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On a class of random perturbations of the hierarchical Laplacian

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Abstract. Let (X,d) be a locally compact separable ultrametric space. Given a measure m on X and a function $C(B)$ defined on the set B of all balls of positive measure of X, we consider the hierarchical Laplacian $L=L_C$. The operator L acts on $L^2(X,m)$. It is essentially self-adjoint and has a pure point spectrum. By choosing a family $\{\varepsilon(B)\}$ of independent identically distributed random variables, we define the perturbed function $C(B,\omega)$ and the perturbed hierarchical Laplacian $L^\omega=L_{C(\omega)}$. We study the arithmetic means $\bar\lambda(\omega)$ of the eigenvalues of L^ω. Under some mild assumptions the normalized arithmetic means $(\bar\lambda-\mathbb{E}\bar\lambda)/\sigma[\bar\lambda]$ converge to $N(0,1)$ in distribution. We also give examples when the normal convergence fails. We prove the existence of an integrated density of states. Introducing an empirical point process N^ω for the eigenvalues of L^ω and assuming that the density of states exists and is continuous, we prove that the finite-dimensional distributions of N^ω converge to those of the Poisson point process. As an example we consider random perturbations of the Vladimirov operator acting on $L^2(X,m)$, where $X=\mathbb{Q}_p$ is the ring of p-adic numbers and m is the Haar measure.

Keywords: ultrametric measure space, field of p-adic numbers, hierarchical Laplacian, fractional derivative, Vladimirov Laplacian, point spectrum, integrated density of states, Bernoulli convolutions, Erdős problem, point process, Poisson convergence.

In memory of N. S. Landkof (1915–2004)

§ 1. Introduction

The concepts of hierarchical lattice and hierarchical distance were proposed by Dyson in his famous papers on the phase transitions in a one-dimensional ferromagnetic model with long-range interactions [1], [2]. The notion of the hierarchical Laplacian L, which is closely related to Dyson's model, has been studied in several mathematical papers [3]–[9]. These papers contain some basic information about L (the spectrum, the Markov semigroup, the resolvent and so on) in the

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case when the state space X is discrete and the hierarchical lattice satisfies some symmetry conditions (homogeneity, self-similarity and so on). These conditions enable one to identify the state space (X, m) with a discrete infinitely generated Abelian group G equipped with a translation-invariant ultrametric in such a way that the Markov semigroup $P^t = \exp\{-tL\}$ acting on $L^2(G, m)$ becomes symmetric, translation-invariant and isotropic. In particular, $\mathsf{Spec}(L)$ is pure point and all eigenvalues have infinite multiplicity.

The main goal of the papers mentioned above was to study the corresponding Anderson Hamiltonian $H = L + V$ (the hierarchical Laplacian L plus a random potential V). There was hope of detecting a spectral bifurcation from the pure point spectrum to a continuous one for such operators, that is, to prove Anderson's famous conjecture. Unfortunately, the opposite was true: under mild technical conditions, the hierarchical Anderson Hamiltonian has a pure point spectrum (the phenomenon of localization; see [9] and [5]). Moreover, the local statistics of the spectrum of H is Poissonian (see [6]), which is always deemed a manifestation of spectral localization (see [10] and [11]).

We introduce a new class of operators: the random hierarchical Laplacians L^ω, which demonstrate new spectral effects. The spectrum of such operators is still pure point (with compactly supported eigenfunctions) but in contrast to the deterministic case there is a continuous density of states. This density detects a spectral bifurcation from the pure point spectrum to a continuous one. Locally, the eigenvalues form a Poissonian point process with intensity given by the density of states. We will show that our assumptions on the random variables in the definition of L^ω are almost definitive. Counterexamples demonstrate that all major results do not hold without such assumptions.

A systematic study of the class of isotropic Markov semigroups defined on an ultrametric measure space (X, d, m) was made in [12] (for discrete X) and in the recent paper [13] (where X may contain both isolated and non-isolated points); see also [8] and [14]. This study was motivated by the theory of *random walks on infinitely generated groups* (a classical subject which can be traced back to the pioneering work of Erdős, Spitzer, Kesten, Molchanov, Lawler and others). The two studies mentioned above turned out to be closely related. Namely, given an isotropic Markov semigroup (P^t) defined on an ultrametric measure space (X, d, m) with minus Markov generator L, one can show that L coincides with the hierarchical Laplacian L_C on (X, d, m) for an appropriate choice function $C(B)$ (see the definition below), and vice versa. Then the general theory developed in [12] and [13] applies. In particular, by canonically modifying the underlying ultrametric d to a new ultrametric d_*, one can describe the set $\mathsf{Spec}(L)$ as

$$\mathsf{Spec}(L) = \mathsf{closure}\left\{ \frac{1}{d_*(x,y)} : x \neq y \right\} \cup \{0\}. \tag{1.1}$$

Since the families of d-balls and d_*-balls coincide, these two ultrametrics generate the same topology and the same hierarchical structure and, in particular, the same class of hierarchical Laplacians. In its turn, equation (1.1) yields the following facts that are crucial for our analysis (see [15]). Let $S \subseteq [0, \infty)$ be any given closed set containing 0 as an accumulation point. Assume that X is non-compact and either all points of X are isolated or S is unbounded. Then the following assertions hold.

1) One can find a proper ultrametric d' on X and a choice function $C(B)$ on the set of balls in (X, d') such that $\mathsf{Spec}(L_C) = S$. The ultrametric d' determines the same topology as d.

2) If d is proper and X admits a partition into d-balls containing infinitely many non-singletons, then there is a choice-function $C(B)$ on the set of balls in (X, d) such that $\mathsf{Spec}(L_C) = S$.

The following very simple example shows that the condition 'X admits a partition into d-balls containing infinitely many non-singletons' cannot be dropped in assertion 2): $X = \mathbb{N}$ and $d(m, n) = \max\{m, n\}$ when $m \neq n$ and $d(m, n) = 0$ otherwise.

In the course of study we assume that (X, d) is a locally compact separable ultrametric space. We recall that a metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}, \tag{1.2}$$

which is obviously stronger than the usual triangle inequality. Usually we also assume that the ultrametric d is *proper*, that is, all closed d-balls are compact.

Let m be a Radon measure on (X, d) such that

(a) $m(B) > 0$ for each ball B of positive diameter,

(b) $m(\{x\}) = 0$ if and only if x is a non-isolated point,

(c) $m(X) = \infty$ if X is non-compact.

Let $\mathcal{B}$ be the set of all balls of positive measure. Our assumptions imply that $\mathcal{B}$ is at most countable. Let $C\colon \mathcal{B} \to (0, \infty)$ be a function satisfying the following assumptions (in short, a choice function):

– for every ball $B \in \mathcal{B}$,

$$\lambda(B) := \sum_{T \in \mathcal{B}\colon B \subseteq T} C(T) < \infty;$$

– for every non-isolated point $x \in X$,

$$\sup\{\lambda(B)\colon B \in \mathcal{B} \text{ and } x \in B\} = \infty.$$

Let $\mathcal{D}$ be the set of all locally constant compactly supported functions. Given the data (X, d, m, C), we define (pointwise) the hierarchical Laplacian

$$L_C f(x) := -\sum_{B \in \mathcal{B}\colon x \in B} C(B)(P_B f - f(x)), \qquad f \in \mathcal{D}, \tag{1.3}$$

where

$$P_B f := \frac{1}{m(B)} \int_B f \, dm.$$

The operator $(L_C, \mathcal{D})$ acts on $L^2 = L^2(X, m)$. It is symmetric and admits a complete system of eigenfunctions $\{f_{B,B'}\}_{B \in \mathcal{B}}$, namely,

$$f_{B,B'} = \frac{1}{m(B)} 1_B - \frac{1}{m(B')} 1_{B'}, \tag{1.4}$$

where the $B \subset B'$ are the nearest neighbouring balls (when $m(X) < \infty$, we also put $f_{X,X'} = 1/m(X)$). The eigenvalue $\lambda(B')$ corresponding to $f_{B,B'}$ is

$$\lambda(B') = \sum_{T \in \mathcal{B}:\ B' \subseteq T} C(T); \tag{1.5}$$

when $m(X) < \infty$, we also put $X' = X \cup \{\varpi\}$ and $\lambda(X') = 0$. In particular, we conclude that $(L_C, \mathcal{D})$ is an essentially self-adjoint operator on L^2. We abuse notation and write $(L_C, \mathsf{Dom}_{L_C})$ for its unique self-adjoint extension. For all this we refer to [15].

Observe that we do not need to specify the ultrametric d to define the functions $C(B)$, $\lambda(B)$ and, in particular, the operator $(L_C, \mathsf{Dom}_{L_C})$. What is needed is the family $\mathcal{B}$ of balls, which can evidently be the same for two different ultrametrics d and d'.

On the other hand, given the data (X, d, m) and choosing the function

$$C(B) = \frac{1}{\mathsf{diam}(B)} - \frac{1}{\mathsf{diam}(B')},$$

where $B \subset B'$ are any two nearest neighbouring balls, we obtain the hierarchical Laplacian $(L_C, \mathsf{Dom}_{L_C})$ satisfying

$$\lambda(B) = \frac{1}{\mathsf{diam}(B)}.$$

We will refer to $(L_C, \mathsf{Dom}_{L_C})$ as the standard hierarchical Laplacian associated with the data (X, d, m).

Let us describe the main body of the paper. In § 2 we specify some spectral properties of the hierarchical Laplacian L_C assuming that the ultrametric measure space (X, d, m) where it acts and the Laplacian itself satisfy certain symmetry conditions (homogeneity, self-similarity). As an example, we consider the operator $\mathfrak{D}^\alpha$ of the p-adic fractional derivative of order $\alpha > 0$. This operator is related to p-adic quantum mechanics. It was introduced by Vladimirov (see [16]–[18]). The operator $\mathfrak{D}^\alpha$ is the hierarchical Laplacian acting on $L^2(\mathbb{Q}_p, m)$, where $\mathbb{Q}_p$ is the field of p-adic numbers and m is the Haar measure. The set $\mathsf{Spec}(\mathfrak{D}^\alpha)$ consists of eigenvalues $p^{k\alpha}$, $k \in \mathbb{Z}$ (each of which has infinite multiplicity) and contains 0 as an accumulation point.

In § 3, given a homogeneous Laplacian L_C and a family $\{\varepsilon(B)\}_{B \in \mathcal{B}}$ of symmetric independent identically distributed random variables, we define a perturbed choice function $C(B, \omega)$ and a perturbed Laplacian $L^\omega = L_{C(\omega)}$. This new operator L^ω is the main object of study in spectral statistics.

For every $\omega \in \Omega$ the operator L^ω is a hierarchical Laplacian and, therefore, has a pure point spectrum. On the other hand, it may be non-homogeneous for some ω. In particular, the set of its eigenvalues may be dense in certain intervals. We study the arithmetical means $\bar{\lambda}_O(\omega)$, $O \in \mathcal{B}$, of the eigenvalues of L^ω. Under some mild assumptions, the normalized arithmetic means $(\bar{\lambda}_O - \mathbb{E}\bar{\lambda}_O)/\sigma[\bar{\lambda}_O]$ converge to $N(0,1)$ in distribution as $O \to \varpi$. We also give examples where the normal convergence fails.

In § 4 we study the problem of the existence of the integrated density of states. It turns out that the integrated density of states, whenever it exists, has a remarkable structure: it can be represented as an infinite convolution of probability measures. More precisely, the integrated density of states coincides with the distribution of a random variable $X = \sum A_k X_k$, where the X_k are independent and identically distributed, and the coefficients $A_k > 0$ satisfy $\sum A_k = 1$. Various properties of probability measures of the form $\mu = \mathbb{P}_X$ (that is, infinite convolutions) have been studied by many authors since the 1930s (see, for example, [19]–[23] and the references therein). This classical issue can be traced back to the pioneering work of Erdős, Kerschner, Lévy, Jessen and Wintner. We apply known results on infinite convolutions to study random perturbations of the Vladimirov operator $\mathfrak{D}^\alpha$ by Bernoulli random variables. It turns out that the integrated density of states has an L^2-density for almost all α with $0 < \alpha \leqslant (\log 2)/(\log p)$ and is purely singular for $\alpha > (\log 2)/(\log p)$.

In the concluding § 5 we apply the results of the previous sections to study the empirical process

$$N_O^\omega = \sum_{B \subseteq O} \delta_{\lambda(B,\omega)}$$

associated with the eigenvalues $\lambda(B,\omega)$ of the perturbed operator L^ω, where δ_a is the probability measure taking the value 1 at $\{a\}$. For every finite interval I we define $N_O(I)\colon \omega \mapsto N_O^\omega(I)$. We show that if $\mathbb{E}N_O(I)$ converges to some value $\lambda = \lambda(I) > 0$ as $O \to \varpi$, then the random variable $N_O(I)$ converges to the Poisson random variable $\mathcal{P}_\lambda$ in distribution as $O \to \varpi$. We provide various examples to illustrate this result.

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§ 2. Homogeneous Laplacians

In this section we specify some spectral properties of the hierarchical Laplacian L_C assuming that the ultrametric measure space (X, d, m) on which it acts and the Laplacian itself satisfy the following symmetry conditions:

(i) the group of isometries of (X, d) acts transitively on X;

(ii) the reference measure m and the choice-function $C(B)$ are invariant under the action of isometries.

The ultrametric measure space (X, d, m) and the hierarchical Laplacian L_C on it are said to be *homogeneous* if they satisfy these conditions.

It follows from condition (i) that (X, d) is either discrete or perfect. We have in mind the following basic examples:

1) $X = \mathbb{Z}_p$ (the ring of p-adic integers);

2) $X = \mathbb{Q}_p$ (the ring of p-adic numbers);

3) $X = \mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\infty)$ (the multiplicative group of p^n th roots of unity endowed with the discrete topology, where n runs through the set of all non-negative integers).

