\lambda + i\mu)||_{B(\mathcal{E},\mathcal{E}^\star)} ||T^\star||_{B(\mathcal{K},\mathcal{E})},$$

and the right-hand side is uniformly bounded when λ belongs to a compact subset of I° and $0 < |\mu| \leqslant 1$. Theorem XIII.30 from [RS] implies then (observe that $D(H) \subset \mathcal{G} \subset \mathcal{E}^{\star\circ}$):

Corollary 7.3. *Under the assumptions of the theorem, each bounded operator* $T : \mathcal{E}^{*\circ} \to \mathcal{K}$ *($\mathcal{K}$ is a Hilbert space) is locally H-smooth on I°.*

As we explained in Section 3, the space $\mathcal{E} = (\mathcal{G}^*, \mathcal{F})_{1/2,1}$ is of cotype 2, because $\mathcal{G}^*$ and $\mathcal{F}$ are Hilbert spaces. Moreover, it has the bounded approximation property. Hence, if $\mathcal{E}_j = (\mathcal{G}_j^*, \mathcal{F}_j)_{1/2,1}$ are two spaces similar to $\mathcal{E}$, Theorem 3.5 of G.Pisier says that every bounded operator $V : \mathcal{E}_1^{*\circ} \to \mathcal{E}_2$ can be written as $V = UT$ where T is a continuous operator from $\mathcal{E}_1^{*\circ}$ to a Hilbert space $\mathcal{K}$ and $U : \mathcal{K} \to \mathcal{E}_2$ is also continuous. Then $U^* = \mathcal{E}_2^* \to \mathcal{K}^* \cong \mathcal{K}$ and if we define $S = U^*|_{\mathcal{E}_2^{*\circ}}$, we get $S \in B(\mathcal{E}_2^{*\circ}, \mathcal{K})$ and $S^* = U$. To conclude, $V = S^*T$ with $T \in B(\mathcal{E}_1^{*\circ}, \mathcal{K})$ and $S \in B(\mathcal{E}_2^{*\circ}, \mathcal{K})$. Now it is straightforward to apply Theorem XIII.31 from [RS] and to prove

Theorem 7.4. *Let H_1, H_2 be two self-adjoint operators on the Hilbert space $\mathcal{H}$ and $I \subset \mathbb{R}$ an open subset. Assume that H_j fulfill the assumptions of the preceding theorem with respect to operators A_j and $\mathcal{G}_j = D(|H_j|^{1/2})$; denote by $\mathcal{E}_j$ the corresponding spaces constructed according to (47). Finally, assume that there is a continuous operator $H_{12} : \mathcal{E}_1^{*\circ} \to \mathcal{E}_2$ such that for all $u_j \in D(H_j)$:*

$$< H_1 u_1 \,|\, u_2 > - < u_1 \,|\, H_2 u_2 > = < H_{12} u_1 \,|\, u_2 > . \tag{90}$$

Let us denote by E_j^c continuous components of the spectral measure of H_j. Then the following wave operators exist:

$$W_1^{\pm} = s - \lim_{t \to \pm\infty} e^{iH_2 t} e^{-iH_1 t} E_1^c(I), \tag{91}$$

$$W_2^{\pm} = s - \lim_{t \to \pm\infty} e^{iH_1 t} e^{-iH_2 t} E_2^c(I), \tag{92}$$

and are complete, i.e.

$$(W_1^{\pm})^* = W_2^{+}, \qquad W_2^{+} W_1^{\pm} = E_1^c(I), \qquad W_1^{\pm} W_2^{\pm} = E_2^c(I). \tag{93}$$

We finish this section with two results which facilitate the application of these theorems. The first lemma is a variant of a result of Mourre and will be used in order to prove the invariance of $\mathcal{G}$ under W. In applications, W does not leave the domain of H invariant, but does leave invariant the domain of an operator H_0 such that $D(|H_0|^{1/2}) = D(|H|^{1/2}) \equiv \mathcal{G}$.

Lemma 7.5. *Let H_0 be a self-adjoint operator in $\mathcal{H}$, $\{W_\alpha\}_{\alpha \in \mathbb{R}}$ a strongly continuous unitary group in $\mathcal{H}$ and $D \subset D(H_0)$ a core of H_0 such that $W_\alpha D \subset D$ for all α, the function $\alpha \mapsto \|W_\alpha u\|_{D(H_0)}$ is locally bounded if $u \in D$, and for each $u, v \in D$, the derivative $|\frac{d}{d\alpha} < v \,|\, W_\alpha^* H_0 W_{+\alpha} u > |_{\alpha=0}$ exists and satisfies an estimate:*

$$\left| \frac{d}{d\alpha} < v \,|\, W_\alpha^* H_0 W_{+\alpha} u > |_{\alpha=0} \right| \leqslant c \|v\| (\|u\|^2 + \|H_0 u\|^2)^{1/2}. \tag{94}$$

Then $D(H_0)$ is invariant under W.

Proof. Clearly, there is a bounded operator $B_\circ : D(H_\circ) \to \mathcal{H}$ such that

$$< v \,|\, B_\circ u > = \frac{d}{d\alpha} < v \,|\, W_\alpha^* H_\circ W_{+\alpha} u > |_{\alpha=0} \qquad \text{if } u, v \in D;$$

in fact, $||B_\circ||_{B(D(H_\circ);\mathcal{H})} \leqslant c$. The group property shows that $\alpha \mapsto < v \,|\, W_{-\alpha} H_\circ W_{+\alpha} u >$ is everywhere derivable with derivative equal to $< v \,|\, W - \alpha B_\circ W_{+\alpha} u >$, which is bounded by $||v||c||W_\alpha u||_{D(H_\circ)}$, so is locally bounded by hypothesis. We get

$$< v \,|\, W_{-\alpha} H_\circ W_{+\alpha} u > = < v \,|\, H_\circ u > + \int_\circ^\alpha < v \,|\, W_{-\tau} B_\circ W_\tau u > d\tau$$

for all $\alpha \in \mathbb{R}$ and $u, v \in D$. Replacing v by $W_{-\alpha} v$, we get

$$< v \,|\, H_\circ W_\alpha u > = < v \,|\, W_\alpha H_\circ u > + \int_\circ^\alpha < v \,|\, W_{\alpha-\tau} B_\circ W_\tau u > d\tau. \tag{95}$$

