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QUALITY IN SOME DISCRETE MINIMIZATION PROBLEMS

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In this note we construct for minimization problems of the form

$$(1) \qquad \min\left\{\sum_{j=1}^{n} c_j x_j \,\Big|\, \sum_{j=1}^{n} g_j x_j = g_0;\ x_j \in \mathbf{Z},\ x_j \geqslant 0\ (j=1,\ \ldots,\ n)\right\}$$

(where the c_j are commensurable and $g_0, g_1, \ldots, g_n$ are elements of an Abelian group G with $g_1, \ldots, g_n$ as generators) an analogue to the duality problem of linear programming.

If $G \subset \mathbf{Z}^m$ for some m, then (1) is a problem of integral linear programming; if G is finite, then we have the so-called Gomori problem.

Let φ be the homomorphism of $\mathbf{Z}^n$ into G defined by $\varphi(x) = \sum_{j=1}^{n} g_j x_j$ $(x = (x_1, \ldots, x_n) \in \mathbf{Z}^n)$, and $(\mathbf{Z}^n)^+ = \{x \in \mathbf{Z}^n : x \geqslant 0\}$. Then φ maps $(\mathbf{Z}^n)^+$ to the semigroup G^+ of all non-negative integral combinations of $g_1, \ldots, g_n$. Consequently, to every function γ on G^+ there corresponds a function $\varphi^*\gamma$ on $(\mathbf{Z}^n)^+$ defined by $(\varphi^*\gamma)(x) = \gamma(\varphi(x))$.

A function γ on G^+ is called subadditive if $\forall g', g'' \in G^+$: $\gamma(g' + g'') \leqslant \gamma(g') + \gamma(g'')$. Let Γ denote the set of subadditive functions on G^+. We call the following problem dual to (1):

$$(2) \qquad \max\ \{\gamma(g_0) \mid \gamma \in \Gamma;\quad \varphi^*\gamma \leqslant c\}$$

(here $\varphi^*\gamma \leqslant c$ means that $(\varphi^*\gamma)(x) \leqslant \Sigma c_j x_j\ \forall x \in (\mathbf{Z}^n)^+$).

LEMMA. *The problem* (2) *is equivalent to*

$$(3) \qquad \max\{\gamma(g_0) \mid \gamma \in \Gamma;\quad \gamma(g_j) \leqslant c_j\ (j = 1,\ \ldots,\ n)\}.$$

Let $c(x) = \Sigma c_j x_j$. Let us assume that the domains of definition of (1) and (3) are non-empty. The following theorem is analogous to the first duality theorem of linear programming.

THEOREM 1. 1) *If x satisfies the constraints of* (1) *and γ those of* (3), *then* $\gamma(g_0) \leqslant c(x)$.

2) *There are x^* and γ^* satisfying the constraints of* (1) *and* (3), *respectively, such that* $\gamma^*(g_0) = c(x^*)$.

PROOF. 1) Let $x \in (\mathbf{Z}^n)^+$ and $\Sigma g_j x_j = g_0$; $\gamma \in \Gamma$ and $\gamma(g_j) \leqslant c_j$ $(j = 1, \ldots, n)$. Then $\gamma(g_0) \leqslant \Sigma\gamma(g_j)x_j \leqslant \Sigma c_j x_j = c(x)$. 2) We consider the graph F whose vertices are the elements of G^+; n arcs go out from each vertex g joining g to $g + g_j$ $(j = 1, \ldots, n)$. Then to every $x \in (\mathbf{Z}^n)^+$ for which $\Sigma g_j x_j = g$ there corresponds in F a path from 0 to g. We define the length of the arc $(g, g + g_j)$ as c_j (independent of g). Let $\gamma^*(g)$ be the length of the shortest path in F from 0 to g (the existence follows from the assumption that the c_j are commensurable). Then γ^* is subadditive (triangle-inequality) and $\gamma^*(g_j) \leqslant c_j$. If x^* is the shortest path from 0 to g_0, then $\gamma^*(g_0) = c(x^*)$, by definition. This proves the theorem.