As noticed in [24], [25], our assumptions imply that the measure space (X, m) can be identified with a locally compact totally disconnected Abelian group G equipped with the Haar measure. Notice that G is not unique. If X is *perfect and non-compact*, then a possible choice of G is the Abelian group

$$G = \lim_{l\to-\infty}\operatorname{ind}\biggl(\prod_{k\geqslant l}\mathbb{Z}(n_k)\biggr), \tag{2.1}$$

where the $\mathbb{Z}(n_k)$ are cyclic groups and $\{n_k\}_{k\in\mathbb{Z}}$ is an appropriately chosen two-sided sequence of integers. The canonical ultrametric structure on G is defined by the following descending sequence of compact subgroups of G:

$$G_l = \prod_{k\geqslant l}\mathbb{Z}(n_k).$$

Namely, the groups G_l, $l \in \mathbb{Z}$, and their cosets $\{a + G_l\}$ form the family $\mathcal{B}$ of all clopen balls.

There is a natural ultrametric structure associated with the two-sided chain of subgroups G_l of G. We define the *absolute value* $|a|$ of an element a of G by the formula

$$|a| = \begin{cases} 0 & \text{if } \ a = 0, \\ m(G_l) & \text{if } \ a \in G_l \setminus G_{l+1}. \end{cases}$$

The absolute value $|a|$ satisfies the ultrametric inequality

$$|a + b| \leqslant \max\{|a|, |b|\}.$$

Clearly, $|a| = |-a|$ and the ultrametric $d(a, b) := |a - b|$ makes (G, m) into a homogeneous ultrametric measure space as defined above. In particular, for every ball B we have

$$m(B) = \mathsf{diam}(B).$$

By choosing the Haar measure m in such a way that $m(G_0) = 1$, we compute $m(G_l)$ for all $l \neq 0$ as follows:

$$m(G_l) = \begin{cases} n_l \cdots n_{-1} & \text{if } \ l < 0, \\ (n_{l-1} \cdots n_0)^{-1} & \text{if } \ l > 0. \end{cases}$$

Recall that in the classical setting $X = \mathbb{Q}_p$ we have $G_0 = \mathbb{Z}_p$, $G_l = p^l\mathbb{Z}_p$ and

$$|a| = p^{-n(a)}, \quad \text{where } \ n(a) = \max\{l \colon a \in G_l\}.$$

The quantity $|a|$ becomes a pseudo-norm, that is,

$$|ab| \leqslant |a|\,|b|.$$

It is a norm ($|ab| = |a|\,|b|$) if p is a prime. This property plays a basic role in p-adic analysis and its applications.

We recall that with every ultrametric space (X, d) there is a canonically associated tree $\mathcal{T}(X)$ (see Fig. 1). Its vertices are metric balls, that is, in our case the cosets

$\{a+G_l\colon a\in G,\, l\in\mathbb{Z}\}$. The ascending sequence of subgroups $\{G_l\colon l=0,-1,-2,\dots\}$ identifies a special boundary point to be denoted by ϖ. We consider the horocycles of the tree with respect to this point. In this case, a *horocycle* is the set of all vertices that are balls of a given diameter or, in other words, cosets with respect to a given subgroup G_l. Thus for every l we have a horocycle $H_l = \{a+G_l\colon a\in G\}$. The boundary $\partial\mathcal{T}(G)$ can be identified with the one-point compactification $G\cup\{\varpi\}$ of G. We refer to [24], [25] and [15] for a complete treatment of the connection between an ultrametric space and the tree of its metric balls. The most complete source of the basic definitions related to the geometry of trees is [26]; see also [27].

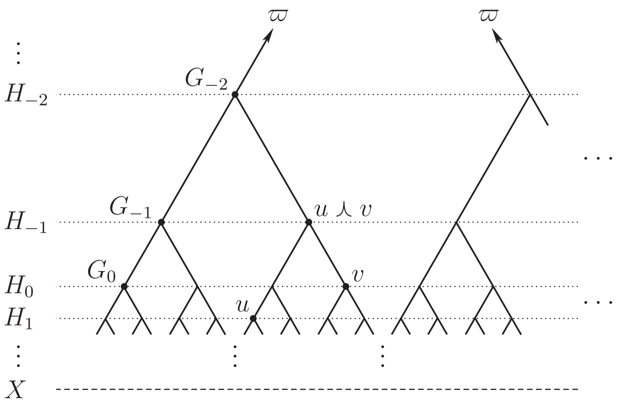

Figure 1. Tree of balls $\mathcal{T}(X)$ with forward degree $n_l = 2$

Let L_C be the homogeneous hierarchical Laplacian (determined by a choice function $C(B)$) on an ultrametric measure space (G,d,m), that is,

$$L_C f(x) = -\sum_{B\in\mathcal{B}\colon x\in B} C(B)(P_B f - f(x)).$$

By the homogeneity property we have $C(A)=C(B)$ for any two balls belonging to the same horocycle H. Of course, the same holds for the eigenvalues $\lambda(A)$ and $\lambda(B)$. We put $c_H = C(B)$ and $\lambda_H = \lambda(B)$ for every ball $B\in H$. When $H=H_k$, we will also write $c_k = c_{H_k}$ and $\lambda_k = \lambda_{H_k}$. In this notation,

$$\lambda_k = \sum_{l\leqslant k} c_l.$$

Each ball $B\in H_k$ has n_k nearest neighbours $B_i\subset B$. The eigenfunctions $f_{B_i,B}$ corresponding to $\lambda(B)$ are of the form

$$f_{B_i,B} = \frac{1_{B_i}}{m(B_i)} - \frac{1_B}{m(B)}.$$

The system of functions $\{f_{B_i,B}\colon i=1,\dots,n_k\}$ is not orthogonal. Its linear span $\mathcal{H}(B)\subset L^2$ has dimension n_k-1. Setting

$$f_i := f_{B_i,B},\quad m := m(B_i),\qquad i=1,\dots,n_k-1,$$

and using the Gram–Schmidt procedure, we obtain an orthonormal basis $\{u_i : i = 1, \dots, n_k - 1\} \subset \mathcal{H}(B)$ such that $u_1 = f_1$ and, for $i \geqslant 2$,

$$u_i = f_1 + \dots + f_{i-1} + (n_k - i + 1) f_i, \qquad \|u_i\|^2 = \frac{(n_k - i)(n_k - i + 1)}{m}.$$

For different balls S and T, the eigenspaces $\mathcal{H}(S)$ and $\mathcal{H}(T)$ are orthogonal. It follows that the eigenspace $\mathcal{H}_k$ corresponding to a horocycle H_k (or, equivalently, to the eigenvalue λ_k) is of the form

$$\mathcal{H}_k = \bigoplus_{B \in H_k} \mathcal{H}(B).$$

Since the system of eigenfunctions $\{f_{B_i,B}\}_{B \in \mathcal{B}}$ is complete, we have

$$\bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k = L^2(G, m).$$

Among the variety of homogeneous hierarchical Laplacians L_C on (G, d, m) we would like to mention a one-parameter family $\{\mathfrak{B}^\alpha\}_{\alpha>0}$. The hierarchical Laplacian $\mathfrak{B}^\alpha$ is defined by the choice function

$$C_\alpha(B) = (\mathsf{diam}(B))^{-\alpha} - (\mathsf{diam}(B'))^{-\alpha}, \tag{2.2}$$

where the $B \subset B'$ are nearest neighbouring balls. Hence, for every ball $B \in \mathcal{B}$, the eigenvalue $\lambda_\alpha(B)$ of $\mathfrak{B}^\alpha$ corresponding to B is

$$\lambda_\alpha(B) = (\mathsf{diam}(B))^{-\alpha}.$$

The eigenvalue $\lambda_\alpha(k)$ of $\mathfrak{B}^\alpha$ corresponding to a horocycle H_k is then computed as follows:

$$\lambda_\alpha(0) = 1, \qquad \lambda_\alpha(k) = \begin{cases} (n_k \cdots n_{-1})^{-\alpha} & \text{if } \ k < 0, \\ (n_{k-1} \cdots n_0)^{\alpha} & \text{if } \ k > 0. \end{cases}$$

We recall (see [13]) that the set $\mathcal{D}$ of compactly supported locally constant functions is in the domain of $\mathfrak{B}^\alpha$. The following remarkable properties can easily be proved:

$$\mathfrak{B}^\beta : \mathcal{D} \to \mathsf{Dom}(\mathfrak{B}^\alpha)$$

and, on $\mathcal{D}$,

$$\mathfrak{B}^\alpha \circ \mathfrak{B}^\beta = \mathfrak{B}^{\alpha+\beta}, \qquad (\mathfrak{B}^\alpha)^\beta = \mathfrak{B}^{\alpha\beta}.$$

Moreover, when $G = \mathbb{Q}_p$, the operator $\mathfrak{D}^\alpha = p^\alpha \mathfrak{B}^\alpha$ coincides with the operator of the fractional derivative of order α, which was defined and studied by means of the Fourier transform in [28], [16], [18] and [29]. In this case, for $f \in \mathcal{D}$,

$$\mathfrak{D}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_G \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} \, dm(y) \tag{2.3}$$

and, therefore,

$$\mathfrak{B}^\alpha f(x) = \frac{1 - p^{-\alpha}}{1 - p^{-\alpha-1}} \int_G \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} \, dm(y).$$

We finally notice that similar identifications (based on the cyclic groups $\mathbb{Z}(n)$ as building blocks) can be carried out in the two remaining cases: when (X,d) is *infinite and discrete*, and when (X,d) is *compact and perfect*. For example, the infinite (non-Abelian) symmetric group S_∞, whose canonical ultrametric structure is defined by the family $\{S_n\}$ of its finite symmetric subgroups S_n, can be identified (as an ultrametric measure space) with the discrete Abelian group $G = \bigoplus_{l>1} \mathbb{Z}(l)$. The canonical ultrametric structure of G is defined by the family $\{G_n\}$ of its finite subgroups $G_n = \prod_{1<l\leqslant n} \mathbb{Z}(l)$. As a second example, we consider a compact perfect ultrametric space $X = \mathbb{Z}_p \subset \mathbb{Q}_p$ (the ring of p-adic integers). It can be identified (as an ultrametric measure space) with the compact Abelian group $G = \prod_{k\geqslant 1} \mathbb{Z}(l_k)$, where all l_k are equal to p. The ultrametric structure is then given by the descending family of small subgroups $G_l = \prod_{k\geqslant l} \mathbb{Z}(l_k) \subset G$.

§ 3. Random perturbations

Let (X,d,m) be a non-compact homogeneous ultrametric space, L_C the homogeneous hierarchical Laplacian on X defined by a choice function $C(B)$, and $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ a family of symmetric independent identically distributed random variables defined on a probability space $(\Omega,\mathbb{P})$ and taking values in some small interval $[-\epsilon,\epsilon] \subset (-1,1)$. We define the perturbed choice function $C(B,\omega)$ and the perturbed hierarchical Laplacian as follows:

$$C(B,\omega) = C(B)(1+\varepsilon(B,\omega))$$

and

$$L^\omega f(x) := L_{C(\omega)} f(x) = -\sum_{B\in\mathcal{B}:\ x\in B} C(B,\omega)(P_B f - f(x)).$$

Clearly, the operator L^ω may be inhomogeneous for some $\omega \in \Omega$. It has a pure point spectrum for all ω, but the structure of the closed set $\mathsf{Spec}(L^\omega)$ can be quite complicated (see [15] for various examples).

Two stationary families. We fix a horocycle $H = H_l$ for some $l \in \mathbb{Z}$ and write $\lambda_H = \lambda_l$ for the eigenvalue of the homogeneous Laplacian L_C corresponding to H. Take $B \in H$ and let $\{B_k\}_{k\leqslant l}$ be the unique geodesic path in $\mathcal{T}(X)$ from ϖ to B. We compute the eigenvalue $\lambda(B,\omega)$ of $L^\omega = L_{C(\omega)}$:

$$\begin{aligned}\lambda(B,\omega) &= \sum_{k\leqslant l} C(B_k,\omega) = \sum_{k\leqslant l} c_k(1+\varepsilon(B_k,\omega))\\ &= \lambda_l\biggl(1+\sum_{k\leqslant l} a_k\varepsilon(B_k,\omega)\biggr) =: \lambda_l(1+U(B,\omega)),\end{aligned}$$

where $a_k = c_k/\lambda_l$ and

$$U(B,\omega) = \sum_{k\leqslant l} a_k\varepsilon(B_k,\omega). \tag{3.1}$$

Notice that $\sum_{k\leqslant l} a_k = 1$ and the family $\{U(B)\}_{B\in H}$ consists of (dependent) identically distributed symmetric random variables with values in some symmetric interval $I \subsetneqq (-1,1)$.

We want to study the families $\{\lambda(B,\omega)\}_{B\in H}$ and $\{U(B)\}_{B\in H}$ of random variables. Since the horocycle $H = H_l$ is fixed, it is useful to identify the balls $B \in H$ with elements $g \in G$ of the (discrete!) Abelian group $G = \bigoplus_{k<l} \mathbb{Z}(n_k)$. Having this identification in mind, we verify in a straightforward way that the family $\{U(B)\}_{B\in H} = \{U(g)\}_{g\in G}$ is stationary, that is, for all $g, g_1, \dots, g_s$ in G we have

$$\big\{U(g+g_1),\dots,U(g+g_s)\big\} \overset{d}{=} \big\{U(g_1),\dots,U(g_s)\big\}.$$

See [30] and [20] for the general theory of stationary processes.