Since D is dense in $\mathcal{H}$, we easily deduce that

$$||H_\circ W_\alpha u|| \leqslant ||H_\circ u|| + \int_\circ^\alpha ||B_\circ W_\tau u||\, d\tau \leqslant ||H_\circ u|| + c \int_\circ^\alpha ||W_\tau u||_{D(H_\circ)}\, d\tau. \tag{96}$$

So, there is a constant c' such that

$$||W_\alpha u||_{D(H_\circ)} \leqslant c'||u||_{D(H_\circ)} + c' \int_\circ^\alpha ||W_\tau u||_{D(H_\circ)}\, d\tau.$$

Using Gronwall's lemma we get

$$||W_\alpha u||_{D(H_\circ)} \leqslant c' e^{c'|\alpha|} ||u||_{D(H_\circ)}. \quad \bullet$$

Remark. Sometimes it is important to know that the $C_\circ$-group induced by W in $D(H_\circ)$ is of polynomial growth. We would like to mention here a sufficient condition for this. Let us keep the notations of the lemma and of its proof, i.e. $B_\circ = i[H_\circ, A]$. As we said, the assumption (94) is equivalent to the boundedness of the operator $B_\circ : D(H_\circ) \to \mathcal{H}$. Assume that there is $\theta < 1$ such that $B_\circ$ is a bounded operator $D(|H_\circ|^\theta) \to \mathcal{H}$. Then the group induced by W in $D(H_\circ)$ is of polynomial growth; more precisely, if we put $m = (1-\theta)^{-1}$, then

$$||W(\alpha)||_{D(H_\circ)} \leqslant c < \alpha >^m . \tag{97}$$

Proof. For each $v \in D(H_\circ)$ we have

$$||B_\circ v|| \leqslant c|| < H_\circ >^\theta v|| \leqslant c|| < H_\circ > v||^\theta ||v||^{1-\theta} = c||v||^\theta_{D(H_\circ)} ||v||^{1-\theta}.$$

Using this in (96) and taking into account the unitarity of W in $\mathcal{H}$, we get

$$||W_\alpha u||_{D(H_\circ)} \leqslant c||u||_{D(H_\circ)} + c||u||^{1-\theta} \int_\circ^\alpha ||W_\tau u||^\theta_{D(H_\circ)}\, d\tau.$$

Now, instead of the usual Gronwall lemma, we use what we have called θ-Gronwall lemma in [BGM]. A particular case of that estimate says that $0 \leqslant \xi(t) \leqslant a + \int_\circ^t \varphi(s)\xi(s)^\theta\, ds$ with $0 < \theta < 1$ implies that

$$\xi(t) \leqslant \left[a^{1-\theta} + (1-\theta) \int_\circ^t \varphi(s)\right]^{(1-\theta)^{-1}}$$

from which (97) follows. $\bullet$

The final result gives a perturbation method of verifying Mourre's estimate.

Proposition 7.6. *Let $H_\circ$ be a self-adjoint operator in the Hilbert space $\mathcal{H}$, $\mathcal{G} = D(|H_\circ|^1/2)$ the form-domain of $H_\circ$, $J \subset \mathbb{R}$ an open subset and A an operator conjugate to $H_\circ$ (in the form-sense) on J. Let $V : \mathcal{G} \to \mathcal{G}^\star$ be a compact, symmetric operator such that $\alpha \mapsto W_\alpha V W_{-\alpha} \in B(\mathcal{G}, \mathcal{G}^\star)$ is norm-derivable (e.g. $V \in C^1(A; \mathcal{G}, \mathcal{G}^\star)$). Let $H = H_\circ + V$ be the form-sum. Then A is conjugate to H on each open subset of J having compact closure in J.*

Proof. Let us remark first that for each $\varphi \in C_\circ^\infty(\mathbb{R})$ the operator $\varphi(H) - \varphi(H_\circ)$ is compact from $\mathcal{G}^\star$ to $\mathcal{G}$. An immediate proof of this is a consequence of the formula:

$$\begin{aligned}\varphi(H) = \sum_{k=0}^{n-1} \frac{1}{\pi k!} \int_{\mathbb{R}} \varphi^{(k)}(\lambda) \mathrm{Im}[i^k R(\lambda + i)]\, d\lambda + \\ + \frac{1}{\pi(n-1)!} \int_0^1 \varepsilon^{n-1}\, d\varepsilon \int_{\mathbb{R}} \varphi^{(n)}(\lambda) \mathrm{Im}[i^n R(\lambda + i\varepsilon)]\, d\lambda\end{aligned} \tag{98}$$

valid for $\varphi \in C_0^n(\mathbb{R})$ and $n \geqslant 2$ (see [BG 1]). On the other hand, $[iV, A]$ is a compact operator $\mathcal{G} \to \mathcal{G}^\star$ because it is the norm-limit of the operators $\alpha^{-1}[W_\alpha^\star V W_\alpha - V]$ as $\alpha \to 0$. Conversely, it follows from the integral representation of this quotient in terms of $[iV, A]$ that if $V \in C^1(A; \mathcal{G}, \mathcal{G}^\star)$ and $[iV, A]$ is compact, then $V \in C_n^1(A; \mathcal{G}, \mathcal{G}^\star)$. Then we write:

$$\begin{aligned}\varphi(H)[iH, A]\varphi(H) = (\varphi(H) - \varphi(H_\circ))[iH, A]\varphi(H) + \\ + \varphi(H_\circ)[iH, A](\varphi(H) - \varphi(H_\circ)) + \\ + \varphi(H_\circ)[iV, A]\varphi(H_\circ) + \\ + \varphi(H_\circ)[iH_\circ, A]\varphi(H_\circ).\end{aligned}$$