COROLLARY. *The x^* and γ^* defined in Theorem* 1 *are a minimum point of* (1) *and a maximum point of* (3), *respectively*.

Let us examine the case when (1) is a problem of integral linear programming. In the numerical methods for solving (1) an important role is played by the cut-off: a hyperplane separating of vertex of the polyhedron $M = \{x \geqslant 0: \Sigma g_j x_j = g_0\}$ at which $\min_M c(x)$ is attained from the set $M \cap (\mathbf{Z}^n)^+$. We say that a cut-off $(f, x) \geqslant f_0$ is stronger than $(f', x) \geqslant f_0'$ if

$$M \cap \{x\colon (f,\ x) \geqslant f_0\} \subset M \cap \{x\colon (f',\ x) \geqslant f_0'\}.$$

THEOREM 2. *For every cut-off we can find a function $\gamma \in \Gamma$ such that the cut-off* $\Sigma\gamma(g_j)x_j \geqslant \gamma(g_0)$ *is at least as strong.*

The proof is similar to that of Theorem 1.

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ON LARGE DIRECT SUMMANDS OF ABELIAN GROUPS

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In his book Fuchs mentions the following conjecture of Irwin: If a primary Abelian group is fully distinguished, that is, if each subgroup of it is distinguished (has an irreducible system of generators), then it has a direct summand that splits into a direct sum of cyclic groups and has the same final rank as the group itself. In this note we generalize the concepts of final rank [1] and of a strictly distinguished group [2] from p-groups to periodic groups. When the final rank is a countable-limit cardinal (that is, if it is of the form $\mathfrak{n} = \sum_{i=1}^{\infty} \mathfrak{n}_i$ with $\mathfrak{n}_i < \mathfrak{n}$ for all i), then our results in the class of periodic groups and mixed groups are stronger than Irwin's conjecture.

All the groups in this note are Abelian.

DEFINITION 1. The *final rank* of a periodic group T is defined as follows: $\operatorname{fin} r(T) = \inf_{n=1,2\ldots} r(nT)$.

DEFINITION 2. A periodic group T is called *strictly distinguished* if nT for every natural number n is distinguished.

LEMMA. *A periodic group T having a direct summand that splits into a direct sum of cyclic groups and has the same final rank as T is strictly distinguished.*

THEOREM 1. *Let T be periodic group whose final rank is a countable-limit cardinal. Then the following assertions are equivalent:*

1. *T is strictly distinguished.*
2. *The basic subgroup B of T has the same final rank as T.*
3. *T has a direct summand that splits into a direct sum of cyclic groups and has the same final rank as T.*

THEOREM 2. *If the periodic part T of a group G is strictly distinguished, $|G/T| \leqslant \operatorname{fin} r(T) = \mathfrak{n}$ and $\mathfrak{n}$ is an uncountable countable-limit cardinal, then G has a direct summand that splits into a direct sum of finite cyclic groups and has the same final rank as T.*

REMARKS. 1. In Theorems 1 nad 2 the condition that the final rank should be a countable-limit cardinal is essential. In fact, for any cardinal numbers $\mathfrak{m}$ and $\mathfrak{n}$ with $\mathfrak{n}^{\aleph_0} \leqslant \mathfrak{m}$ there exists a p-group of cardinal $\mathfrak{m}$ and final rank $\mathfrak{n}^{\aleph_0}$ such that the final rank of any direct summand of it that splits into a direct sum of cyclic groups is 0. Conditions 1 and 2 of Theorem 1 are equivalent even for an arbitrary periodic group T.

2. In Theorem 1 the condition of periodicity and in Theorem 2 the uncountability of the final rank are essential. This is shown by Example III in [3].

3. It is interesting to compare Theorem 1 with Theorem 5 in [4].

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