One can easily compute the correlation functions $\mathcal{K}_U(g,g') = \mathbb{E}U(g)U(g')$. Namely, we have

$$\mathcal{K}_U(g,g') = \mathcal{K}_U(0,g-g') =: \mathcal{K}_U(g-g')$$

and

$$\mathcal{K}_U(g) = \sum_{k\geqslant |g|} a_{l-k}^2, \tag{3.2}$$

where

$$|g| = \min\bigg\{n\colon g \in \{0\} \times \bigoplus_{-n\leqslant k<l} \mathbb{Z}(n_k) \subset G\bigg\}.$$

Since the function $\mathcal{K}_U(g)$ is positive definite, Bochner's theorem yields a finite measure $\mathcal{F}_U$ (the spectral measure) on the compact group $\widehat{G} = \prod_{k<l} \mathbb{Z}(n_k)$ such that

$$\mathcal{K}_U(g) = \int_{\widehat{G}} \langle g,\gamma\rangle \, d\mathcal{F}_U(\gamma).$$

We have

$$\int_G \mathcal{K}_U(g)\, dg = a_l^2 + n_{l-1}a_{l-1}^2 + n_{l-1}n_{l-2}a_{l-2}^2 + \cdots .$$

In particular, if the function $\mathcal{K}_U$ is integrable, then the spectral measure $\mathcal{F}_U$ is absolutely continuous with respect to the Haar measure on $\widehat{G}$ and admits a continuous density $F_U(\gamma)$, which can be computed as the inverse Fourier transform of $\mathcal{K}_U(g)$:

$$F_U(\gamma) = a_l^2 + n_{l-1}a_{l-1}^2 1_{A(G_{l-1})}(\gamma) + n_{l-1}n_{l-2}a_{l-2}^2 1_{A(G_{l-2})}(\gamma) + \cdots .$$

Here $A(G_{l-i}) \subset \widehat{G}$ is the annihilator of the subgroup G_{l-i}, that is,

$$A(G_{l-i}) = \{0\} \times \prod_{k<l-i} \mathbb{Z}(n_k).$$

Since $\lambda(B,\omega) = \lambda_H(1+U(B,\omega))$ for $B \in H$, the family of random variables

$$\lambda(g,\omega) = \lambda_H(1+U(g,\omega))$$

is also stationary. In particular, its correlation function $\mathcal{K}_\lambda(g)$ satisfies

$$\mathcal{K}_\lambda(g) = \lambda_H^2 \mathcal{K}_U(g), \qquad g \in G.$$

We will use this property in our further calculations.

The law of large numbers. Choose a reference point $o \in X$ (for example, the neutral element 0 in our identification $X \equiv G$ of X and G) and consider the family $\mathcal{O}$ of all balls O with centre o. We fix a horocycle H and study the limiting behaviour (as $O \to \varpi$ along $\mathcal{O}$) of the family of arithmetic means

$$\bar{\lambda}_H(O,\omega) = \frac{1}{|\mathcal{B}_H(O)|} \sum_{B \in \mathcal{B}_H(O)} \lambda(B,\omega),$$

where $\mathcal{B}_H(O)$ is the set of all balls B in O that lie in the horocycle H, and $|\mathcal{B}_H(O)|$ stands for the cardinality of the finite set $\mathcal{B}_H(O)$.

We recall that the eigenvalues $\lambda(A)$ and $\lambda(B)$ of the homogeneous Laplacian L_C coincide for any two balls A and B belonging to the same horocycle H. Their common value is denoted by λ_H.

Theorem 3.1. *For any given horocycle H, as $O \to \varpi$ we have*

$$\bar{\lambda}_H(O,\omega) \to \lambda_H \quad \textit{a. e.}$$

Proof. Let $B \in H_l$ for some $l \in \mathbb{Z}$, and let $\{B_k\}_{k \leqslant l}$ be the unique infinite geodesic path from ϖ to B in $\mathcal{T}(X)$. We have already computed the eigenvalue $\lambda(B,\omega)$ of $L_{C(\omega)}$ corresponding to the ball B:

$$\lambda(B,\omega) = \lambda_l(1 + U(B,\omega)),$$

where $a_k = c_k/\lambda_l$ and

$$U(B,\omega) = \sum_{k \leqslant l} a_k \varepsilon(B_k,\omega). \tag{3.3}$$

We now compute the arithmetic mean $\bar{\lambda}_{H_l}(O,\omega)$ assuming that $O \in H_L$ and $L \ll l$. To simplify the notation, we put $\bar{\lambda}_L(\omega) := \bar{\lambda}_{H_l}(O,\omega)$. Then

$$\begin{aligned}\bar{\lambda}_L(\omega) &= \frac{1}{|\mathcal{B}_H(O)|} \sum_{B \in \mathcal{B}_H(O)} \lambda(B,\omega) \\ &= \frac{\lambda_l}{n_{l-1} \cdots n_L} \sum_{B \in \mathcal{B}_{H_l}(O)} (1 + U(B,\omega)) = \lambda_l(1 + \overline{U}_L(\omega)),\end{aligned} \tag{3.4}$$

where

$$\overline{U}_L(\omega) = \frac{1}{n_{l-1} \cdots n_L} \sum_{B \in \mathcal{B}_{H_l}(O)} U(B,\omega). \tag{3.5}$$

Hence, to prove the theorem it remains to establish that $\overline{U}_L(\omega) \to 0$ a.s. as $L \to -\infty$. Let $\{O_k\}_{k\leqslant l}$ be the unique geodesic path from ϖ to O. Substituting (3.3) into (3.5), we obtain

$$\begin{aligned}\overline{U}_L &= \frac{1}{n_{l-1}\cdots n_L}\sum_{B\in H_l:\, B\subset O}\sum_{k\leqslant l} a_k\varepsilon(B_k) = \frac{1}{n_{l-1}\cdots n_L}\sum_{k\leqslant l} a_k \sum_{B\in H_l:\, B\subset O}\varepsilon(B_k)\\ &= \frac{1}{n_{l-1}\cdots n_L}\biggl(a_l\sum_{B\in H_l:\, B\subseteq O}\varepsilon(B) + a_{l-1}n_{l-1}\sum_{B\in H_{l-1}:\, B\subseteq O}\varepsilon(B)\\ &\qquad + a_{l-2}n_{l-1}n_{l-2}\sum_{B\in H_{l-2}:\, B\subseteq O}\varepsilon(B) + \cdots + a_L n_{l-1}n_{l-2}\cdots n_L\varepsilon(O_L)\biggr)\\ &\qquad + a_{L-1}\varepsilon(O_{L-1}) + a_{L-2}\varepsilon(O_{L-2}) + \cdots .\end{aligned}$$

We introduce two random variables:

$$\mathcal{U}_L = \frac{a_l}{n_{l-1}\cdots n_L}\sum_{B\in H_l:\, B\subseteq O}\varepsilon(B) + \frac{a_{l-1}}{n_{l-2}\cdots n_L}\sum_{B\in H_{l-1}:\, B\subseteq O}\varepsilon(B) + \cdots + a_L\varepsilon(O_L) \tag{3.6}$$

and

$$\mathcal{V}_L = a_{L-1}\varepsilon(O_{L-1}) + a_{L-2}\varepsilon(O_{L-2}) + \cdots . \tag{3.7}$$

Then $\mathcal{U}_L$ and $\mathcal{V}_L$ are independent, have zero mean and

$$\overline{U}_L = \mathcal{U}_L + \mathcal{V}_L. \tag{3.8}$$

Moreover $\mathcal{V}_L \to 0$ uniformly as $L \to -\infty$. Hence we are left with showing that, as $L \to -\infty$,

$$\mathcal{U}_L \to 0 \quad \text{a.s.}$$

Put $\sigma = \sqrt{\mathsf{Var}[\varepsilon]}$ and $\sigma[\mathcal{U}_L] = \sqrt{\mathsf{Var}[\mathcal{U}_L]}$. We calculate $\sigma[\mathcal{U}_L]$ using (3.6):

$$(\sigma[\mathcal{U}_L])^2 = \sigma^2\biggl(\frac{a_l^2}{n_{l-1}\cdots n_L} + \frac{a_{l-1}^2}{n_{l-2}\cdots n_L} + \cdots + a_L^2\biggr). \tag{3.9}$$

Since all the n_k are greater than or equal to 2, it follows that

$$\begin{aligned}\sum_{L\leqslant l}(\sigma[\mathcal{U}_L])^2 &= \sigma^2 a_l^2\biggl(1 + \frac{1}{n_{l-1}} + \frac{1}{n_{l-1}n_{l-2}} + \cdots\biggr)\\ &\qquad + \sigma^2 a_{l-1}^2\biggl(1 + \frac{1}{n_{l-2}} + \frac{1}{n_{l-2}n_{l-3}} + \cdots\biggr) + \cdots\\ &\leqslant 2\sigma^2(a_l^2 + a_{l-1}^2 + \cdots) < 2\sigma^2(a_l + a_{l-1} + \cdots)^2 = 2\sigma^2.\end{aligned}$$

Now the desired result follows from Chebyshev's inequality and the Borel–Cantelli lemma. □

The central limit theorem. We now study the limiting behaviour (as $O \to \varpi$) of the normalized arithmetic means

$$\Lambda_H(O) = \frac{\bar{\lambda}_H(O) - \lambda_H}{\sigma[\bar{\lambda}_H(O)]}.$$

Recall that for every random variable Y we denote its mean-square deviation $\sqrt{\mathsf{Var}[Y]}$ by $\sigma[Y]$. In the course of our study we shall assume that the following condition holds:

$$\frac{1}{\kappa} \leqslant C(B)(\mathsf{diam}(B))^{\delta/2} \leqslant \kappa \tag{3.10}$$

for all balls $B \in \mathcal{B}$ and some $\delta, \kappa > 0$.

It is easy to see that (3.10) is equivalent to the following condition:

$$\frac{1}{2\kappa} \leqslant \lambda(B)(\mathsf{diam}(B))^{\delta/2} \leqslant 2\kappa. \tag{3.11}$$

Clearly, conditions (3.10) and (3.11) hold with $\delta = 2\alpha$ for the operator $\mathfrak{B}^\alpha$ introduced in the previous section (see (2.2)). Indeed, in this case we have

$$\lambda(B)(\mathsf{diam}(B))^{\delta/2} = 1.$$

Let $N(0,1)$ be the standard normal random variable. The main result of this subsection is the following theorem.

Theorem 3.2. *Assume that $\delta \geqslant 1$. Then, as $O \to \varpi$, we have*

$$\Lambda_H(O) \to N(0,1) \quad \textit{in distribution.}$$

Proof. Let $H = H_l$ and $O = O_L \in H_L$ for some $L \ll l$. As in the proof of Theorem 3.1, we fix l and let L tend to $-\infty$. To simplify the notation, we set $\Lambda_H(O) =: \Lambda_L$. By (3.4) we have

$$\Lambda_L = \frac{\overline{U}_L}{\sigma[\overline{U}_L]}.$$

Hence it remains to show that, as $L \to -\infty$,

$$\frac{\overline{U}_L}{\sigma[\overline{U}_L]} \to N(0,1) \quad \text{in distribution.} \tag{3.12}$$

As in (3.8), we write $\overline{U}_L = \mathcal{U}_L + \mathcal{V}_L$. Since $\mathcal{U}_L$ and $\mathcal{V}_L$ are independent, we have

$$(\sigma[\overline{U}_L])^2 = (\sigma[\mathcal{U}_L])^2 + (\sigma[\mathcal{V}_L])^2.$$

Claim 3.3. *For l fixed and $L \to -\infty$,*

$$(\sigma[\mathcal{V}_L])^2 \asymp \sigma^2(n_{L-1}n_L \cdots n_0)^{-\delta} \tag{3.13}$$

and

$$(\sigma[\mathcal{U}_L])^2 \asymp \begin{cases} \sigma^2(n_L \cdots n_0)^{-1} & \text{if } \delta > 1, \\ -\sigma^2 L(n_L \cdots n_0)^{-1} & \text{if } \delta = 1, \\ \sigma^2(n_L \cdots n_0)^{-\delta} & \text{if } \delta < 1, \end{cases} \tag{3.14}$$

where $x \asymp y$ means that the ratio x/y is uniformly bounded above and below.

To prove (3.13), we use (3.7). Since the random variables $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ are independent and identically distributed, we have

$$(\sigma[\mathcal{V}_L])^2 = \sigma^2(a_{L-1}^2 + a_{L-2}^2 + \cdots) = \frac{\sigma^2}{\lambda_l^2}(c_{L-1}^2 + c_{L-2}^2 + \cdots),$$

and (3.10) yields that

$$\begin{aligned}(\sigma[\mathcal{V}_L])^2 &\asymp \sigma^2((n_{L-1}n_L\cdots n_0)^{-\delta} + (n_{L-2}n_{L-1}n_L\cdots n_0)^{-\delta} + \cdots)\\ &= \sigma^2(n_{L-1}n_L\cdots n_0)^{-\delta}(1 + (n_{L-2})^{-\delta} + (n_{L-2}n_{L-3})^{-\delta} + \cdots)\end{aligned}$$

for every fixed l. Since all n_k are greater than or equal to 2, we obtain (3.13).

To prove (3.14), we use (3.9) and (3.10). Since l is fixed, we have

$$\begin{aligned}(\sigma[\mathcal{U}_L])^2 &= \sigma^2\biggl(\frac{a_l^2}{n_{l-1}\cdots n_L} + \frac{a_{l-1}^2}{n_{l-2}\cdots n_L} + \cdots + a_L^2\biggr)\\ &= \frac{\sigma^2}{n_{l-1}\cdots n_L}(a_l^2 + a_{l-1}^2 n_{l-1} + \cdots + a_L^2 n_{l-1}\cdots n_L)\\ &\asymp \frac{\sigma^2}{n_0\cdots n_L}\biggl(1 + \Bigl(\frac{a_{l-1}}{a_l}\Bigr)^2 n_{l-1} + \cdots + \Bigl(\frac{a_L}{a_l}\Bigr)^2 n_{l-1}\cdots n_L\biggr)\\ &\asymp \frac{\sigma^2}{n_0\cdots n_L}\bigl(1 + (n_{l-1})^{1-\delta} + \cdots + (n_{l-1}\cdots n_L)^{1-\delta}\bigr).\end{aligned}$$

The desired relation (3.14) follows.

Now let $\delta \geqslant 1$. By Claim 3.3,

$$\sigma[\overline{U}_L] \sim \sigma[\mathcal{U}_L] \quad \text{as} \quad L \to -\infty.$$

Hence, to prove (3.12) it remains to show that

$$\frac{\mathcal{U}_L}{\sigma[\mathcal{U}_L]} \to N(0,1) \quad \text{in distribution.}$$

Claim 3.4. *Assume that $\delta \geqslant 1$ and that l is fixed. Then*

$$\lim_{L\to-\infty}\max_{L\leqslant k\leqslant l}\frac{\sigma a_k}{\sigma[\mathcal{U}_L]\prod_{L\leqslant i\leqslant k-1} n_i} = 0 \tag{3.15}$$

(*with the agreement that* $\prod_{i\in\varnothing} b_i := 1$).