The first three terms in the right-hand side are compact, so there is $K_\varphi : \mathcal{G}^\star \to \mathcal{G}$ compact such that

$$\varphi(H)[iH, A]\varphi(H) = \varphi(H_\circ)[iH_\circ, A]\varphi(H_\circ) + K_\varphi.$$

Choose a compact subset $J_\circ$ of J and $\varphi \in C_\circ^\infty(J)$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi(x) = 1$ on $J_\circ$. Since A is conjugate to $H_\circ$ on J, there erist $m > 0$ and a new compact operator $K : \mathcal{G}^\star \to \mathcal{G}$ such that

$$\varphi(H)[iH, A]\varphi(H) \geqslant m\varphi(H_\circ)^2 + K = m\varphi(H)^2 + m(\varphi(H_\circ)^2 - \varphi(H)^2) + K.$$

The sum of the last two terms being compact, the proof of the proposition is finished. •

8. Applications to Pseudo-Differential Operators

From now on we work in the framework of Section 5. Our purpose is to study operators of the form $H = H_\circ + V$ with $H_\circ = h(P)$ and V an operator which is small in some sense with respect to $H_\circ$. We shall find local conjugate operators for $H_\circ$ and then we shall formulate conditions on V such as to be able to apply Proposition 7.6.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be of class C^1. Our choice of local conjugate operators A for $H_\circ = h(P)$ is motivated by the following heuristic argument. Clearly:

$$i[h(P), Q_j] = (\partial_j h)(P).$$

Hence, if we denote h' the vector field $(\partial_1 h, \dots, \partial_n h) : \mathbb{R}^n \to \mathbb{R}$, we get

$$i[h(P), \frac{1}{2}(h'(P)Q + Qh'(P))] = |h'(P)|^2 \tag{99}$$

with $h'(P)Q = \sum_{j=1}^{n} (\partial_j h)(P)Q_j$, etc. If $h'(k)$ grows more rapidly than $|k|$ at infinity, some problems can arise in the definition of the self-adjoint operator associated to $\frac{1}{2}(h'(P)Q + Qh'(P))$ in $\mathcal{H}$. Hence it is convenient to use a cut-off function $\zeta : \mathbb{R}^n \to \mathbb{R}$ and to take

$$F = \zeta h' : \mathbb{R}^n \to \mathbb{R}^n, \tag{100}$$

$$A = \frac{1}{2}(F(P)Q + QF(P)) = QF(P) - \frac{1}{2}f(P), \tag{101}$$

where $f = \operatorname{div} F$. Particular cases of this choice appear in [M], [A], [BMP]. Clearly,

$$[ih(P), A] = \zeta(P)|h'(P)|^2. \tag{102}$$

If $I \subset \mathbb{R}$ is an open set, then the spectral projection of the self-adjoint operator $h(P)$ associated to I is $E_\circ(I) = \chi_I \circ h(P) = \chi_{\Omega(I)}(P)$, where χ_M is the characteristic function of a set M and $\Omega(I) = h^{-1}(I)$. Hence

$$E_\circ(I)[ih(P), A]E_\circ(I) = \zeta(P)\,|h'(P)|^2 \chi_{\Omega(I)}(P). \tag{103}$$

If we demand this to be $\geqslant mE_\circ(I)$ for some $m > 0$, we get a condition of the form:

$$\zeta(k)|h'(k)|^2 \geqslant m > 0 \qquad \text{if } h(k) \in I. \tag{104}$$

Before going into the formal aspects of the theory, let us make some comments on the physical interpretation of the conditions which will appear below. The variable $k \in \mathbb{R}^n$ is interpreted as the momentum of the physical system, so that $h(k)$ is the kinetic energy of the system when the momentum is k. From the classical hamiltonian equations we see that $h'(k)$ is the velocity of the system when the momentum is k. The same interpretation can be obtained uhen one considers the quantum equation of motion

$$e^{ih(P)t}Qe^{-ih(P)t} = Q + ih'(P)t$$

because Q is the position observable. Hence a condition of the type

$$\inf\{|h'(k)| \mid k \in \mathbb{R} \quad \text{such that} \quad h(k) \in I\} > 0 \tag{105}$$

can be read in physical terms as follows: when the kinetic energy belongs to I, the velocity is bounded below by a strictly positive constants. It is clear from (102) that we shall have troubles in constructing a conjugate operator at kinetic energies λ such that there is a momentum k with $h(k) = \lambda$ and $h'(k) = 0$. These are the critical values of the function h and are called *threshold* energies in the physical literature. The condition which we shall have to put in order to make a simple spectral analysis of H can be stated in physical terms as follows: if the kinetic energy belongs to a compact disjoint from thresholds, then the velocity is bounded below by a strictly positive constant. An assumption of the form $|h(k)| \to \infty$ (resp. $|h(k)| + |h'(k)| \to \infty$) as $|k| \to \infty$ means: if the kinetic energy (resp. the kinetic energy *and* the velocity) is bounded by a constant, then the momentum is also bounded by a constant.

Let us pass now to the technical details. In the next lemma we describe properties of the operators of the form (101).