Indeed, define $\epsilon = \min\{1, \delta - 1\}$ and consider k such that $L \leqslant k \leqslant l$. Since l is fixed, we can assume that $k \leqslant 0$. We denote the fraction on the right-hand side of (3.15) by $A(\delta)$. By Claim 3.3, when $\delta > 1$,

$$\begin{aligned}(A(\delta))^2 &\asymp \frac{\sigma^2(n_k\cdots n_0)^{-\delta}}{\sigma^2(n_L\cdots n_0)^{-1}(n_L\cdots n_{k-1})^2}\\ &= \frac{1}{(n_k\cdots n_0)^{\delta-1}(n_L\cdots n_{k-1})} \leqslant \frac{1}{(n_L\cdots n_0)^{\epsilon}},\end{aligned}$$

and when $\delta = 1$,

$$(A(\delta))^2 \asymp -\frac{\sigma^2(n_k\cdots n_0)^{-1}}{\sigma^2 L(n_L\cdots n_0)^{-1}(n_L\cdots n_{k-1})^2} = -\frac{1}{L(n_L\cdots n_{k-1})} \leqslant -\frac{1}{L}.$$

The result follows.

Observe that when $\delta < 1$, we obtain

$$\max_{L\leqslant k\leqslant l} \frac{\sigma a_k}{\sigma[\mathcal{U}_L]\prod_{L\leqslant i\leqslant k-1} n_i} \geqslant \frac{\sigma a_L}{\sigma[\mathcal{U}_L]} \geqslant \frac{c\sigma(n_L\cdots n_0)^{-\delta/2}}{\sigma(n_L\cdots n_0)^{-\delta/2}} = c$$

for some $c > 0$. In particular, (3.15) does not hold in this case.

Let ϕ and Φ be the characteristic functions of the random variables $\varepsilon = \varepsilon(B)$ and $\mathcal{U}_L/\sigma[\mathcal{U}_L]$ respectively. By (3.6),

$$\Phi(x) = \prod_{L\leqslant k\leqslant l} \phi\left(\frac{a_k x}{\sigma[\mathcal{U}_L]n_L\cdots n_{k-1}}\right)^{n_L\cdots n_{k-1}}$$

(with the agreement that $n_L\cdots n_{k-1} = 1$ when $k = L$).

Since the random variable ε has two moments, we can write

$$\phi(z) = 1 - \frac{1}{2}(\sigma z)^2(1+\beta(\sigma z)), \tag{3.16}$$

where $\beta(z) \to 0$ as $z \to 0$. We put

$$\Delta_k = \frac{\sigma a_k}{\sigma[\mathcal{U}_L]n_L\cdots n_{k-1}}.$$

Observe that $\sum_{L\leqslant k\leqslant l} n_L\cdots n_{k-1}\Delta_k^2 = 1$. Now, applying (3.16), we obtain

$$\begin{aligned}\log\Phi(x) &= \sum_{L\leqslant k\leqslant l} n_L\cdots n_{k-1}\log\left[1-\frac{x^2}{2}\Delta_k^2(1+\beta(\Delta_k x))\right]\\ &\sim -\frac{x^2}{2}\left[\sum_{L\leqslant k\leqslant l} n_L\cdots n_{k-1}\Delta_k^2(1+\beta(\Delta_k x))\right]\\ &= -\frac{x^2}{2}\left[1+\sum_{L\leqslant k\leqslant l} n_L\cdots n_{k-1}\Delta_k^2\beta(\Delta_k x)\right].\end{aligned}$$

Finally, the desired result follows from Claim 3.4 and the inequality

$$\sum_{L\leqslant k\leqslant l} n_L\cdots n_{k-1}\Delta_k^2\beta(\Delta_k x) \leqslant \max_{L\leqslant k\leqslant l}\beta(\Delta_k x). \qquad \square$$

The operator $\mathfrak{B}^\alpha$. As an example we consider the space $X = \mathbb{Q}_p$ with its standard ultrametric structure, which is defined by the descending sequence of compact subgroups $G_l = p^l\mathbb{Z}_p$. Let $\mathfrak{B}^\alpha$ be the homogeneous Laplacian determined by the choice function (2.2), and let $\mathfrak{B}^\alpha(\omega)$ be its random perturbation by the symmetric independent identically distributed random variables $\{\varepsilon(B)\}_{B\in\mathcal{B}}$, which is defined and studied above. We have already noticed that $\mathfrak{B}^\alpha$ satisfies the condition (3.11) with $\alpha = \delta/2$. In particular, for every $\alpha \geqslant 1/2$ the normalized arithmetic means $\Lambda^\alpha_H(O)$ of eigenvalues of $\mathfrak{B}^\alpha(\omega)$ converge in distribution to the standard normal random variable $N(0,1)$ as $O \to \varpi$. In this subsection we study the issue of convergence of the normalized arithmetic means $\Lambda^\alpha_H(O)$ assuming that $0 < \alpha < 1/2$.

Theorem 3.5. *For every α, $0 < \alpha < 1/2$, there is a random variable Λ^α_H such that as $O \to \varpi$ we have*

$$\Lambda^\alpha_H(O) \to \Lambda^\alpha_H \quad \textit{in distribution.}$$

The random variable Λ^α_H is not Gaussian. It has a C^∞-distribution function $\mathcal{F}_{\Lambda^\alpha_H}$ belonging to $D(2)$ (the domain of attraction of the normal law). The function $\mathcal{F}_{\Lambda^\alpha_H}$ is unimodal whenever the common distribution function $\mathcal{F}_\varepsilon$ of the independent identically distributed random variables $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ is unimodal.

Proof. We follow verbatim the proof of Theorem 3.2. There is no loss of generality in assuming that $H = H_0$ and $O = O_L \in H_L$ for some $L < -1$. To simplify the notation, we set $\Lambda_L := \Lambda^\alpha_H(O)$. Since $\Lambda_L = \overline{U}_L/\sigma[\overline{U}_L]$, where $\overline{U}_L$ is the random variable defined in (3.5), we can write $\overline{U}_L = \mathcal{U}_L + \mathcal{V}_L$ and

$$(\sigma[\overline{U}_L])^2 = (\sigma[\mathcal{U}_L])^2 + (\sigma[\mathcal{V}_L])^2.$$

Since $\lambda_k = p^{\alpha k}$ and $c_k = (p^\alpha - 1)p^{\alpha(k-1)}$, we obtain

$$a_k = \frac{c_k}{\lambda_0} = (p^\alpha - 1)p^{\alpha(k-1)}, \qquad k \leqslant 0.$$

The data above enable us to estimate $\sigma[\mathcal{U}_L]$ and $\sigma[\mathcal{V}_L]$ at $-\infty$. We have

$$\begin{aligned}(\sigma[\mathcal{U}_L])^2 &= \sigma^2\left(\frac{a_0^2}{p^{-L}} + \frac{a_{-1}^2}{p^{-L-1}} + \cdots + a_L^2\right) = \sigma^2(p^\alpha - 1)^2 \sum_{0\leqslant l\leqslant -L} p^{L+l-2\alpha(l+1)} \\ &\sim \frac{\sigma^2(p^\alpha-1)^2}{1-p^{2\alpha-1}} p^{2\alpha(L-1)} = \frac{\sigma^2}{1-p^{2\alpha-1}} a_L^2 \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}(\sigma[\mathcal{V}_L])^2 &= \sigma^2(a_{L-1}^2 + a_{L-2}^2 + \cdots) = \sigma^2(p^\alpha - 1)^2 \sum_{l\geqslant -L+1} p^{-2\alpha(l+1)} \\ &\sim \frac{\sigma^2(p^\alpha-1)^2}{1-p^{-2\alpha}} p^{2\alpha(L-2)} = \frac{\sigma^2}{1-p^{-2\alpha}} a_{L-1}^2. \end{aligned} \tag{3.18}$$

Let $\{\varepsilon_i\}_{i\geqslant 0}$ be independent identically distributed random variables that are independent of $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ and have the same common distribution as $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. By (3.7) and (3.18), the random variable $\mathcal{V}_L/\sigma[\mathcal{V}_L]$ converges in distribution to the random variable

$$V = \sqrt{1-p^{-2\alpha}}\left(\frac{\varepsilon_0}{\sigma[\varepsilon_0]} + p^{-\alpha}\frac{\varepsilon_1}{\sigma[\varepsilon_2]} + \cdots + p^{-k\alpha}\frac{\varepsilon_k}{\sigma[\varepsilon_k]} + \cdots\right).$$

By Cramér's theorem, V is not Gaussian.

Let $\{\varepsilon_{ij}\}_{i,j\geqslant 0}$ be independent identically distributed random variables that are independent of both $\{\varepsilon_i\}_{i\geqslant 0}$ and $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ and have the same common distribution as $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. Define the random variables

$$S_k = \sum_{0\leqslant j\leqslant p^k-1} \varepsilon_{kj}, \qquad k = 0, 1, 2, \dots .$$

By (3.6) and (3.17) the random variable $\mathcal{U}_L/\sigma[\mathcal{U}_L]$ converges in distribution to the random variable

$$U = \sqrt{1-p^{2\alpha-1}} \sum_{k\geqslant 0} p^{(2\alpha-1)k/2} \frac{S_k}{\sigma[S_k]}.$$

By Cramér's theorem, U is not Gaussian.

Finally, the random variable

$$\Lambda_L = \frac{\overline{U}_L}{\sigma[\overline{U}_L]} = \frac{\sigma[\mathcal{U}_L]}{\sigma[\overline{U}_L]} \frac{\mathcal{U}_L}{\sigma[\mathcal{U}_L]} + \frac{\sigma[\mathcal{V}_L]}{\sigma[\overline{U}_L]} \frac{\mathcal{V}_L}{\sigma[\mathcal{V}_L]}$$

converges in distribution to the random variable

$$\Lambda = \sqrt{\frac{1-p^{-2\alpha}}{1-p^{-1}}}\, U + \sqrt{\frac{p^{-2\alpha}-p^{-1}}{1-p^{-1}}}\, V.$$

Since U and V are independent and non-Gaussian, Λ is also non-Gaussian.

Given an arbitrary random variable X, we write $\Phi_X(\xi) = \mathbb{E}\exp\{i\xi X\}$ for its characteristic function. Since the random variables $\{\varepsilon_{kj}\}$ are independent and identically distributed, the function $\Phi_{\varepsilon_{kj}}$ is independent of i, j. We put $\Phi = \Phi_{\varepsilon_{kj}}$. By (3.16), for every ϵ, $0<\epsilon<1$, there is a $\delta>0$ such that

$$\begin{aligned}
|\Phi_U(\xi)| &= \prod_{k\geqslant 0}\left|\Phi_{S_k}\left(\frac{p^{(2\alpha-1)k/2}}{\sigma[S_k]}\xi\right)\right| = \prod_{k\geqslant 0}\left|\Phi\left(\frac{p^{(2\alpha-1)k/2}}{\sigma[S_k]}\xi\right)\right|^{p^k} \\
&\leqslant \prod_{k:\ p^{(2\alpha-1)k/2}\xi/\sigma[S_k]<\delta}\left|\Phi\left(\frac{p^{(2\alpha-1)k/2}}{\sigma[S_k]}\xi\right)\right|^{p^k} \\
&\leqslant \prod_{k:\ p^{(2\alpha-1)k/2}\xi/\sigma[S_k]<\delta}\left(1-\frac{\xi^2}{2}\,\frac{\sigma^2 p^{(2\alpha-1)k}(1-\epsilon)}{(\sigma[S_k])^2}\right)^{p^k}.
\end{aligned}$$

Since $(\sigma[S_k])^2 = \sigma^2 p^k$, we obtain

$$|\Phi_U(\xi)| \leqslant \exp\left\{-\frac{\xi^2}{2}(1-\epsilon)\sum_{k:\ p^{(\alpha-1)k}\xi/\sigma<\delta} p^{(2\alpha-1)k}\right\} \leqslant \exp\{-A\xi^\beta\}$$

for some $A>0$, $\beta=(1-\alpha)^{-1}\in(1,2)$ and for all $\xi>\delta\sigma$. When $\xi\leqslant\delta\sigma$, we get

$$|\Phi_U(\xi)| \leqslant \exp\{-B\xi^2\}$$

for some $B > 0$. Thus, for all ξ we have

$$|\Phi_U(\xi)| \leqslant \exp\{-C\min\{\xi^2, \xi^\beta\}\}$$

for some $C > 0$.

Since the random variables U and V are independent and $\Lambda = \lambda_1 U + \lambda_2 V$, we have

$$\Phi_\Lambda(\xi) = \Phi_U(\lambda_1\xi)\Phi_V(\lambda_2\xi).$$

In particular, Φ_Λ satisfies an inequality similar to that for Φ_U. This proves that the distribution function $\mathcal{F}_\Lambda$ of Λ belongs to C^∞. Since Λ has a second moment, $\mathcal{F}_\Lambda$ belongs to $D(2)$, that is, to the domain of attraction of the normal law.

We finally assume that the common distribution function $\mathcal{F}_\varepsilon$ of the independent identically distributed random variables $\{\varepsilon_{ij}\}$ is unimodal. Since a convolution of the symmetric unimodal distribution functions $\mathcal{F}_U$ (resp. $\mathcal{F}_V$) is symmetric and unimodal, so is $\mathcal{F}_\Lambda$. □

§ 4. The integrated density of states

Let L_C be the homogeneous hierarchical Laplacian, and let $L_{C(\omega)}$ be a random perturbation of L_C as defined and studied in the previous section. Given an ultrametric ball $O \in \mathcal{B}$ and a horocycle $H \in \mathcal{T}$, we write $\mathcal{B}_H(O)$ for the set of all balls $B \subseteq O$ belonging to H.

Let δ_a be the probability distribution concentrated at $a \in \mathbb{R}$. We fix a horocycle H and study the limiting behaviour of the normalized empirical process

$$\mathcal{M}_O^\omega = \frac{1}{|\mathcal{B}_H(O)|}\sum_{B\in\mathcal{B}_H(O)} \delta_{\lambda(B,\omega)} \tag{4.1}$$

as O tends to ϖ.

Theorem 4.1. *There is a probability measure $\mathcal{M}$ such that, as O tends to ϖ, we have for almost all $\omega \in \Omega$*

$$\mathcal{M}_O^\omega \to \mathcal{M} \quad \textit{in the Bernoulli topology.} \tag{4.2}$$

Proof. As in the proof of Theorem 3.1, for every $B \in H$ we write

$$\lambda(B,\omega) = \lambda_H(1 + U(B,\omega)),$$

where $U(B,\omega)$ is defined in (3.3). The random variables $\{U(B)\}_{B\in H}$ are identically distributed (and dependent). We denote their common distribution by $\mathcal{N}$. Since the horocycle H is fixed, we are left with the study of the normalized empirical process

$$\mathcal{N}_O^\omega = \frac{1}{|\mathcal{B}_H(O)|}\sum_{B\in\mathcal{B}_H(O)} \delta_{U(B,\omega)}.$$

We claim that, as O tends to ϖ, we have for almost all $\omega \in \Omega$

$$\mathcal{N}_O^\omega \to \mathcal{N} \quad \text{in the Bernoulli topology.} \tag{4.3}$$

Indeed, there is no loss of generality in assuming that $H = H_0$ and $O \in H_L$. Then we write $\mathcal{N}_O = \mathcal{N}_L^\omega$. Let Φ_L^ω be the characteristic function of the probability measure $\mathcal{N}_L^\omega$:

$$\Phi_L^\omega(\theta) = \frac{1}{n_L n_{L+1} \cdots n_{-1}} \sum_{B \in \mathcal{B}_H(O)} \exp\{i\theta U(B,\omega)\}. \tag{4.4}$$

We denote the characteristic function of the measure $\mathcal{N}$ by Φ. Then

$$\mathbb{E}\Phi_L^\omega(\theta) = \Phi(\theta).$$

Let us show that, for all θ,

$$\Phi_L^\omega(\theta) \to \Phi(\theta) \quad \text{a. s.} \tag{4.5}$$

Since all the probability measures $\mathcal{N}_L^\omega$, $\omega \in \Omega$, are supported by some finite interval $[-a, a]$, the family of functions $\{\Phi_L^\omega\}$ is equicontinuous. Hence the exceptional null-set in (4.5) can be chosen to be the same for all θ. Thus, given that (4.5) holds, the claim (4.3) follows by the Lévy continuity theorem.

For every $B \in \mathcal{B}_H(O)$ let $\{B_k\}_{k\leqslant 0}$ be the unique infinite geodesic path from ϖ to B in the tree $\mathcal{T}$ of balls. We write

$$U(B,\omega) = \left(\sum_{L\leqslant k\leqslant 0} + \sum_{k<L}\right) a_k \varepsilon(B_k,\omega) =: U_L(B,\omega) + V_L(B,\omega).$$

Since for any balls B and B' in $\mathcal{B}_H(O)$,

$$V_L(B,\omega) = V_L(B',\omega) =: V_L(\omega),$$

we have

$$\Phi_L^\omega(\theta) = \frac{\exp\{i\theta V_L(\omega)\}}{n_L n_{L+1} \cdots n_{-1}} \sum_{B \in \mathcal{B}_H(O)} \exp\{i\theta U_L(B,\omega)\}.$$

As $L \to -\infty$, uniformly in $\omega \in \Omega$,

$$\exp\{i\theta V_L(\omega)\} \to 1 \quad \text{for all} \quad \theta.$$

Hence we are left with the study of the limiting behaviour of the random variables

$$\omega \mapsto \Psi_L^\omega(\theta) = \frac{1}{n_L n_{L+1} \cdots n_{-1}} \sum_{B \in \mathcal{B}_H(O)} \exp\{i\theta U_L(B,\omega)\}.$$

Note that the random variables $\{U_L(B,\omega)\}_{B\in\mathcal{B}_H(O)}$ are dependent and identically distributed. We denote their common characteristic function by $\Psi_L(\theta)$. We have

$$\mathbb{E}\Psi_L^\omega(\theta) = \Psi_L(\theta).$$

Let us show that, for every fixed θ,

$$\sum_{L<0} (\sigma[\Psi_L^\omega(\theta)])^2 < \infty. \tag{4.6}$$

As soon as (4.6) is established, the Chebyshev inequality and the Borel–Cantelli lemma yield that

$$\Psi_L^{\omega}(\theta) - \Psi_L(\theta) \to 0 \quad \text{a.s.}$$

Since the random variables $\Phi_L^{\omega}(\theta) - \Psi_L^{\omega}(\theta)$ and the functions $\Phi(\theta) - \Psi_L(\theta)$ tend to zero pointwise as $L \to -\infty$, we will finally get (4.5).

We now prove (4.6). If X, Y are independent random variables with $|X| = 1$ and $|Y| \leqslant 1$, then

$$(\sigma[XY])^2 \leqslant (\sigma[Y])^2 + 2(1 - |\mathbb{E}X|). \tag{4.7}$$

Let ϕ be the common characteristic function of the independent identically distributed random variables $\{\varepsilon(B)\}_{B\in\mathcal{B}}$. Substituting the data

$$X = \exp\{i\theta a_L \varepsilon(B_L)\}, \qquad Y = \frac{1}{n_L n_{L+1} \cdots n_{-1}} \sum_{B\in\mathcal{B}_H(O)} \exp\{i\theta U_{L+1}(B)\},$$

into (4.7), we obtain

$$(\sigma[\Psi_L(\theta)])^2 \leqslant (\sigma[Y])^2 + 2\bigl(1 - |\phi(a_L\theta)|\bigr) \leqslant (\sigma[Y])^2 + a_L^2\theta^2.$$

Let $\{O_l\colon 0 \geqslant l \geqslant -\infty\}$ be an infinite geodesic path in $\mathcal{T}$ with $O_0 \in H$, $O_L = O$ and $O_{-\infty} = \varpi$. We write $\{O_{L+1}^i\colon 1 \leqslant i \leqslant n_L,\}$, where $O_{L+1}^1 = O_{L+1}$, for the sequence of those n_L ultrametric balls that belong to the horocycle H_{L+1} and are subballs of the ball $O = O_L$. Put

$$Y = \frac{1}{n_L} \sum_{i=1}^{n_L} Y_i,$$

where

$$Y_i = \frac{1}{n_{L+1} \cdots n_{-1}} \sum_{B\in\mathcal{B}_H(O_{L+1}^i)} \exp\{i\theta U_{L+1}(B)\}.$$

The random variables $\{Y_i\colon 1 \leqslant i \leqslant n_L\}$ are independent and identically distributed, and $Y_1 = \Psi_{L+1}(\theta)$. As a result, we obtain the following inequality:

$$(\sigma[\Psi_L(\theta)])^2 \leqslant \frac{1}{n_L}(\sigma[\Psi_{L+1}(\theta)])^2 + a_L^2\theta^2,$$

which evidently proves (4.6). □

Definition 4.2. The measure $\mathcal{M}$ defined in (4.2) is called the *integrated density of states* corresponding to the horocycle H. If $\mathcal{M}$ is absolutely continuous with respect to the Lebesgue measure, that is,

$$\mathcal{M}(I) = \int_I \mathfrak{m}(\tau)\, d\tau,$$

then the function $\mathfrak{m}(\tau)$ is called the *density of states* corresponding to the horocycle H.

The issues of existence, uniqueness, C^∞-smoothness and so on for the density of states $\mathfrak{m}(\tau)$ related to (C, ϵ) are basic in various applications (see Theorem 5.1 in § 5).

Remark 4.3. We recall that the measures $\mathcal{M}$ and $\mathcal{N}$ (defined in (4.2) and (4.3) respectively) are related by the equation

$$\mathcal{M} = \mathcal{N} \circ \vartheta^{-1},$$

where $\vartheta\colon x \mapsto ax + b$ with $a = b = \lambda_H$. In particular, $\mathcal{M}$ is absolutely continuous with respect to the Lebesgue measure if and only if $\mathcal{N}$ is.

The measure $\mathcal{N}$ has a remarkable feature: it belongs to the class $\mathfrak{I}$ of probability measures that can be represented as the distribution of a random variable U of the form

$$U = \sum_{k \geqslant 0} b_k \varepsilon_k, \tag{4.8}$$

where $\{\varepsilon_k\}_{k\geqslant 0}$ are symmetric independent identically distributed random variables with values in some finite interval $I \subset \mathbb{R}^1$, and the numbers $b_k > 0$ satisfy $\sum b_k = 1$.

Various properties of $\mathfrak{I}$-distributions (infinite convolutions) have been studied by many authors since the 1930s (see, for example, [19]–[23], [31] and the references therein). We mention two remarkable properties of $\mathfrak{I}$-distributions. The first was found by Lévy (1937), and the second by Jessen and Wintner (1935) (see, for example, [19], Theorems 3.7.6 and 3.7.7 respectively).

(a) The Lebesgue decomposition of any $\mathfrak{I}$-distribution $\mathcal{N}$ contains no discrete component.

(b) If $\{\varepsilon_k\}$ in (4.8) are discrete, then $\mathcal{N}$ is either singular or absolutely continuous (with respect to the Lebesgue measure).

Examples of singular $\mathfrak{I}$-distributions will be given below (infinite Bernoulli convolutions). We first consider a simple class of absolutely continuous $\mathfrak{I}$-distributions. Let $\mathcal{N}$ be an $\mathfrak{I}$-distribution of the form (4.8). Suppose that the common characteristic function ϕ of independent identically distributed random variables $\{\varepsilon_k\}$ satisfies

$$|\phi(x)| \leqslant x^{-D} \quad \text{at} \quad \infty$$

for some $D > 0$. Then, clearly, the characteristic function $\Phi(x)$ of the measure $\mathcal{N}$, being an infinite product of characteristic functions, satisfies

$$|\Phi(x)| \leqslant x^{-B} \quad \text{at} \quad \infty$$

for all $B > 0$ and, therefore, $\mathcal{N}$ admits a C^∞-density with respect to the Lebesgue measure. This observation can be refined as follows.

Proposition 4.4. *Let $\mathcal{N}$ be an $\mathfrak{I}$-distribution. Assume that the common characteristic function ϕ of independent identically distributed random variables $\{\varepsilon_k\}$ tends to zero at infinity and*

$$b_k \geqslant C \exp\{-Dk\}$$

for some $C, D > 0$. Then $\mathcal{N}$ admits a C^∞-density with respect to the Lebesgue measure.

Proof. Let Φ be the characteristic function of $\mathcal{N}$. For a given $\epsilon > 0$ choose $N = N(\epsilon) > 1$ in such a way that $|\phi(z)| \leqslant \epsilon$ for all $z \geqslant N$ and write

$$|\Phi(x)| \leqslant \prod_{k\colon b_k x \geqslant N} \phi(b_k x) \leqslant \exp\left\{-\#\{k\colon b_k x \geqslant N\} \log \frac{1}{\epsilon}\right\}.$$

By hypothesis,

$$\#\{k\colon b_k x \geqslant N\} \geqslant \log\left(\frac{Cx}{N}\right)^{\frac{1}{D}},$$

whence

$$|\Phi(x)| \leqslant Ax^{-B} \quad \text{at} \quad \infty$$

with

$$A = \left(\frac{N}{C}\right)^{\frac{1}{D}\log\frac{1}{\epsilon}}, \qquad B = \frac{1}{D}\log\frac{1}{\epsilon}.$$

Since $\epsilon > 0$ can be chosen arbitrarily small, the result follows. □

Various examples of characteristic functions ϕ of singular distributions satisfying the hypotheses of Proposition 4.4 are given in [19], Ch. 3, and [20], Chs. 6 and 7. Here is an example due to Kerschner (1936): if a is a rational number, $0 < a < 1/2$ and a is not equal to $1/n$ for an integer $n \geqslant 3$, then the function

$$\phi(x) = \prod_{k=1}^{\infty} \cos(xa^k)$$

is the characteristic function of a singular symmetric $\mathfrak{I}$-distribution and we have

$$|\phi(x)| \leqslant \frac{1}{(\log x)^{\gamma}} \quad \text{at} \quad +\infty \tag{4.9}$$

for some $\gamma > 0$.

Certain applications of Proposition 4.4 which we have in mind concern homogeneous ultrametric spaces X such that the sequence $\{n_H\}$ of forward degrees (defined by the tree $\mathcal{T}(X)$ of balls) is bounded and the homogeneous hierarchical Laplacian L_C on X satisfies the condition (3.11). For example, take $\mathbb{Q}_p$ for X and let L_C be the operator of the factional derivative introduced in (2.2).

By modifying the proof of Proposition 4.4, one can obtain results that are applicable to unbounded sequences $\{n_H\}$, for example, when X is the infinite symmetric group $S_\infty = \bigcup_{n\in\mathbb{N}} S_n$. In this case n_H is equal to l for the horocycle H consisting of the symmetric group S_l and its cosets aS_l.

Proposition 4.5. *Assume that the common characteristic function ϕ of independent identically distributed random variables $\{\varepsilon_k\}$ satisfies (4.9) and $b_k \geqslant C/k!$ for some $C > 0$. Then the characteristic function $\Phi(x)$ of the corresponding $\mathfrak{I}$-distribution $\mathcal{N}$ satisfies*

$$|\Phi(x)| \leqslant x^{-(\gamma-\epsilon)} \quad \textit{at} \quad +\infty$$

for any ϵ, $0 < \epsilon < \gamma$. In particular, if $\gamma > 1$, then the distribution $\mathcal{N}$ admits a C^k-density with $k \leqslant \gamma - 1$.