Lemma 8.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function, i.e. $F(k) - F(p)| \leqslant c|k-p|$ for some constant c and all $k,p \in \mathbb{R}^n$. Then the operator A defined by (101) is essintially self-adjoint in $\mathcal{H}$ of $\mathcal{S}$. The group $W_\alpha = e^{iA\alpha}$ is equal to $\mathcal{F}^\star \hat{W}_\alpha \mathcal{F}$, where $\hat{W}_\alpha$ is explicitly given by the formula:*

$$(\hat{W}_\alpha u)(k) = (\det \nabla \xi_\alpha(k))^{1/2} u(\xi_\alpha(k));$$

here $\xi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the flow in $\mathbb{R}^n$ assjciated to the vector field - F, i.e. we have for all $\alpha \in \mathbb{R}$ and $k \in \mathbb{R}^n$

$$\frac{d}{d\alpha}\xi_\alpha(k) = -F(\xi_\alpha(k)), \xi_0(k) = k.$$

Proof. It is straightforward to prove that the family of operators $\hat{W}_\alpha$ is in fact a strongly continuous one-parameter unitary group in $\mathcal{H}$. If we denote D the space of Lipschitz functions with compact support in $\mathbb{R}^n$, it is clear that $W_\alpha D = D$ for all α. Moreover, D is contained in the domain of the infinitesimal generator $\hat{A}$ of $\hat{W} = e^{i\hat{A}\alpha}$ and for $u \in D$

$$\hat{A}u = -\frac{1}{2}(F(Q)P + PF(Q))u. \tag{106}$$

A well-known lemma of Nelson (Theorem 1.9 in [D]) implies that D is a core for $\hat{A}$. Since the closure of D in the domain of $\hat{A}$ (provided with the graph-norm) is equal to the closure of $\mathcal{S}$ in the same space, we get that $\mathcal{S}$ is a core for $\hat{A}$ too. Finally, since $\mathcal{F} \star Q\mathcal{F} = P, \mathcal{F} \star P\mathcal{F} = -Q$ and the generator of W_α is $A = \mathcal{F}^\star \hat{A} \mathcal{F}$ all the assertions of the lemma are proved. •

Remark. It is clear that $\xi_\alpha(k) = k$ if $k \notin \operatorname{supp} F$.

If $h : \mathbb{R}^n \to \mathbb{R}$ is any Borel function, then we easily see that

$$W_\alpha h(P) W_{-\alpha} = h(\xi_\alpha(P)) \tag{107}$$

as self-adjoint operators in $\mathcal{H}$. Using this or directly the definition (101) we get at a formal level

$$i[h(P), A] \equiv -\frac{d}{d\alpha} W_\alpha h(P) W_{-\alpha}|_{\alpha=0} = (F\nabla h)(P) \equiv (Fh')(P). \tag{108}$$

For a rigorous proof, we use Lemma 7.5 and get

Lemma 8.2. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 with polynomially bounded derivative. Let $\{W_\alpha\}$ be as in Lemma 8.1 and assume that*

$$|\mathbb{F}(k)h'(k)| \leqslant c < h(k) > \tag{109}$$

for some constant $c < \infty$ and all $k \in \mathbb{R}^n$. Then the domain of $h(P)$ in $\mathcal{H}$ (i.e.$\mathcal{H}^h$) is left invariant by W and the commutator

$$i[h(P), A] = (Fh')(P) \tag{110}$$

is a bounded operator $\mathcal{H}^h \to \mathcal{H}$.

For the rest of this section, we fix a real function $h \in C^\infty(\mathbb{R}^n)$ with the following properties:

1) $lh(k)| \to \infty$ as $|k| \to \infty$;

2) there is an integer $m \geqslant 1$ such that the derivatives of h of order $\geqslant m$ are bounded;

3) $\sum_{|\alpha| \leqslant m} |h^{(\alpha)}(k)| \leqslant c < h(k) >$ for some $c < \infty$.

All hypoelliptic polynomials (which depend effectively on all the variables) of degree m have these properties. The spaces $\mathcal{H}^h$ and $\mathcal{G}$ are defined in Section 5. So $\mathcal{G}$ is the form-domain of the hamiltonian $H_0 = h(P)$, while $\mathcal{H}^h$ is its domain. Observe that for any bounded open $I \subset \mathbb{R}$, $\Omega(I) = h^{-1}(I)$ is a bounded open set in $\mathbb{R}^n$. Let us fix a bounded open interval I whose closure does not contain critical values of h. Let $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta = 1$ on I and such that $\operatorname{supp}\theta$ is disjoint from critical values of h. It is clear that $|h'(k)| \geqslant \text{const.} > 0$ if $\theta(h(k)) \neq 0$ and $\theta(h) \equiv \theta \circ h \in C_0^\infty(\mathbb{R}^n)$. We shall take A as in (101) with

$$F(k) = \frac{\theta(h(k))}{|h'(k)|^2} h'(k). \tag{111}$$

Then $F \in C_0^\infty(\mathbb{R}^n)$ and

$$[iH_0, A] = \theta(h(P)), \tag{112}$$

$$[[H_0, A], A] = -\theta(h(P))\theta'(h(P)) \tag{113}$$

are both bounded operators in $\mathcal{H}$, so that H_0 is of class $C^2(A; \mathcal{G}, \mathcal{G}^\star)$ (much more in fact). Also:

$$E_0(I)[iH_0, A]E_0(I) = \theta(h(P))\chi_I(h(P)) = E_0(I) \tag{114}$$

so that A is strictly conjugate to H_0 on I.

We now study $H = H_0 + V$ by using Proposition 7.6. So, we require that V be a compact symmetric operator $\mathcal{G} \to \mathcal{G}^\star$ and belong to $C^1(A; \mathcal{G}, \mathcal{G}^\star)$. Then there is a unique self-adjoint operator H in $\mathcal{H}$ such that $D(|H|^{1/2}) = D(|H_0|^{1/2}) = \mathcal{G}$ and $H = H_0 + V$ as operators $\mathcal{G} \to \mathcal{G}^\star$ (H is the so-called form-sum of H_0 and V). The conclusions of Theorem 7.2 will hold for H on I, the conjugate operator being the A defined above.

We shall now impose some explicit and simple conditions on V in order to satisfy the regulatity assumption $V \in C^1(A; \mathcal{G}, \mathcal{G}^\star)$. We decompose V is a long-range part V_L and a shortrange part V_S and treat them differently, as explained in Section 5.