Proof. By hypothesis,

$$n(t) := \#\{k\colon b_k \geqslant t\} \geqslant \#\left\{k\colon \frac{C}{k!} \geqslant t\right\} \sim \frac{\log(1/t)}{\log\log(1/t)} \quad \text{at} \quad 0.$$

For $x > 1$ we put $\overline{\Phi}(x) = (\log x)^\gamma$ and choose $r(x)$ in such a way that

$$\log r(x) \sim \frac{\log x}{\log\log x} \quad \text{at} \quad +\infty.$$

For sufficiently large x we define

$$A(x) := n\left(\frac{r(x)}{x}\right)\log\overline{\Phi}(r(x)).$$

By hypothesis,

$$\liminf_{x\to+\infty}\frac{A(x)}{\gamma\log x} \geqslant 1. \tag{4.10}$$

We now estimate the characteristic function $\Phi(x)$ of the $\mathfrak{I}$-distribution $\mathcal{N}$:

$$\begin{aligned}|\Phi(x)| = \prod_k |\phi(b_k x)| &\leqslant \prod_{k\colon b_k x \geqslant r(x)} |\phi(b_k x)| \leqslant \prod_{k\colon b_k x\geqslant r(x)} (\overline{\Phi}(b_k x))^{-1} \\ &\leqslant (\overline{\Phi}(r(x)))^{-n(\frac{r(x)}{x})} = \exp\{-A(x)\}.\end{aligned}$$

Using (4.10), we obtain the desired result. □

Infinite Bernoulli convolutions. Let $\{\varepsilon_k\}_{k\geqslant 0}$ be independent identically distributed random variables taking the values ± 1 with equal probability $1/2$. We define a one-parameter family of random variables

$$U_\lambda = \sum_{k\geqslant 0}\lambda^k\varepsilon_k, \qquad 0<\lambda<1. \tag{4.11}$$

Let $\mathcal{N}_\lambda$ be the distribution of U_λ. The measure $\mathcal{N}_\lambda$ is an infinite convolution product of the discrete measures $(\delta_{-\lambda^k}+\delta_{\lambda^k})/2$. It is called an *infinite Bernoulli convolution.* The characteristic function Φ_λ of $\mathcal{N}_\lambda$ can be represented as a convergent infinite product,

$$\Phi_\lambda(\theta) = \prod_{k=0}^{\infty}\cos(\lambda^k\theta).$$

We describe the results of some papers on infinite Bernoulli convolutions (see the survey [21] of Solomyak for a complete bibliography).

Jessen and Wintner (1935) showed that $\mathcal{N}_\lambda$ is either absolutely continuous, or purely singular, depending on λ.

Kerschner and Wintner (1935) observed that $\mathcal{N}_\lambda$ is singular for $\lambda\in(0,2^{-1})$ since it is supported on a Cantor set of zero Lebesgue measure.

Wintner (1935) noted that $\mathcal{N}_\lambda$ is uniform on $[-2,2]$ for $\lambda = 2^{-1}$. Moreover, if $\lambda = 2^{-1/k}$ for an integer $k\geqslant 1$, then $\mathcal{N}_\lambda$ is absolutely continuous with a C^{k-1}-density.

Erdős (1939) showed that $\mathcal{N}_\lambda$ with $\lambda \in (2^{-1}, 1)$ is singular if $1/\lambda$ is a PV number (an algebraic integer whose Galois conjugates are strictly less than one in modulus, the golden ratio $(1+\sqrt{5})/2$ being an example). No other $\lambda \in (2^{-1}, 1)$ with singular $\mathcal{N}_\lambda$ are known.

Solomyak (1995) proved a conjecture of Garsia (1962) that $\mathcal{N}_\lambda$ is absolutely continuous for almost all $\lambda \in (2^{-1}, 1)$. A stronger conjecture (saying that this holds for all but countably many $\lambda \in (2^{-1}, 1)$) is still very much open.

Bernoulli perturbations of $\mathfrak{B}^\alpha$. As an example, we consider the homogeneous Laplacian $\mathfrak{B}^\alpha$ introduced in (2.2). The operator $\mathfrak{B}^\alpha$ acts on $L^2(\mathbb{Q}_p, m)$, where $\mathbb{Q}_p$ is the ring of p-adic numbers (p is not necessarily a prime) and m is the Haar measure:

$$\mathfrak{B}^\alpha f(x) = \frac{1-p^{-\alpha}}{1-p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{f(x)-f(y)}{|x-y|^{1+\alpha}}\, dm(y).$$

Let $\mathfrak{B}^\alpha(\omega)$ be the perturbation of $\mathfrak{B}^\alpha$ by the symmetric independent identically distributed Bernoulli random variables $\{\theta\varepsilon(B)\}_{B\in\mathcal{B}}$, $0<\theta<1$. Let $\mathcal{N}_\lambda$ be the infinite Bernoulli convolution with $\lambda = p^{-\alpha}$. Set

$$a = \theta(1-p^{-\alpha})\lambda_H, \qquad b = \lambda_H, \quad \vartheta(x) = ax+b.$$

Since the eigenvalue λ_H of $\mathfrak{B}^\alpha$ corresponding to the horocycle $H = H_l$ is equal to $p^{\alpha l}$, the following result is obtained using Theorem 4.1 and the properties of infinite Bernoulli convolutions listed above.

Theorem 4.6. *Let $\mathcal{M}_\alpha$ be the integrated density of states associated with the operator $\mathfrak{B}^\alpha(\omega)$ by Theorem 4.1 and Definition 4.2. Let $\mathcal{N}_\lambda$ be the infinite Bernoulli convolution with $\lambda = p^{-\alpha}$. Then*

$$\mathcal{M}_\alpha = \mathcal{N}_\lambda \circ \vartheta^{-1}.$$

In particular, with respect to the Lebesgue measure, $\mathcal{M}_\alpha$ is

– *singular for all $\alpha > (\log 2)/(\log p)$;*

– *uniform for $\alpha = (\log 2)/(\log p)$;*

– *absolutely continuous with an L^2-density for almost all α in the interval $0 < \alpha \leqslant (\log 2)/(\log p)$.*

Moreover, when $\alpha = (\log 2)/(k \log p)$, $k \in \mathbb{N}$, it admits a C^{k-1}-density.

§ 5. Poisson convergence

We fix a horocycle H. The eigenvalues $\lambda(B,\omega)$, $B \in \mathcal{B}_H(O)$, may themselves be represented by the following *empirical process*:

$$N_O^\omega(I) = \sum_{B\in\mathcal{B}_H(O)} \delta_{\lambda(B,\omega)}(I), \qquad I \in \mathcal{B}(\mathbb{R}).$$

Here the *intensity measure* $\mu_O(I)$ (the expected number of points $\lambda(B,\omega)$, $B \in \mathcal{B}_H(O)$, occurring in the set I) is calculated by the formula

$$\mu_O(I) = \mathbb{E}N_O(I) = |\mathcal{B}_H(O)|\mathbb{P}\bigl(\omega\in\Omega\colon \lambda(B,\omega)\in I\bigr).$$

Recall that the right-hand side is independent of $B \in \mathcal{B}_H(O)$. We fix numbers $c, \tau_0 > 0$ and consider a small interval

$$I = \left\{\tau \colon |\tau - \tau_0| \leqslant \frac{c}{2|\mathcal{B}_H(O)|}\right\}. \tag{5.1}$$

Assume that the density of states $\mathfrak{m}(\tau)$ (see Definition 4.2) exists and is continuous at $\tau = \tau_0$ and that $\mathfrak{m}(\tau_0) > 0$. Then

$$\lim_{O \to \varpi} \mu_O(I) = c\mathfrak{m}(\tau_0) =: \lambda_c > 0. \tag{5.2}$$

In particular, if the random variables $\lambda(B, \omega)$, $B \in \mathcal{B}_H(O)$, were independent and identically distributed, then (5.2) would yield the classical convergence of $N_O = N_O^\omega(I)$ to the Poisson random variable $\mathcal{P}_\lambda$ with intensity $\lambda = \lambda_c$. More precisely, in the case of independent identically distributed random variables we would have (see [32]–[34]) that

$$\|\mathcal{L}(N_O) - \mathcal{L}(\mathcal{P}_\lambda)\|_{\mathrm{TV}} \leqslant \frac{\min\{\lambda, \lambda^2\}}{2|\mathcal{B}_H(O)|}.$$

Recall that $\mathcal{L}(X)$ stands for the law of a random variable X and $\|\mu - \nu\|_{\mathrm{TV}}$ is the distance between μ and ν in total variation.

However, in our case the random variables $\lambda(B, \omega)$, $B \in \mathcal{B}_H(O)$, are *dependent*, whence the classical theory is not directly applicable and needs some justifications and complements. The basic ingredients in our study are the stationarity property of the family $\{\lambda(B, \omega)\}_{B \in H}$ and certain bounds for its correlation functions (see § 3). We will prove the following theorem.

Theorem 5.1. *Assume that condition* (3.11) *holds with* $\delta > 2$ *and the common law of the independent identically distributed random variables* $\{\varepsilon(B)\}$ *admits a bounded density. Then, in the notation introduced above, as* $O \to \varpi$ *we have*

$$\mathcal{L}(N_O) \to \mathcal{L}(\mathcal{P}_\lambda) \quad \textit{in the Bernoulli topology.}$$

Before proving this, we note that under these hypotheses the density of states $\mathfrak{m}(\tau)$ exists and belongs to C^∞ (see Proposition 4.4). In particular, the number $\lambda = \lambda_c$ is well defined by the formula (5.2) for every $c > 0$. Furthermore, for every $B \in H$ the eigenvalue $\lambda(B, \omega)$ may be written in the form

$$\lambda(B, \omega) = \lambda_H(1 + U(B, \omega)), \tag{5.3}$$

where

$$U(B, \omega) = \sum_{B \subseteq B_k} a_k \varepsilon(B_k, \omega), \qquad a_k = \frac{C(B_k)}{\lambda_H}. \tag{5.4}$$

Since $\sum a_k = 1$ and $|\varepsilon(B, \omega)| \leqslant \epsilon$ for all B, ω and for some ϵ, $0 < \epsilon < 1$, we have

$$|U(B, \omega)| \leqslant \sup_{k, \omega} |\varepsilon(B_k, \omega)| = \epsilon.$$

The common distribution function $\mathcal{N}(t)$ of the family $\{U(B,\omega)\}_{B\in H}$ is absolutely continuous and its density $\mathfrak{n}(t)$ is related to the integrated density of states $\mathfrak{m}(\tau)$ by the formula

$$\mathfrak{n}(t) = \lambda_H \mathfrak{m}(\lambda_H t + \lambda_H).$$

In particular, the function $\mathfrak{n}(t)$ with support in $[-\epsilon,\epsilon]$ is continuous and strictly positive at $t_0 = \tau_0/\lambda_H - 1$.

Let $\widetilde{N}_O^\omega$ be the empirical process defined by the family $\{U(B,\omega)\}_{B\in\mathcal{B}_H(O)}$. We choose an interval

$$\widetilde{I} = \left\{t\colon |t-t_0| \leqslant \frac{\widetilde{c}}{2|\mathcal{B}_H(O)|}\right\}, \qquad \widetilde{c} = \frac{c}{\lambda_H},$$

and put $\widetilde{N}_O := \widetilde{N}_O^\omega(\widetilde{I})$. It follows from (5.3) that

$$\mathbb{P}\{N_O = k\} = \mathbb{P}\{\widetilde{N}_O = k\}$$

and, therefore,

$$\lim_{O\to\varpi} \mathbb{P}\{N_O = k\} = \lim_{O\to\varpi} \mathbb{P}\{\widetilde{N}_O = k\}.$$

Having all these observations in mind, we will prove the evident U-version of Theorem 5.1.

Fix a horocycle H and let O tend to ϖ. Clearly, when studying the family $U(B,\omega)_{B\in H}$, we can replace the original ultrametric space X by a certain discrete ultrametric space. For example, X may be replaced by the discrete Abelian group

$$G = \bigoplus_{k\geqslant 1} \mathbb{Z}(n_k)$$

whose canonical ultrametric structure is defined by the family $\{G_l\}_{l\geqslant 0}$ of finite subgroups

$$G_0 = \{0\}, \qquad G_l = \prod_{1\leqslant k\leqslant l} \mathbb{Z}(n_k).$$

With this agreement in mind, we see that $H = H_0$ is the set of all singletons, H_1 is the set of all ultrametric balls of the form $g+G_1$ and so on. When $B=\{g\}$, we shall write $U_g(\omega)$ instead of $U(B,\omega)$. For every $B_i \supseteq B$ we set $\varepsilon(B_i,\omega) =: \varepsilon_{ig}(\omega)$. Thus we define a stationary family $\{U_g\}_{g\in G}$:

$$U_g(\omega) = \sum_{i=0}^{\infty} a_i \varepsilon_{ig}(\omega).$$

Let $Z_l^c(\omega)$ be the number of elements $U_g(\omega)$, $g\in G_l$, lying in the interval

$$I_l^c = \left\{t\colon |t-t_0| \leqslant \frac{c}{2\pi_l}\right\}, \qquad \pi_l = n_1\cdots n_l.$$

We set $\lambda_c = c\mathfrak{n}(t_0)$ and prove that

$$\lim_{l\to\infty} \mathbb{P}\{Z_l^c = k\} = \frac{(\lambda_c)^k}{k!}\exp\{-\lambda_c\}. \tag{5.5}$$

Writing Z_l^c in the form

$$Z_l^c(\omega) = \sum_{g \in G_l} \delta_{U_g(\omega)}(I_l^c),$$

we have

$$\lim_{l\to\infty} \mathbb{E} Z_l^c = \lim_{l\to\infty} \pi_l \mathbb{P}(U_g \in I_l^c) = \lambda_c.$$

It follows that the family of measures $\{\mathcal{L}(Z_l^c)\}_{l\in\mathbb{N}}$ is *tight*. Hence it is relatively compact in the weak topology. Let $\mathcal{L}(Z)$ be an accumulation point of $\{\mathcal{L}(Z_l^c)\}_{l\in\mathbb{N}}$. We claim that the *random variable* Z is *infinitely divisible*. Indeed, using the ultrametric structure of G, we can write

$$Z_l^c = \sum_{g\in\mathbb{Z}(n_l)} \tau_g(Z_{l-1}^{c/n_l}),$$

where

$$\tau_g(Z_{l-1}^{c/n_l}) := \sum_{a\in G_{l-1}} \delta_{U_{g+a}}(I_{l-1}^{c/n_l}).$$

The random variables $\tau_g(Z_{l-1}^{c/n_l})$, $g \in \mathbb{Z}(n_l)$, are identically distributed and dependent, but their dependence becomes weaker as l tends to infinity. Namely, for every $g \in G_l$ we can write

$$U_g = \sum_{i=0}^{l-1} a_i \varepsilon_{ig} + \sum_{i=l}^{\infty} a_i \varepsilon_{ig} =: \widetilde{U}_g + K_l. \tag{5.6}$$

The common part K_l of the random variables U_g is independent of the family $\{\widetilde{U}_g\}$ and may be estimated as

$$|K_l(\omega)| \leqslant \epsilon \sum_{i=l}^{\infty} a_i = O(a_l) = O(\pi_l^{-\delta/2}).$$

In particular, assuming that $\delta > 2$, we obtain

$$k_l := \sup_{\omega} |K_l(\omega)| = o(\pi_l^{-1}). \tag{5.7}$$

Let us compute the characteristic function

$$\Phi_Z(\gamma) = \mathbb{E}\exp\{-\gamma Z\}, \qquad \gamma \geqslant 0,$$

of the random variable $Z = Z_l^c$. We have

$$Z_l^c = \sum_{a\in\mathbb{Z}(n_l)} \sum_{g\in G_{l-1}} \delta_{U_{a+g}}(I_l^c) = \sum_{a\in\mathbb{Z}(n_l)} \tau_a(Z_{l-1}^{c/n_l}),$$

whence

$$\begin{aligned}\Phi_{Z_l^c}(\gamma) &= \mathbb{E}\big[\mathbb{E}(\exp\{-\gamma Z_l^c\} \mid K_l)\big] = \mathbb{E}\big[(\mathbb{E}(\exp\{-\gamma Z_{l-1}^{c/n_l}\} \mid K_l))^{n_l}\big] \\ &= \int d\mu_l(k)\big(\mathbb{E}(\exp\{-\gamma Z_{l-1}^{c/n_l}\} \mid K_l = k)\big)^{n_l},\end{aligned}$$

where μ_l is the law of K_l. Using (5.6), we compute

$$\mathbb{E}\big(\exp\{-\gamma Z_{l-1}^{c/n_l}\} \mid K_l = k\big) = \mathbb{E}\exp\Big\{-\gamma \sum_{g\in G_{l-1}} \delta_{\widetilde{U}_g}(I_{l-1}^{c/n_l} - k)\Big\}.$$

It follows from (5.7) that $\epsilon_l := \pi_l k_l = o(1)$. Hence for all $g \in G_{l-1}$ and $k \in [-k_l, k_l]$ we have

$$\{U_g \in I_{l-1}^{(c-2\epsilon_l)/n_l}\} \subseteq \{\widetilde{U}_g \in I_{l-1}^{c/n_l} - k\} \subseteq \{U_g \in I_{l-1}^{(c+2\epsilon_l)/n_l}\}$$

and

$$(\Phi_{Z_{l-1}^{(c+2\epsilon_l)/n_l}}(\gamma))^{n_l} \leqslant \Phi_{Z_l^c}(\gamma) \leqslant (\Phi_{Z_{l-1}^{(c-2\epsilon_l)/n_l}}(\gamma))^{n_l}. \tag{5.8}$$

Case 1: $n_l = n$ for some infinite sequence $\{l_k\}$. Then, along this sequence, we have

$$\mathbb{E}Z_{l-1}^{(c\pm 2\epsilon_l)/n} = \pi_{l-1}\mathbb{P}(U_g(\omega) \in I_{l-1}^{(c\pm2\epsilon_l)/n}) \to \frac{\lambda_c}{n},$$

whence the family $\{\mathcal{L}(Z_{l-1}^{(c\pm2\epsilon_l)/n})\}$ is tight. We recall that $\{\mathcal{L}(Z_l^c)\}$ is also tight. Choose a sequence $\{l'_k\} \subset \{l_k\}$ along which we have

$$Z_l^c \to Z, \quad Z_{l-1}^{(c\pm2\epsilon_l)/n} \to Z^\pm \quad \text{in distribution.}$$

Since

$$Z_{l-1}^{(c-2\epsilon_l)/n}(\omega) \leqslant Z_{l-1}^{(c+2\epsilon_l)/n}(\omega)$$

for all $\omega \in \Omega$, with strict inequality if and only if ω belongs to the event

$$\Omega_l = \big\{U_g(\omega) \in I_{l-1}^{(c+2\epsilon_l)/n} \setminus I_{l-1}^{(c-2\epsilon_l)/n} \text{ for some } g \in G_{l-1}\big\}$$

whose probability is estimated by the formula

$$\mathbb{P}(\Omega_l) \leqslant 2|G_{l-1}|\frac{4\epsilon_l}{\pi_l}(\mathfrak{n}(t_0) + o(1)) = O(\epsilon_l) \to 0 \quad \text{as} \quad l \to \infty,$$

we must have $Z^+ = Z^-$ a.s. Put $Z' = Z^\pm$. Passing to the limit in the inequalities (5.8), we obtain

$$\Phi_Z(\gamma) = (\Phi_{Z'}(\gamma))^n. \tag{5.9}$$

Using the same procedure as before, we have

$$\Phi_{Z'}(\gamma) = (\Phi_{Z''}(\gamma))^n$$

and so on. Hence Z is infinitely divisible, as required.

Case 2: $n_l \to \infty$. We partition G_l into disjoint subsets A_i, $i = 1, 2, 3$, consisting of G_{l-1}-cosets such that $|A_1| = |A_2| = [\pi_l/2]$. Put

$$Z_{l,i}^c = \sum_{g\in A_i} \delta_{U_g(\omega)}(I_l^c), \qquad i = 1, 2, 3,$$

whence

$$Z_l^c = Z_{l,1}^c + Z_{l,2}^c + Z_{l,3}^c.$$

Note that, as $l \to \infty$,

$$\mathbb{E}Z_{l,3}^c \leqslant \frac{1}{n_l}\mathbb{E}Z_l^c \to 0, \qquad \mathbb{E}Z_{l,i}^c \to \frac{\lambda_c}{2}, \quad i = 1, 2.$$

In particular, $Z_{l,3}^c \to 0$ in probability. Clearly, $Z_{l,1}^c$ and $Z_{l,2}^c$ have the same distributions. The families of laws $\{\mathcal{L}(Z_{l,i}^c)\}$ are tight. Arguing as in Case 1, we find a subsequence along which $Z_l^c \to Z$, $Z_{l,1}^c \to Z'$ and $Z_{l,2}^c \to Z''$ in distribution, with $\mathcal{L}(Z') = \mathcal{L}(Z'')$. The equation (5.9) holds for $n = 2$, whence Z is infinitely divisible, as desired.

Since the random variable Z is non-negative and integer-valued, its characteristic function takes the form

$$\Phi_Z(\gamma) = \exp\left\{-a\gamma - \int (1 - e^{-\gamma x})\, m(dx)\right\}, \qquad \gamma \geqslant 0, \tag{5.10}$$

where m is a finite measure on $\mathbb{N}$ and $a \in \mathbb{N} \cup \{0\}$. Since the range of Z is the whole of $\mathbb{N} \cup \{0\}$, we must have $a = 0$. Note that

$$\mathbb{E}Z = \int x\, m(dx), \tag{5.11}$$

$$\mathbb{E}Z^2 = \int x^2\, m(dx) + \left(\int x\, m(dx)\right)^2. \tag{5.12}$$

We claim that the measure m is concentrated at $x = 1$. Indeed, suppose that the following inequality holds:

$$\limsup_{l\to\infty} \mathbb{E}(Z_l^c)^2 \leqslant \lambda_c + (\lambda_c)^2. \tag{5.13}$$

Then the family $\{Z_l^c\}$ is uniformly integrable, so that along some subsequence (l_k) we have

$$\mathbb{E}Z = \lim \mathbb{E}Z_l^c = \lambda_c. \tag{5.14}$$

Furthermore, by Fatou's lemma and (5.13),

$$\mathbb{E}Z^2 \leqslant \limsup_{l\to\infty} \mathbb{E}(Z_l^c)^2 \leqslant \lambda_c + (\lambda_c)^2,$$

whence by (5.12) and (5.14) we obtain

$$\int x^2\, m(dx) \leqslant \lambda_c = \int x\, m(dx).$$

Since m is concentrated on $\mathbb{N}$, it follows that $m = \lambda_c \delta_{\{1\}}$. Hence Z is Poisson with parameter λ_c. This proves (5.5). It remains to prove (5.13).

There is no loss of generality in assuming that $n_l \equiv n$. Let $g \wedge g'$ be the confluent of g and g', that is, the smallest ball in G which contains both g and g'. We have

$$\mathbb{E}(Z_l^c)^2 = \sum_{g,g' \in G_l} \mathbb{P}(U_g \in I_l^c, U_{g'} \in I_l^c) = \sum_{0 \leqslant j \leqslant l} \sum_{g \wedge g' \in \mathcal{B}_{H_j}(G_l)} \mathbb{P}(U_g \in I_l^c, U_{g'} \in I_l^c)$$
$$= \sum_{1 \leqslant j \leqslant l} n^{l-j} \sum_{g \wedge g' = G_j} \mathbb{P}(U_g \in I_l^c, U_{g'} \in I_l^c) + n^l \mathbb{P}(U_0 \in I_l^c).$$

For any two pairs (g, g') and (f, f') with $g \wedge g' = f \wedge f'$ we have

$$\mathbb{P}(U_g \in I_l^c, U_{g'} \in I_l^c) = \mathbb{P}(U_f \in I_l^c, U_{f'} \in I_l^c).$$

Therefore we obtain

$$\sum_{g \wedge g' = G_j} \mathbb{P}(U_g \in I_l^c, U_{g'} \in I_l^c) = n^j (n^j - n^{j-1}) \mathbb{P}(U_{g_j} \in I_l^c, U_{g_j'} \in I_l^c),$$

where g_j, g_j' are chosen in such a way that $g_j \wedge g_j' = G_j$. It follows that

$$\mathbb{E}(Z_l^c)^2 = J + J',$$

where

$$J = (n-1) \sum_{1 \leqslant j \leqslant l} n^{l+j-1} \mathbb{P}(U_{g_j} \in I_l^c, U_{g_j'} \in I_l^c), \qquad J' = n^l \mathbb{P}(U_0 \in I_l^c).$$

Choosing $\{\varepsilon_i\}$ and $\{\varepsilon_i'\}$ to be two independent families of independent identically distributed random variables having the same distribution as $\{\varepsilon(B)\}_{B \in \mathcal{B}}$, we write

$$U_{g_j} = \sum_{i=0}^{j-1} a_i \varepsilon_i + \sum_{i=j}^{\infty} a_i \varepsilon_i =: \widetilde{U}_j + K_j$$

and, similarly,

$$U_{g_j'} = \sum_{i=0}^{j-1} a_i \varepsilon_i' + \sum_{i=j}^{\infty} a_i \varepsilon_i =: \widetilde{U}_j' + K_j.$$

We already know that

$$\lim_{l \to \infty} n^l \mathbb{P}(U_0 \in I_l^c) = \lambda_c, \tag{5.15}$$

whence we are left with showing that

$$\limsup_{l \to \infty} J \leqslant (\lambda_c)^2. \tag{5.16}$$

To simplify the notation, we put

$$\mathcal{P}_{l-i} = \mathbb{P}(U_{g_{l-i}} \in I_l^c, U_{g_{l-i}'} \in I_l^c).$$

Choose $m,\ 0<m<l$, and split the sum in J into two parts:

$$J=(n-1)\sum_{0\leqslant i<l} n^{2l-i-1}\mathcal{P}_{l-i}$$
$$=(n-1)n^{2l}\biggl(\sum_{0\leqslant i\leqslant m}+\sum_{m<i<l}\biggr)n^{-(i+1)}\mathcal{P}_{l-i}=:J_m+J^m.$$

Write

$$U_{g_{l-i}}=a_0\varepsilon_0+A_{l-i}+K_{l-i}$$

and, similarly,

$$U_{g'_{l-i}}=a_0\varepsilon'_0+A'_{l-i}+K_{l-i}.$$

Since ε_0, ε'_0, A_{l-i}, A'_{l-i}, K_{l-i} are independent, we can write $\mathcal{P}_{l-i}$ in the form

$$\int \mathbb{P}(a_0\varepsilon_0\in I_l^c-a-k)\mathbb{P}(a_0\varepsilon'_0\in I_l^c-a'-k)\,d\mu(a)\,d\mu(a')\,d\nu(k), \tag{5.17}$$

where μ is the common distribution of the independent identically distributed random variables A_{l-1}, A'_{l-1}, and ν is the distribution of K_{l-i}.