We shall first deal with the short-range part, a symmetric operator $V_S : \mathcal{G} \to \mathcal{G}^\star$. In the first part of Theorem 6.3 we shall take $\Lambda = \langle Q\rangle$. Since $A = F(P)Q - \frac{i}{2}f(P)$ and

$$A^2 = (F(P)Q)(F(P)Q) - \frac{i}{2}F(P)Qf(P) - \frac{1}{2}f(P)F(P)Q + \frac{1}{4}f^2(P) =$$
$$= \sum_{j=1}^{n} F_j(P)F(P)Q_jQ + i\sum_{j=1}^{n} F_j(P)(\partial_j F)(P)Q -$$
$$-\frac{i}{2}F(P)f(P)Q + \frac{1}{2}F(P)(\nabla f)(P) - \frac{i}{2}f(P)F(P)Q + \frac{1}{4}f^2(P)$$

we get $D(\Lambda;\mathcal{G}^\star) \subset D(A;\mathcal{G}^\star)$ and $D(\Lambda^2;\mathcal{G}^\star) \subset D(A^2;\mathcal{G}^\star)$. The estimate $\|(\Lambda + ir)^{-1}\|_{B(\mathcal{G}^\star)} \leqslant c(1+r)^{-1}$ follows from the last inequality Section 5. It is also trivial to verity that the norm of $(\Lambda + ir)(\Lambda + r)^{-1}$ in $\mathcal{G}$ and $\mathcal{G}^\star$ is bounded uniformly in r. Hence Theorem 6.3 shows that a sufficient condition in order to have $V_S \in \mathcal{C}^1(A;\mathcal{G},\mathcal{G}^\star)$ is

$$\int_0^\infty [\|(\frac{\langle Q\rangle}{\langle Q\rangle + r})^2 V_S\|_{\mathcal{X}} + \|\frac{\langle Q\rangle}{\langle Q\rangle + r} V_S \frac{\langle Q\rangle}{\langle Q\rangle + r}\|_{\mathcal{X}}]dr < \infty. \tag{115}$$

We shall prove that the second term here is bounded by the first one. Let $\psi(x) = \frac{\langle x\rangle}{\langle x\rangle + r}$. Then is in $S^0(\mathbb{R}^n)$ and satisties an estimate of the form $|\psi^{(\alpha)}(x)| \leqslant c_\alpha \langle x\rangle^{-|\alpha|}$ with c_α independent of $r \geqslant 0$. Let $T \in \mathcal{X}$ be symmetric; we shall find an estimate $\|\psi(Q)T\psi(Q)\|_{\mathcal{X}} \leqslant M\|\psi^2(Q)T\|_{\mathcal{X}}$ with M independent of r and T. This is equivalent to

$$|\langle \psi u|T\psi v\rangle| \leqslant M\|\psi^2(Q)T\|_{\mathcal{X}}\|u\|_{\mathcal{G}}\|v\|_{\mathcal{G}}$$

for all $u,\ v \in \mathcal{S}$. We shall prove this by complex interpolation. Let $z = x + iy \in \mathbb{C}$ with $0 \leqslant x \leqslant 2$ and

$$f(z) = \langle \psi^{z^*} u|T\psi^{2-z}v\rangle e^{(z-1)^2}.$$

Then f is a continuous function for $0 \leqslant x \leqslant 2$ and holomorphic in $0 < x < 2$. One may prove that $\|\psi^z(Q)\|_{B(\mathcal{G})} \leqslant c_r e^{c_r|z|}$ by using (68). Or one may use the estimate

$$\|\psi^z\|_{B(\mathcal{G})} = \|e^{z ln\psi}\|_{B(\mathcal{G})} \leqslant \exp[|z|\|\ln\psi\|_{B(\mathcal{G})}]$$

and observe that $\ln\psi(Q)$ is a bounded operator in $\mathcal{G}$ for each $r \geqslant 0$. Any way, we get $|f(z)| \leqslant \text{const}\cdot e^{-y^2/2}$. By the maximum modulus principle:

$$|f(1)| = |\langle \psi u|T\psi v >| \leqslant \max\{\sup_{y\in\mathbb{R}}|f(iy)|, \sup_{y\in\mathbb{R}}|f(2+iy)|\} =$$
$$= e\max\{\sup_{y\in\mathbb{R}} e^{-y^2}|\langle \psi^{ij}u|T\psi^2\psi^{iy}v\rangle|, \sup_{y\in\mathbb{R}} e^{-y^2}|\langle \psi^{iy}u|\psi^2T\psi^{iy}v\rangle|\} \leqslant$$
$$\leqslant e[\sup_{y\in\mathbb{R}}{}^{e^{-y^2}}\|\psi(Q)^{iy}\|_{B(\mathcal{G})}]\|T\psi^2\|_{\mathcal{X}}\|u\|_{\mathcal{G}}\|v\|_{\mathcal{G}}$$

because $(\psi^2 T)^\star = T\psi^2$. Using the group property of ψ^{iy}, it is easy to show that the expression in brackets is bounded by $N\exp(\ln N)^2$, with

$$\sup_{-1\leqslant y\leqslant 1} \|\psi(Q)^{iy}\|_{B(\mathcal{G})} = N.$$

Hence we just have to show that N is bounded by a constant independent of r. We have $\psi^{iy} = e^{iy\ln\psi}$ and observe that $\varphi \equiv \ln\psi = \ln\langle Q\rangle - \ln(\langle Q\rangle + r)$, so that $\partial_j\varphi \in S^{-1}$ and $|\varphi^{(\alpha)}(x)| \leqslant c_\alpha\langle x\rangle^{-|\alpha|}$ for $|\alpha| \geqslant 1$ with c_α independent or r. Since $\partial_j\psi^{iy} = iy\psi^{iy}\partial_j\varphi$, we obtain by induction $|\partial^\alpha\psi^{iy}(x)| \leqslant c_\alpha\langle x\rangle^{-|\alpha|}$ for $|\alpha| \geqslant 0$ with c_α independent of $r(|y| \leqslant 1)$. Using (68), we get

$$\|\frac{\langle Q\rangle}{\langle Q\rangle + r}T\frac{\langle Q\rangle}{\langle Q\rangle + r}\|_{\mathcal{X}} \leqslant M\|(\frac{\langle Q\rangle}{\langle Q\rangle + r})^2 T\|_{\mathcal{X}}$$

with M independent of $r \geqslant 0$ (and of T) for T symmetric.

We have isolated one class of perturbations of class $C^1(A;\mathcal{G},\mathcal{G}^\star)$ of H_0:

Definition 8.3. We say that a symmetric operator $V_S : \mathcal{G} \longrightarrow \mathcal{G}^\star$ is of short-range if

$$\int_1^\infty \|(\frac{\langle Q\rangle}{\langle Q\rangle + r})^2 V_S\|_{\mathcal{X}} dr < \infty. \tag{117}$$

Remark. From the developments of Sections 3 and 6 we obtain the following equivalent condition in order that a symmetric operator $V_S : \mathcal{G} \longrightarrow \mathcal{G}^\star$ be of short-range: there is a function $\eta \in C_0^\infty(\mathbb{R}^n)$, such that $\eta(x) > 0$ in some region $0 < a < |x| < b < \infty$ and $\eta(x) = 0$ otherwise, such that

$$\int_1^\infty \|\eta(\frac{Q}{r})V_S\|_{\mathcal{X}} dr < \infty. \tag{118}$$

Let us consider now the long-range part of V, which is a symmetric operator V_L : $\mathcal{G} \longrightarrow \mathcal{G}^\star$. We have $\mathcal{S} \subset \mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^\star \subset \mathcal{S}^\star$, all the embeddings being dense. Then we obviously have in $B(\mathcal{S},\mathcal{S}^\star)$:

$$[A, V_L] = \sum_{j=1}^n \{[Q_j, V_L]F_j(P) + Q_j[F_j(P), V_L]\} - \frac{i}{2}[f(P), V_L]. \tag{119}$$

The group W leaves $\mathcal{S}$ invariant and the group induced in $\mathcal{S}$ is strongly continuous. Moreover, $\mathcal{S}$ is invariant under A. So, for any operator $T \in B(\mathcal{G},\mathcal{G}^\star)$ it is easy to show that $T \in C^1(A;\mathcal{G},\mathcal{G}^\star)$ if and only if $[A,T]$ (which is well defined on $\mathcal{S}$ with values on $\mathcal{S}^\star$) belongs to $B(\mathcal{G},\mathcal{G}^\star)$ (see Section 1 in [ABG] for details). Then, for $g \in \mathcal{S}$ it is straightforward to show that

$$[g(P), T] = \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^n \int_0^1 dt \int_{\mathbb{R}^n} dx e^{iP\chi t}[P_j, T]e^{+iPx(1-t)}(\mathcal{F}\partial_j g)(x)$$

as operators $\mathcal{S} \longrightarrow \mathcal{S}^\star$. Hence, if $[P_j, T] \in B(\mathcal{G}, \mathcal{G}^\star)$ for all j, then also $[g(P), T] \in B(\mathcal{G}, \mathcal{G}^\star)$ and

$$\|[g(P), T]\|_{\mathfrak{X}} \leqslant \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} \|[P_j, T]\|_{\mathfrak{X}} \|\mathcal{F}\partial_j g\|_{L^1(\mathbb{R}^n)}. \tag{120}$$

In order to apply the theorem of Section 6 to the terms of (119), we shall have to estimate operators of the form $\varphi(Q)[g(P), T]$. Since $\varphi(Q)e^{iPxt} = e^{iPxt}\varphi(Q - xt)$, we get:

$$\|\varphi(Q)[g(P), T]\|_{\mathfrak{X}} \leqslant \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} \int_0^1 dt \int_{\mathbb{R}^n} dx \|\varphi(Q - xt)[P_j, T]\|_{\mathfrak{X}} |\mathcal{F}\partial_j g(x)|. \tag{121}$$

As we have seen in the short-range case, we may take for the operator Λ in the first part of Theorem 6.2 the operator $\langle Q \rangle$. Taking into account (119), we get

$$\int_1^\infty \|\frac{\langle Q \rangle}{\langle Q \rangle + r}[A, V_L]\|_{\mathfrak{X}} \frac{dr}{r} \leqslant \sum_{j=1}^{n} c \int_1^\infty \|\frac{\langle Q \rangle}{\langle Q \rangle + r}[Q_j, V_L]\|_{\mathfrak{X}} \frac{dr}{r} +$$
$$+ \sum_{j=1}^{n} \int_1^\infty \|\frac{\langle Q \rangle Q_j}{\langle Q \rangle + r}[F_j(P), V_L]\|_{\mathfrak{X}} + \frac{1}{2} \int_1^\infty \|\frac{\langle Q \rangle}{\langle Q \rangle + r}[f(P), V_L]\|_{\mathfrak{X}} \frac{dr}{r}$$

where c is a finite constant $(\sup_{j,x} |F_j(x)|)$. We would like to give more explicit bounds for the last two terms. Observe that $\frac{x_j}{\langle x \rangle}$ is a function of class S^0, so it is a bounded operator in $\mathcal{G}^\star$ (cf. end of Section 5). Moreover, F_j and f are of class $C_0^\infty(\mathbb{R}^n)$. So the last two terms above are dominated by terms of the form

$$\int_1^\infty \|\frac{\langle Q \rangle^a}{\langle Q \rangle + r}[g(P), V_L]\|_{\mathfrak{X}} \frac{dr}{r}$$

with $a = 2$, resp. 1 and $g \in C_0^\infty(\mathbb{R}^n)$. Using (121) we obtain

$$\|\frac{\langle Q \rangle^a}{\langle Q \rangle + r}[g(P), V_L]\|_{\mathfrak{X}} \leqslant \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} \int_0^1 dt \int_{\mathbb{R}^n} dx |\mathcal{F}\partial_j g(x)| \| \frac{\langle Q - xt \rangle^a}{\langle Q - xt \rangle + r}[P_j, V_L]\|_{\mathfrak{X}}.$$