We now assume that the common distribution function of $\{\varepsilon(B)\}_{B\in\mathcal{B}}$ admits a bounded density $\varepsilon(x)$. Then the representation (5.17) for $\mathcal{P}_{l-i}$ yields that

$$\mathcal{P}_{l-i}\leqslant n^{-2l}\|\varepsilon\|_\infty^2\frac{c^2}{a_0^2}$$

and, therefore,

$$J^m\leqslant (n-1)n^{2l}\sum_{m<i<l} n^{-(i+1)}n^{-2l}\|\varepsilon\|_\infty^2\frac{c^2}{a_0^2}$$
$$<\|\varepsilon\|_\infty^2\frac{c^2}{a_0^2}(n-1)\sum_{i>m} n^{-(i+1)}=n^{-(m+1)}\|\varepsilon\|_\infty^2\frac{c^2}{a_0^2}\to 0\quad\text{as}\quad m\to\infty.$$

To estimate J_m, we choose $\theta,\ 0<\theta<1$, and apply the same procedure of decoupling as before:

$$\mathcal{P}_{l-i}=\int(\mathbb{P}(\widetilde{U}_{l-i}\in I_l^c-k))^2\,d\nu(k)\leqslant\sup_{|k|\leqslant\theta/n^l}(\mathbb{P}(\widetilde{U}_{l-i}\in I_l^c-k))^2+\mathbb{P}\biggl(|K_{l-i}|>\frac{\theta}{n^l}\biggr)$$
$$\leqslant(\mathbb{P}(U_g\in I_l^{c+4\theta}))^2+\mathbb{P}\biggl(|K_{l-i}|>\frac{\theta}{n^l}\biggr).$$

Hence,

$$J_m\leqslant(\mathbb{P}(U_g\in I_l^{c+4\theta}))^2(n-1)n^{2l}\sum_{0\leqslant i\leqslant m} n^{-(i+1)}$$
$$+(n-1)n^{2l}\sum_{0\leqslant i\leqslant m} n^{-(i+1)}\mathbb{P}\biggl(|K_{l-i}|>\frac{\theta}{n^l}\biggr).$$

Furthermore, as $l \to \infty$ with fixed m and θ, we have

$$(\mathbb{P}(U_g \in I_l^{c+4\theta}))^2 (n-1) n^{2l} \sum_{0 \leqslant i \leqslant m} n^{-(i+1)} \to (\lambda_{c+4\theta})^2 (1 - n^{-m}).$$

Fix $p > 1$, $m < l$ and apply Chebyshev's inequality with $i \leqslant m$:

$$\mathbb{P}\biggl(|K_{l-i}| > \frac{\theta}{n^l}\biggr) \leqslant \theta^{-p} n^{pl} \|\varepsilon\|_\infty^p \biggl(\sum_{k \geqslant l-m} a_k\biggr)^p.$$

We now assume that condition (3.11) holds with $\delta/2 = 1 + \gamma$, $\gamma > 0$. By (3.11), as $l \to \infty$ with fixed m, we have

$$\frac{1}{a_{l-m}} \sum_{k \geqslant l-m} a_k = O(1).$$

By choosing p sufficiently large (so that $p\gamma > 2$), we get

$$\begin{aligned}(n-1) n^{2l} \sum_{0 \leqslant i \leqslant m} n^{-(i+1)} \mathbb{P}\biggl(|K_{l-i}| > \frac{\theta}{n^l}\biggr) &\leqslant n^{2l} \max_{0 \leqslant i \leqslant m} \mathbb{P}\biggl(|K_{l-i}| > \frac{\theta}{n^l}\biggr) \\ &\leqslant C n^{2l+pl} a_{l-m}^p \leqslant C' n^{-l\beta},\end{aligned}$$

where $C, C', \beta > 0$ are independent of l. Finally, all the above yields that

$$\limsup_{l \to \infty} J \leqslant (\lambda_{c+4\theta})^2 (1 - n^{-m}).$$

Letting θ tend to zero and m to infinity, we get the desired inequality (5.16). □

Binomial perturbations of $\mathfrak{B}^\alpha$. Let ε be the common distribution of the independent identically distributed random variables $\{\varepsilon(B)\}_{B \in \mathcal{B}}$. In Theorem 5.1 one cannot completely omit the hypothesis that ε has bounded density. More precisely, we will show that *if ε contains a discrete component while the other hypotheses of Theorem 5.1 hold, then the Poisson convergence may fail.*

As an example, we choose the operator $\mathfrak{B}^\alpha$ of fractional derivative and consider its perturbation $\mathfrak{B}^\alpha(\omega)$ defined by independent identically distributed symmetrized binomial random variables. In other words, we take $\varepsilon = B_1 + \dots + B_n$, where $\{B_i\}$ are independent identically distributed symmetric Bernoulli random variables, that is, each B_i takes the values ± 1 with probability $1/2$.

The condition (3.11) holds with $\delta > 2$ if $\alpha = \delta/2 > 1$. By our choice, the common distribution function $\mathcal{N}(t)$ of the random variables $U_g(\omega)$, $g \in G$, is equal to the n-fold convolution of the distribution function $\mathcal{N}_\lambda(t)$, $\lambda = p^{-\alpha}$, of the infinite Bernoulli convolution defined in (4.11). Since $\alpha > 1$, we have $0 < \lambda < 1/2$. Therefore $\mathcal{N}_\lambda(t)$ is purely singular. On the other hand, by Proposition 6.1 in [23], the Fourier transform $\Phi_\lambda(x)$ of the function $\mathcal{N}_\lambda(t)$ satisfies

$$|\Phi_\lambda(x)| \leqslant \frac{C}{1 + |x|^\gamma}$$

for some $C = C(\lambda) > 0$ and $\gamma = \gamma(\lambda) > 0$ and for almost all λ, $0 < \lambda < 1$. By choosing $\alpha > 1$ in such a way that the number $\lambda = p^{-\alpha}$ does not belong to the exceptional set and taking a sufficiently large $n = n(\alpha)$, we see that the Fourier transform $\Phi(x)$ of the function $\mathcal{N}(t)$ satisfies

$$|\Phi(x)| = |\Phi_\lambda(x)|^n \leqslant \frac{C'}{1+|x|^2}$$

for some $C' > 0$. In particular, according to our choice, $\mathcal{N}(t)$ is absolutely continuous and has a continuous density. This shows that the density of states *exists* and is a *continuous* function, whereas ε is *discrete*. Thus $\mathfrak{B}^\alpha(\omega)$ with appropriately chosen $\alpha > 1$ and $\varepsilon(\omega)$ is as desired.

We now return to our general setting and prove that the Poisson convergence fails. There is no loss of generality in assuming that $\varepsilon(\{0\}) =: p_0 > 0$. We also assume that all forward degrees n_j are the same and are equal to n. Using the notation in the proof of Theorem 5.1, we write

$$U_g = \sum_{i=0}^{\infty} a_i \varepsilon_{ig} = a_0\varepsilon_{0g} + \widetilde{U}_g$$

and

$$Z_l^c = \sum_{g\in G_l} 1_{\{U_g\in I_l^c\}} \geqslant \sum_{g\in G_l} 1_{\{\widetilde{U}_g\in I_l^c\}} 1_{\{\varepsilon_{0g}=0\}} = \sum_{g\in G_l/G_1} 1_{\{U_g\in I_{l-1}^{c/n}\}} \sum_{a\in g} 1_{\{\varepsilon_{0a}=0\}}. \tag{5.18}$$

For every $g \in G_l/G_1$ we define a random variable

$$\mathcal{B}_g = \sum_{a\in g} 1_{\{\varepsilon_{0a}=0\}}.$$

Then $\{\mathcal{B}_g\}_{g\in G_l/G_1}$ are independent identically distributed binomial random variables with parameters (p_0, n). Furthermore, setting

$$\widetilde{Z}_{l-1}^{c/n} = \sum_{g\in G_l/G_1} 1_{\{U_g\in I_{l-1}^{c/n}\}}, \qquad \mathcal{Z}_l^c = \sum_{g\in G_l/G_1} 1_{\{U_g\in I_{l-1}^{c/n}\}} \sum_{a\in g} 1_{\{\varepsilon_{0a}=0\}},$$

we obtain

$$\mathcal{Z}_l^c \overset{d}{=} \sum_{j\geqslant 0}^{\tau} \mathcal{B}_j,$$

where $\{\mathcal{B}_i\}_{i=0}^\infty$ are independent identically distributed binomial random variables (with parameters (p_0, n)) which are independent of $\tau = \widetilde{Z}_{l-1}^{c/n}$.

Assume that the Poisson convergence holds. Then, as in the proof of Theorem 5.1, we can choose a subsequence $\{l_k\}$ such that along this subsequence $Z_l^c \to Z^c$ and $\widetilde{Z}_{l-1}^{c/n} \to \widetilde{Z}^{c/n}$, where Z^c and $\widetilde{Z}^{c/n}$ are Poisson random variables with intensities λ_c and $\lambda_{c/n}$ respectively. In particular, by (5.18) we have

$$\mathbb{P}(Z^c \geqslant 2) \geqslant \mathbb{P}(\widetilde{Z}^{c/n} \geqslant 1)\mathbb{P}(\mathcal{B}_0 \geqslant 2).$$

We obtain a contradiction since the left-hand side of this inequality has order c^2 as $c \to 0$ and the right-hand side has order c.

Bibliography

[1] F. J. Dyson, "The dynamics of a disordered linear chain", *Phys. Rev.* (2) **92**:6 (1953), 1331–1338.

[2] F. J. Dyson, "Existence of a phase transition in a one-dimensional Ising ferromagnet", *Comm. Math. Phys.* **12**:2 (1969), 91–107.

[3] A. Bovier, "The density of states in the Anderson model at weak disorder: a renormalization group analysis of the hierarchical model", *J. Statist. Phys.* **59**:3-4 (1990), 745–779.

[4] E. Kritchevski, "Hierarchical Anderson model", *Probability and mathematical physics*, CRM Proc. Lecture Notes, vol. 42, Amer. Math. Soc., Providence, RI 2007, pp. 309–322.

[5] E. Kritchevski, "Spectral localization in the hierarchical Anderson model", *Proc. Amer. Math. Soc.* **135**:5 (2007), 1431–1440.

[6] E. Kritchevski, "Poisson statistics of eigenvalues in the hierarchical Anderson model", *Ann. Henri Poincaré* **9**:4 (2008), 685–709.

[7] S. Molchanov, "Hierarchical random matrices and operators. Application to Anderson model", *Multidimensional statistical analysis and theory of random matrices* (Bowling Green, OH 1996), VSP, Utrecht 1996, pp. 179–194.

[8] S. Albeverio and W. Karwowski, "A random walk on p-adics – the generator and its spectrum", *Stochastic Process. Appl.* **53**:1 (1994), 1–22.

[9] J. Pearson and J. Bellissard, "Noncommutative Riemannian geometry and diffusion on ultrametric Cantor sets", *J. Noncommut. Geom.* **3**:3 (2009), 447–480.

[10] M. Aizenman and S. A. Molchanov, "Localization at large disorder and at extreme energies: an elementary derivation", *Comm. Math. Phys.* **157**:2 (1993), 245–278.

[11] N. Minami, "Local fluctuation of the spectrum of a multidimensional Anderson tight binding model", *Comm. Math. Phys.* **177**:3 (1996), 709–725.

[12] A. Bendikov, A. Grigor'yan, and C. Pittet, "On a class of Markov semigroups on discrete ultra-metric spaces", *Potential Anal.* **37**:2 (2012), 125–169.

[13] A. Bendikov, A. Grigoryan, C. Pittet, and W. Woess, "Isotropic Markov semigroups on ultra-metric spaces", *Uspekhi Mat. Nauk* **69**:4(418) (2014), 3–102; English transl., *Russian Math. Surveys* **69**:4 (2014), 589–680.

[14] J. Kigami, "Dirichlet forms and associated heat kernels on the Cantor set induced by random walks on trees", *Adv. Math.* **225**:5 (2010), 2674–2730.

[15] A. Bendikov and P. Krupski, "On the spectrum of the hierarchical Laplacian", *Potential Anal.* **41**:4 (2014), 1247–1266.

[16] V. S. Vladimirov, "Generalized functions over the field of p-adic numbers", *Uspekhi Mat. Nauk* **43**:5(263) (1988), 17–53; English transl., *Russian Math. Surveys* **43**:5 (1988), 19–64.

[17] V. S. Vladimirov and I. V. Volovich, "p-adic Schrödinger-type equation", *Lett. Math. Phys.* **18**:1 (1989), 43–53.

[18] V. S. Vladimirov, I. V. Volovich, and E. I. Zelenov, *p-adic analysis and mathematical physics*, Fizmatlit, Moscow 1994; English transl., Ser. Soviet East European Math., vol. 1, World Sci. Publ., River Edge, NJ 1994.

[19] E. Lukacs, *Characteristic functions*, 2nd ed., Hafner, New York 1970; Russian transl., Nauka, Moscow 1979.

[20] C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Grundlehren Math. Wiss., vol. 238, Springer-Verlag, New York–Berlin 1979.

[21] B. Solomyak, "Notes on Bernoulli convolutions", *Fractal geometry and applications: a jubilee of B. Mandelbrot*, Proc. Sympos. Pure Math., vol. 72, Part 1, Amer. Math. Soc., Providence, RI 2004, pp. 207–230.

[22] Y. Peres and B. Solomyak, "Absolute continuity of Bernoulli convolutions, a simple proof", *Math. Res. Lett.* **3**:2 (1996), 231–239.

[23] Y. Peres, W. Schlag, and B. Solomyak, "Sixty years of Bernoulli convolutions", *Fractal geometry and stochastics*, vol. II (Greifswald/Koserow 1998), Progr. Probab., vol. 46, Birkhäuser, Basel 2000, pp. 39–65.

[24] M. Del Muto and A. Figà-Talamanca, "Diffusion on locally compact ultrametric spaces", *Expo. Math.* **22**:3 (2004), 197–211.

[25] M. Del Muto and A. Figà-Talamanca, "Anisotropic diffusion on totally disconnected abelian groups", *Pacific J. Math.* **225**:2 (2006), 221–229.

[26] P. Cartier, "Fonctions harmoniques sur un arbre", Convegno di Calcolo delle Probabilità, INDAM (Rome 1971), Sympos. Math., vol. IX, Academic Press, London 1972, pp. 203–270.

[27] W. Woess, *Denumerable Markov chains. Generating functions, boundary theory, random walks on trees*, EMS Textbk. Math., Eur. Math. Soc., Zürich 2009.

[28] M. H. Taibleson, *Fourier analysis on local fields*, Princeton Univ. Press, Princeton, NJ; Univ. of Tokyo Press, Tokyo 1975.

[29] A. N. Kochubei, *Pseudo-differential equations and stochastics over non-Archimedian fields*, Monogr. Textbooks Pure Appl. Math., vol. 244, Marcel Dekker, New York 2001.

[30] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Nauka, Moscow 1965; English transl., Wolters–Noordhoff, Groningen 1971.

[31] B. Solomyak, "On the random series $\sum \pm\lambda^n$ (an Erdös problem)", *Ann. of Math.* (2) **142**:3 (1995), 611–625.

[32] M. R. Leadbetter, G. Lindgren, and H. Rootzén, *Extremes and related properties of random sequences and processes*, Springer Ser. Statist., Springer-Verlag, New York–Berlin 1983.

[33] S. Chatterjee, P. Diaconis, and E. Meckes, "Exchangeable pairs and Poisson approximation", *Probab. Surv.* **2** (2005), 64–106.

[34] R. Arratia, L. Goldstein, and L. Gordon, "Two moments suffice for Poisson approximations: the Chen–Stein method", *Ann. Probab.* **17**:1 (1989), 9–25.

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