Let us denote $y = -xt$ and $V_L^{(j)} = [P_j, V_L]$. Then:

$$\|\frac{\langle Q + y \rangle^a}{\langle Q + y \rangle + r} V_L^{(j)}\|_{\mathfrak{X}} \leqslant \|\frac{\langle Q + y \rangle^a}{\langle Q \rangle^a}\|_{B(\mathcal{G}^\star)} \|\frac{\langle Q \rangle + r}{\langle Q + y \rangle + r}\|_{B(\mathcal{G}^\star)} \|\frac{\langle Q \rangle^a}{\langle Q \rangle + r} V_L^{(j)}\|_{\mathfrak{X}}.$$

In order to estimate the first two factors in the right-hand side, we use the inequality (68) from Section 5. The functions

$$\Psi_y(x) = \frac{\langle x + y \rangle^a}{\langle x \rangle^a}, \quad \Psi_{y,x}(x) = \frac{\langle x \rangle + r}{\langle x + y \rangle + r}$$

are symbols of class S^0 and their derivatives satisfy estimates of the form:

$$|\partial^\alpha \Psi_y(x)| + |\partial^\alpha \Psi_{y,x}(x)| \leqslant c_\alpha \langle y\rangle^{N(\alpha)} \langle x\rangle^{-|\alpha|}$$

with c_α independent of $r \geqslant 0$. Hence, there is $N < \infty$ such that

$$\| \frac{\langle Q+y\rangle^a}{\langle Q+y\rangle + r} V_L^{(j)} \|_{\mathfrak{X}} \leqslant c\langle y\rangle^N \| \frac{\langle Q\rangle^a}{\langle Q\rangle + r} V_L^{(j)} \|_{\mathfrak{X}}$$

from which we obviously get (since $\partial_j g \in S$):

$$\| \frac{\langle Q\rangle^a}{\langle Q\rangle + r} [g(P), V_L] \|_{\mathfrak{X}} \leqslant c(g) \sum_{j=1}^{n} \| \frac{\langle Q\rangle^a}{\langle Q\rangle + r} [P_j, V_L] \|_{\mathfrak{X}}.$$

This suggests the introduction of the following class of perturbations:

Definition 8.4. *A symmetric operator $V_L : \mathcal{G} \longrightarrow \mathcal{G}^\star$ is said to be of long-range if*

$$\sum_{j=1}^{n} \int_1^\infty \{ \| \frac{\langle Q\rangle}{\langle Q\rangle + r} [Q_j, V_L] \|_{\mathfrak{X}} + \| \frac{\langle Q\rangle^2}{\langle Q\rangle + r} [P_j, V_L] \|_{\mathfrak{X}} \} \frac{dr}{r} \leqslant \infty. \tag{122}$$

Remark. The methods of Sections 3 and 6 show that a symmetric operator $V_L \mathcal{G} \longrightarrow \mathcal{G}^\star$ is of longrange if and only if there is a function $\xi \in C^\infty(\mathbb{R}^n)$ with $\xi(x) = 0$ near zero and $\xi(x) = 1$ near infinity such that

$$\sum_{j=1}^{n} \int_1^n \| \xi(\frac{Q}{r} [Q_j, V_L] \|_{\mathfrak{X}} + \| |Q| \xi(\frac{Q}{r}) [P_j, V_L] \|_{\mathfrak{X}} \} \frac{dr}{r}.$$

Taking into account the results of Section 7, we obtain the following theorem:

Theorem 8.5. *Let $\mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class C^∞ such that all its derivatives of order $\geqslant m$ are bounded (for some integer $m \geqslant 1$), $\lim_{k\longrightarrow\infty} |h(k)| = \infty$ and*

$$\sum\nolimits_{|\alpha|\leqslant m} |h^{(\alpha)}(k)| \leqslant c(1 + |h(k)|) \quad \forall k \in \mathbb{R}^n.$$

Let $\mathcal{G} = \{u \in \mathcal{H} | \, |h(P)^{1/2} u \in \mathcal{H}\}$ be the form domain of the self-adjoint operator $\mathbb{H}_0 = h(P)$ in $\mathcal{H} = L^2(\mathbb{R}^n)$. We equip $\mathcal{G}$ with the graph-norm and identify $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^\star$. Assume that $V : \mathcal{G} \longrightarrow \mathcal{G}^\star$ is a compact symmetric operator of the form $V = V_S + V_L$ where V_S, V_L are symmetric operators $\mathcal{G} \longrightarrow \mathcal{G}^\star$ of shortl-range, resp. long-range, in the sense of the definitions given above. Let **H** *be the self-adjoint operator in $\mathcal{H}$ defined by the form-sum $H_0 + V$. Then:*

If I is an open, bounded real interval which is at a strictly positive distance from the critical values of h, then H has at most a finite number of eigenvalues in I (counting multiplicities), no singularly continuous spectrum, and the function $z \longrightarrow (H - z)^{-1} \in B(\mathcal{G}^\star_{1/2,1}, \mathcal{G}_{-1/2,\infty})$, which a priori is defined and holomorphic for $\operatorname{Im} z \neq 0$, extends to a weak*-continuous function

$$I^0_\pm = \{z \in \mathbb{C} | \operatorname{Re} z \in I \quad \text{and is not eigenvalue of} \quad H; \pm \operatorname{Im} z \geqslant 0\}.$$

A simple application of the second theorem from Section 7 is

Theorem 8.6. *Assume that the conditions of the preceding theorem are fulfilled. Let $U : \mathcal{G} \to \mathcal{G}^\star$ be a symmetric, compact short-range operator such that $U \in B(\mathcal{G}^0_{-1/2,\infty}, \mathcal{G}^\star_{1/2,1})$. Define $\tilde{H} = H + U$ (form-sum). Then the relative wave operators of H and $\tilde{H}$ exist and are complete on I (see section 7 for a detailed statement).*

Remark. If $U : \mathcal{G} \longrightarrow \mathcal{G}^\star$ is a local operator (e.g. a differential operator) then one can show that U is short-range in the sense of our definition above if and only if $U \in B(\mathcal{G}^0_{-1/2,\infty}, \mathcal{G}^\star_{1/2,1})$. For the proof, use the method of Hörmander, theorem 14.4.2 [H].

If h is an elliptic symbol of degree $2s > 0$, i.e. $|h(k)| \geqslant c|k|^{2s}$ for some $c > 0$ and k is outside a compact, then $\mathcal{G} = \mathcal{H}^s$, $\mathcal{G}^\star = \mathcal{H}^{-s}$ and we have the limiting absorption principle in the space $B(\mathcal{H}^{-s}_{1/2,1}, \mathcal{H}^{+s}_{1/2,\infty})$. For the Schrödinger case, $s = 1$, we get a very sharp result both in the spectral analysis and in the spectral theory of such operators. Observe that the long-range part may be non-local. One should compare our results with those of Lavine [L]. Moreover, the limiting absorption principle proved in Chapter 30 of [H] where only the elliptic case is treated is largely covered by the preceding theorem (we allow much stronger local singularities and much slower decay at infinity). We insist on the fact that, due to the non-local character of the longrange part, it is not clear that Enss' method is applicable in our case.

Let us write explicitly the assumptions of V_L and V_S in case h is an elliptic symbol of degree $2s > 0$. They must be symmetric, compact operators $\mathcal{H}^s \longrightarrow \mathcal{H}^{-s}$ (where $\mathcal{H}^t = \mathcal{H}^t(\mathbb{R}^n)$) are usual Sobolev spaces) and such that there are functions ξ, η of class C^∞ on $\mathbb{R}^n$ with $\xi(x) = 0$ near zero, $\xi(x) = 1$ near infinity. $\eta(x) > 0$ in some region $0 < a < |x| < b < \infty$ and $\eta(x) = 0$ otherwise, with

$$\sum_{j=1}^{n} \int_1^\infty [\|\xi(\frac{Q}{r})[Q_j, V_L]\|_{B(\mathcal{H}^s,\mathcal{H}^{-s}} + \| |Q|\xi(\frac{Q}{r})[P_j, V_L]\|_{B(\mathcal{H}^s,\mathcal{H}^{-s})}] \frac{dr}{r} + \\ + \int_1^\infty \|\eta(\frac{Q}{r}) V_S\|_{B(\mathcal{H}^s,\mathcal{H}^{-s})} dr < \infty. \tag{124}$$

The following can be observed from the proof we have given: if h is a spherically symmetric function (i.e. $h(k)$ depends only on $|k|$) then $|Q|[P_j, V_L]$ be replaced by $[QP, V_L]$. In fact, in this case A will contain only a radial derivative.

Our final remark refers to Dirac operators. We use the notations and formalism of [$BG3$]. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^3, E; dx)$ where **E** is a 4-dimensional complex Hilbert space. The hamiltonian is:

$$H_0 = \alpha P + m\beta = \sum_{j=1}^{3} \alpha_j P_j + m\beta$$

where α_j, β are Dirac matrices and m is a strictly positive real number. Let $\mu(k) = (m^2 + k^2)^{1/2}$ and

$$\pi_\pm(k) = (2\mu(k))^{-1}[\mu(k) \pm (\alpha \cdot k + m\beta)].$$

Then $\pi_\pm(k)$ are orthogonal projections in E with $\pi_+(k) + \pi_-(k) = 1$ and $\pi_+(k) \cdot \pi_-(k) = 0$. Moreover,

$$\alpha \cdot k + m\beta = \mu(k)\pi_+(k) - \mu(k)\pi_-(k).$$

Hence, $\mu : \mathbb{R}^3 \to [m, \infty)$ is an elliptic symbol of degree $1, \pi_\pm : \mathbb{R}^3 \to B(E)$ are symbols of class S^0 and $H_0 = \mu(P)\pi_+(P) - \mu(P)\pi_-(P)$. The limiting absorption principle for H_0 be trivially established using the elliptic case with $s = 1/2$ because

$$(H_0 - z)^{-1} = (\mu(P) - z)^{-1}\pi_+(P) - (\mu(P) + z)^{-1}\pi_-(P)$$

so that we are reduced to the elliptic symbol $h(k) = \mu(k)$ of degree $2s = 1$. So we get the limiting absorption principle for H_0 in $B(\mathcal{H}^{-1/2}_{1/2,1}, \mathcal{H}^{1/2}_{-1/2,\infty})$ and this is optimal.

In order to treat perturbations $H = H_0 + V$ one needs a relatively explicit expression for the conjugate operator. We shall indicate here only the formal idea. Observe that $\Pi_\pm = \pi_\pm(P)$ are orthogonal projections in $\mathcal{H}$ such that $\Pi_+ + \Pi_- = 1, \Pi_+\Pi_- = 0$ and $H_0 = \mu(P)(\Pi_+ - \Pi_-)$. Let $a = \frac{1}{2}(F(P)Q + QF(P))$ for some $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $A = \Pi_+ a \Pi_+ - \Pi_- a \Pi_-$ (formally). Then

$$[iH_0, A] = i[\mu(P)\Pi_+, A] - i[\mu(P)\Pi_-, A] = i[\mu(P)\Pi_+, \Pi_+ a\Pi_+] + \\ + i[\mu(P)\Pi_-, \Pi_- a\Pi] = \Pi_+[i\mu(P), a]\Pi_+ + \Pi_-[i\mu(P), a]\Pi_-,$$

which, by a convenient choice of a, will provide locally conjugate operators for H.

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